# The Effect of Coefficient of Restitution for Two Collision Models

ALACI STELIAN[1],
FILOTE CONSTANTIN[2], CIORNEI FLORINA-CARMEN[1]
[1]Mechanics and Technologies Department
[2]The Computers, Electronics and Automation Department
"Stefan cel Mare" University
University Street, 13, Suceava
ROMANIA
[1]alaci@fim.usv.ro

*Abstract:* - The paper considers two models used in dynamic analysis of multibody system, describing the collision behaviour. One of the models accepts the internal friction work and the other, the plastic deformation work, but the common parameter is the coefficient of restitution. The effect of the coefficient of restitution is presented comparatively, in graphical manner, upon the normal approach, relative velocity versus time dependencies and phase maps. Following the different hypotheses, the dynamical system evolutions are significantly different. The two models studied represent the boundaries of a domain within which the actual system behaviour could be placed and the necessity of a complex model as amalgamation of the two models considered is emphasized.

*Key-Words:* - coefficient of restitution, phase maps, plastic deformation

## 1 Introduction

The collision phenomenon is a frequent mechanical phenomenon. Any mechanical interaction assumes bringing into contact the boundary surfaces of two bodies. If this approach is made with a very slow velocity, one of the conditions required by the static mechanical contact, [1], is fulfilled and the stress and strain fields from the contacting bodies can be obtained using the relations provided by contact mechanics. The stresses reach considerable values, [2], [3] if the initial contact is made theoretically in a point or line.

In practical application, the most frequent situations happen with considerable relative velocity and the electrostatics' conditions are broken. The time parameter appears in the relations describing the behaviour of the two bodies, the model belongs to the elastodynamic domain and the study of the phenomena becomes more intricate. For engineering applications, a series of simplifying assumptions were accepted for a rapid estimation of impact phenomena effects. The monograph due to Brach, [4] is a reference work where the hypothesis of rigid colliding bodies is accepted. Based on an opposite assumptions are the Timoshenko's works, [5], who considers perfectly elastic colliding bodies, Lankarani, [6] who elaborated a perfect plastic collision model and Flores, [7] who thought a model for which the kinetical energy variation is retrieved as work of internal friction.

## 2 Coefficient of Restitution (COR) for Unidimensional Collision

The simplest collision model is considered next, for the case of centric impact of two balls, Fig. 1. In the present paper, regardless of the considered model, the coefficient of restitution is considered in kinematical manner, as defined by Newton, [8]:

$$e = -\frac{\bar{n} \cdot (\bar{v}_2'' - \bar{v}_1'')}{\bar{n} \cdot (\bar{v}_2' - \bar{v}_1')} \tag{1}$$

where $\bar{n}$ is the unit vector of the normal in the initial contact point, $\bar{v}$ is the velocity, by $(\,'\,)$ and $(\,''\,)$ are denoted the parameters corresponding to the initial and the final moment of collision, respectively.

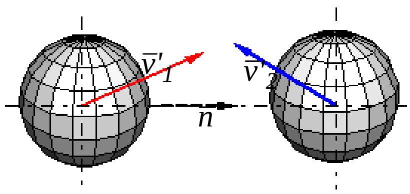


Fig.1. Collision of two spheres

The definition (1) is appropriate for a large variety of collision situations, but the effect of the value of coefficient of restitution differs, depending on the adopted model of collision. To illustrate this effect, two of the models from literature are considered, the Flores and the Lankarani model, respectively.

## 3 Two Models for Describing the Dynamical Behaviour of Systems with Collisions

The two models envisaged differ by the nature of the work corresponding to the variation of kinetical energy of the system.

*A. Variation of Kinetical Energy Retrieved as Work of Internal Friction*

The model describing the collision force variation between the two balls assuming that the kinetical energy variation is completely transformed into work of internal friction is proposed by Flores, [7].

This approach was initially adopted by Lankarani, [9], who developed the differential equation describing the dynamical behaviour of the system.

