Study on the Polyhedral Triangular Tessellation of the Sphere

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Abstract:- In the last years the polyhedral triangular tessellation of the sphere was start from a regular polyhedron, especially from one with triangular faces. The frequently used tessellation method consists in the subdivision of the face of the regular polyhedron and the projection of the nodes of the resulting net on the sphere. This method is easy to be compute and that is why computer programmes have been develop from it.
There are other tessellation methods, which are difficult to compute, but enable a better approximation of the sphere, or a higher aesthetic value. For these methods the authors worked out a library of tessellated faces of regular polyhedra with triangular faces. The methods have as base the subdivision of the plane polyhedral triangles or the subdivision of the projection of these triangles on the circumscribed sphere.
The library contains 2 to 16-frequency tessellated icosahedral, octahedral and tetrahedral triangles. It also contains patterns to the spherical wrapping up of the polyhedral patch surface.

Key- Words-: regular polihedra, sphere, tessellation, subdivision methods, wrapping up patterns, library.

1 Introduction

The word tessellation originates from the ancient Greek word *tessares*, meaning four or four-cornered. The Latin word *tessera* means cube or die and*tessella* refers to small squares laid in a mosaic. From these ancient words derive similar terms in various practical applications.

In mathematics, tessellation refers to the study of "tiling" or how regular shapes can be placed to fill an infinite space with no gaps and no overlapping shapes. This is a mathematical discipline which has been evolving since the early 17th century and formally recognized in the 19th century.

Polyhedral triangularly tessellated surfaces inscribed to the sphere are used in different engineering fields. In the mechanical engineering they are used for containers, lids, casings, protecting covers, support structures, packages and, not least, for decorative parts. In the most of these applications the tessellation starts from the three regular polyhedra with triangular faces, especially from the icosahedron. The most used tessellation method consists in the face subdivision of a regular polyhedron in smaller triangles using division points on the edges of the face. Then the nodes of the resulting net are project on the circumscribed sphere. These projected points are the vertices of the tessellated polyhedral surface. As this method is easy to be compute, computer programmes have being developed starting from it.

2 Tesselation methods

Other tessellation methods, which enable, either a better approximation of the sphere, or an improved aesthetic aspect, are difficult to compute. For these methods the authors worked out a library of tessellated icosahedral, octahedral and tetraedral triangles. The used tessellation methods are presented in Table 1, which also contains general recommendations for choosing.

There are three categories of such tessellation methods:

1. Division of triangle lines and subdivision of the triangle are made on the plan triangle.

2. Division of triangle lines is made on the spherical triangle and subdivision is made on the plan triangle.

3. Division of triangle lines and subdivision of the triangle are made on the spherical triangle.

The spherical triangle is the projection of the triangular face of the regular polyhedron on the circumscribed sphere. The projection centre is the centre of the sphere.

The division made on the spherical triangle leads to a better approximation of the sphere and the subdivision of the spherical triangle to an improved aesthetic aspect.

The library contains wire-frame and 3D-face constructions for division frequencies between 2 and 16.

Fig. 1 and 2 show two of the library pages.

The library also contains patterns to the spherical wrapping up of the polyhedral surface to cover the whole sphere. Fig. 3 shows these patterns.

Fig. 4 shows an example of a polyhedral patch surface obtained from an 11-frequency tessellated octahedral triangle using the tripetal subdivision of the spherical triangle. Fig. 5 shows a patched polyhedral surface obtained from an 11-frequency tessellated octahedral triangle using the concentric subdivision of the spherical triangle.

For the tessellation of polyhedral caps they usually start from polyhedral pyramids. In this case the spherical triangle must be the projection of the face of the polyhedral pyramid on the circumscribed sphere. This projection should be made from the central point of the base of the pyramid. The library also contains pages with tessellated spherical triangles corresponding to the polyhedral pyramids. The used subdivision methods are the two sided and the tripetal subdivision.

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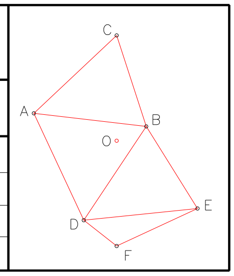
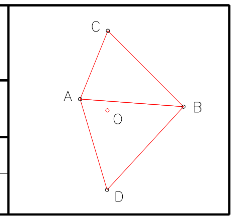
ICOSAHEDRAL COVERING				
CURRENT NUMBER	SYMMETRY OPERATION	OBJECT	SYMMETRY ELEMENT	OPERATION ORDER
1	MIRROR 3D	TRIANGLE ABC	PLANE OAB	2
2	MIRROR 3D	TRIANGLE ABD	PLANE OBD	2
3	MIRROR 3D	TRIANGLE BDE	PLANE ODE	2
4	ARRAY 3D	ZONE CADFEBC	AXIS OC	5



OCTAHEDRAL COVERING				
CURRENT NUMBER	SYMMETRY OPERATION	OBJECT	SYMMETRY ELEMENT	OPERATION ORDER
1	MIRROR 3D	TRIANGLE ABC	PLANE OAB	2
2	ARRAY 3D	ZONE CADBC	AXIS OC	4



TETRAHEDRAL COVERING				
CURRENT NUMBER	SYMMETRY OPERATION	OBJECT	SYMMETRY ELEMENT	OPERATION ORDER
1	MIRROR 3D	TRIANGLE ABC	PLANE OAB	2
2	MOVE	TRIANGLE ABD	$O - O_1$	–
3	ARRAY 3D	TRIANGLE ABD	AXIS OC	3
4	MOVE	TRIANGLE $A_1B_1D_1$	$O_1 - O$	–

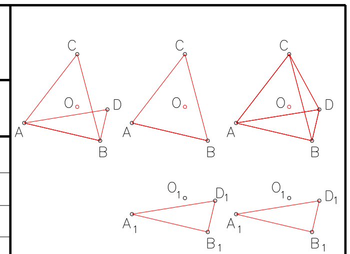
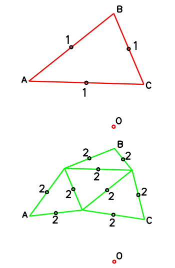
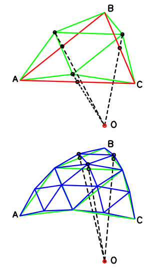
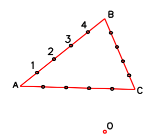
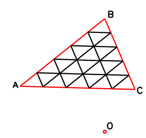
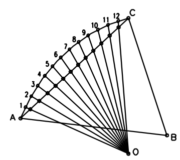