The collision process has two phases, the compression and restitution. Lankarani considered that the work of internal friction has equal values for the two collision phases. This assumption led to a model applicable for a narrow range of collisions, more precisely, for materials exhibiting quasielastic behaviour. The cause of this restriction is the fact that, after impact, the coefficient of restitution given by the model was greater than the COR from initial moment. This aspect becomes more pregnant as the collision is described by a smaller COR. Therefore, the model could be applied only for elastic materials, with $e > 0.9$.

The main problem occurring both in Lankarani's model and Flores's one, is finding the damping coefficient, but this is a frequent major challenge in engineering applications, [10]. Another problem met in models of damped nonlinear systems is the requirement of closing in origin the hysteresis loop. As shown, [11], using Coulombian friction, the hysteresis graph does not pass through origin. Hunt and Crossley, [12] used a dashpot with variable damping coefficient and thus it was ensured the closing of the hysteresis loop. The energy assumption used by Lankarani was eliminated by Flores and the following equation was obtained:

$$F = Kx^n \left[ 1 + \frac{8(1-e)}{5e} \frac{v}{v'_0} \right] \tag{2}$$

where $v$ is the velocity between the initial contacting points at current time, and $v'_0$ is the initial impact velocity; $K$ is a constant considering the elastic characteristics and local geometry from the vicinity of contact point:

$$K = \frac{4}{3\pi(\eta_1 + \eta_2)} \sqrt{\frac{R_1 R_2}{R_1 + R_2}} \tag{3}$$

$$\eta_{1,2} = \frac{1 - \nu_{1,2}^2}{E_{1,2}} \tag{4}$$

where $R_{1,2}$ are the radii of the two balls, $E_{1,2}$, $\nu_{1,2}$ are the Young moduli and the Poisson coefficients of the two materials, respectively. The model can also be applied for bodies with geometries differing from the spherical one, but the radii of the balls must be replaced by the principal curvature radii of the bodies in the contact point. The linear contact of two cylinders is the only situation when this relation cannot be applied. For this case, an analytical model cannot be developed due to the lack of analytical dependence between the normal approach and contact force. This situation is largely observed in practice and the necessity of solving such problems led to an empirical solution for the dependence relating the normal force to normal approach between the two cylinders. Norden, [13], proposed a series of relations to be applied for different particular cases of linear contact. Machado, [14], quoting Brandlein, illustrates a relation force-approach of the following form:

$$F = Kx^{1.08} \tag{5}$$

where K has the same significance as in (3).

The relation obtained by Flores differs from Lankarani model only by the expression of damping coefficient. Equation (2) ensures equal values for post impact velocities, both given by the model and by the COR definition, for any value of the coefficient of restitution used as input value. The Flores model proves that the work of internal forces during compression phase is greater than the work from restitution phase. The difference between these values increases with diminished COR.

*B. Variation of Kinetical Energy Retrieved as Work of Plastic Deformation*

The model considering complete transformation of kinetical energy variation into work of plastic deformation, proposed by Lankarani, [6], is presented in Fig. 2.

For the case of this model, the loading curve has the expression:

$$F = K\,x^{3/2} \tag{6}$$

and for the restitution phase, the force variation is described by:

$$F = F_m \left( \frac{x - x_p}{x_m - x_p} \right)^{3/2} \qquad (7)$$

where K is the same constant considered by Flores in (2). The maximum normal approach is denoted $x_m$, the final plastic deformation is $x_p$ and the maximum impact force is $F_m$.

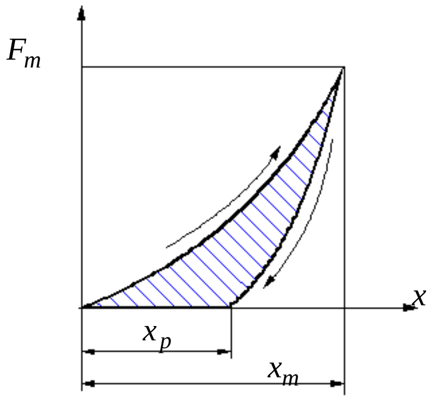


Fig. 2. Hysteresis loop for collision with plastic deformation, [6]

The presented models are the extreme cases between which any actual collisions can be considered.

## 4 Comparison between the Two Models

The collision between a steel ball of radius $R = 0.015\text{m}$ that falls free from a height of $0.4\text{m}$ and the frontal surface of a fixed cylinder is studied. The authors integrated the equations describing the two models, (2),(6) and (7) and the results are plotted in Fig. 3, relative displacement variation with time and in Fig. 4, relative velocity variation with time, for different values of coefficient of restitution. From Fig. 3 and Fig. 4 it can be noticed the important influence of COR upon variation with time of displacement and relative velocity. One must emphasise that for COR value $e = 0.999$, the hysteresis loops corresponding to the two models are practically the same and at the same time, overlapping the loading-unloading curves from the perfect elastic model, due to Timoshenko.

For the perfect elastic model, based on the approach-normal force relation from Hertzian theory, Timoshenko shows that the periods for compression and restitution phases are the same, denoted by $t_H$, and also finds the values of maximum approach $x_H$ and maximum impact force. The values $x_H$ and $t_H$ were used for obtaining dimensionless parameters in abscise from the analysis presented in Fig. 3 and Fig. 4 respectively. The curves traced by continuous line correspond to Flores model and the dashed lines refer to the Lankarani model. For both models, the red lines are plotted for compression phase and the blue curves model the restitution phase.The maximum approach, Fig. 3, is always greater for the model exhibiting plastic deformation and although for quasielastic collisions the hysteresis loops are identical, the plots for approach variation versus time overlap only for the compression phase. The impact time period increases for the model with damping while for the model with plastic deformation, decreases and is limited to the Hertzian value. For quasielastic collisions, the relative velocity variation, Fig. 4, is practically the same for compression phase. For the restitution phase, the velocity is greater for the model with damping. With decreasing COR value, for compression phase the velocity variation presents the same shape for plastic deformation model but for the damped model, the velocity presents a higher gradient. For the restitution phase, the velocity gradient is higher for the model with plastic deformation. The phase diagrams, Fig. 5, are identical for both models, in shape of elliptical arcs, as Flores presumed, [7], for a quasiellastic collision. With COR decrease, the two diagrams split and the differences are obvious; both models present stability points and the damping model has this point in the axes origin.

## 5 Conclusions

The paper studies two boundary models describing the dynamic behaviour of two balls in collision: one considers that the lost energy is entirely recovered as internal friction work and the other one, regards as the lost energy is completely converted into work of plastic deformation. The dynamical behaviour of the two systems is described by nonlinear differential equations. These equations were integrated in order to compare for the two models the effect of the coefficient of restitution upon the time dependencies of relative velocity and normal approach and for plotting the phase diagrams. The conclusion reached is that system behaviour is completely dissimilar for the two hypothesis concerning energy losses. In tangible cases none of these energetic losses is removed and thus intermediary model thinking about both phenomena is necessary.

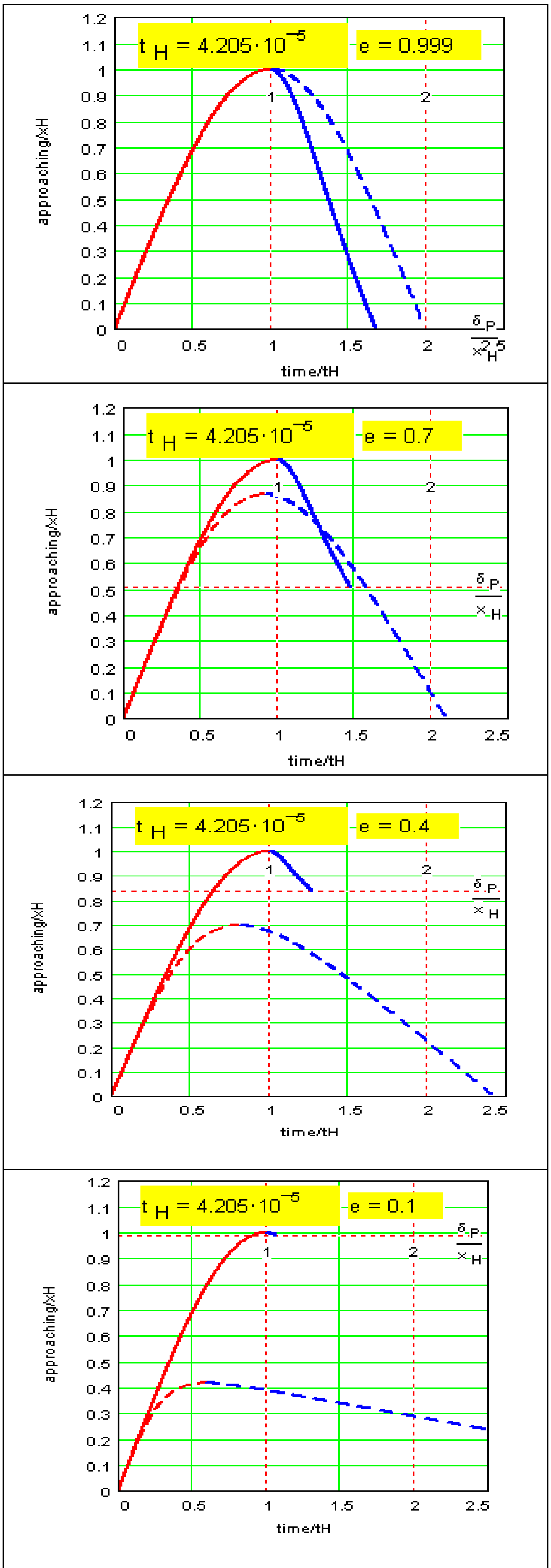

Fig.3. Relative displacement versus time

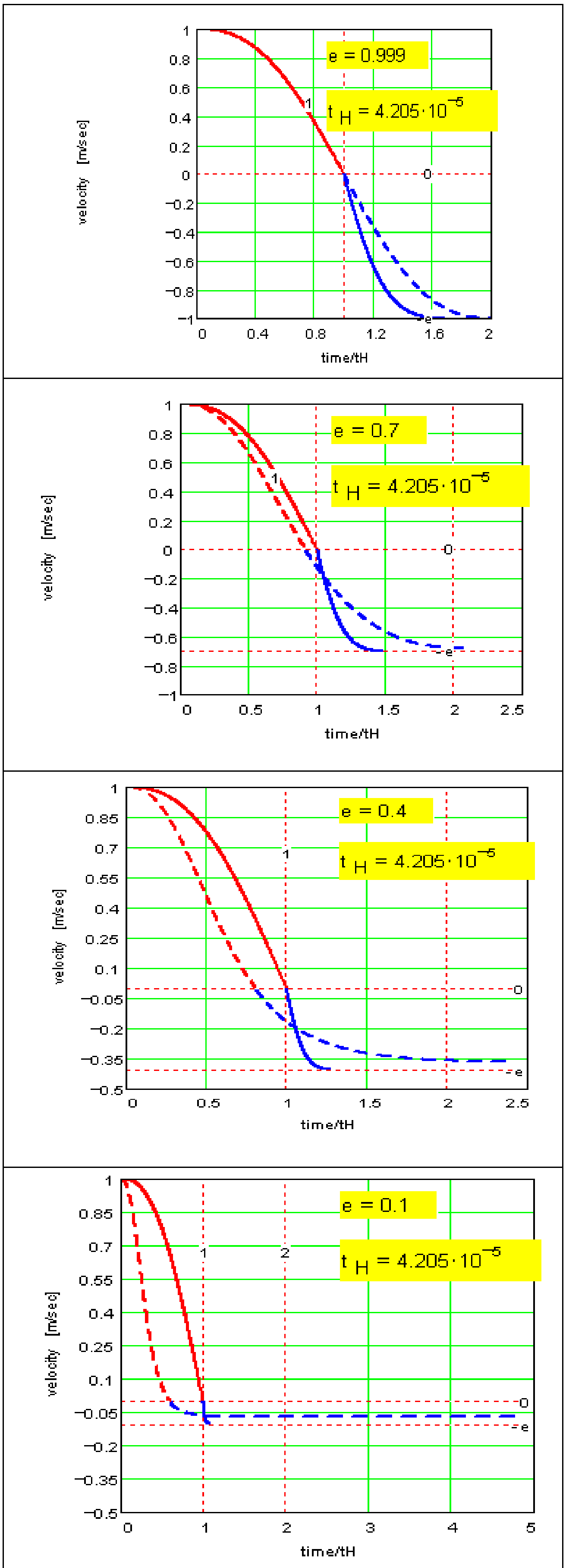

Fig.4. Relative velocity variation with time

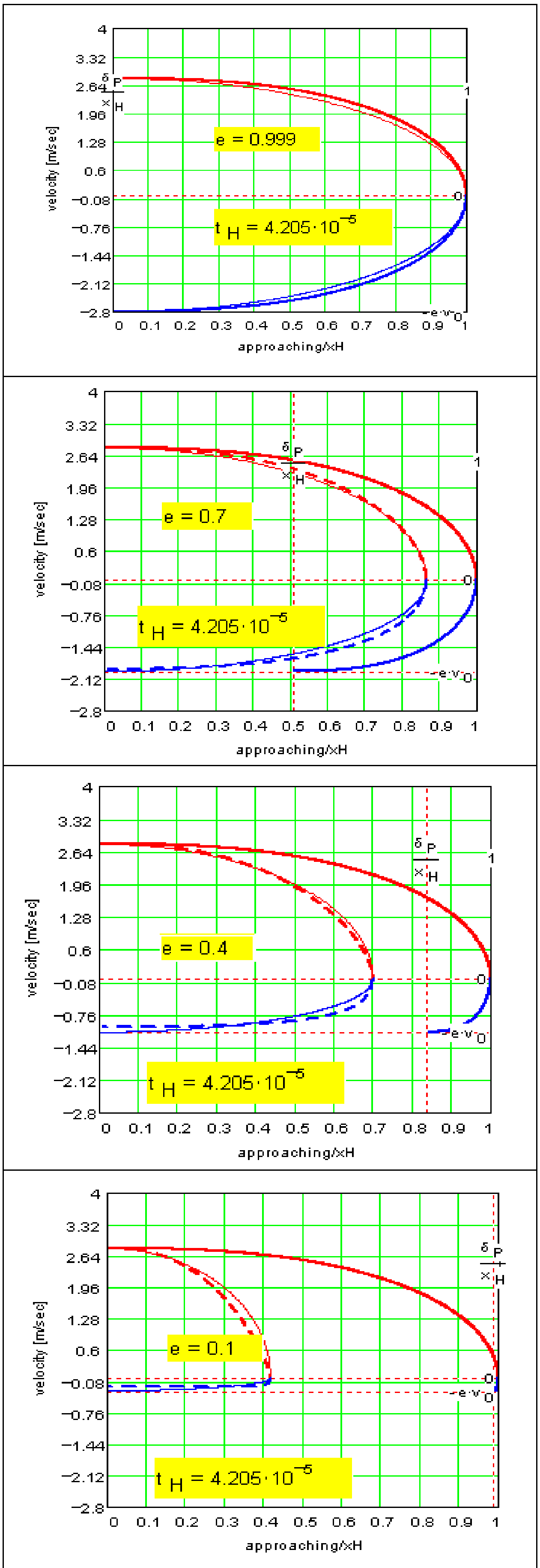


Fig.5. Phase diagram

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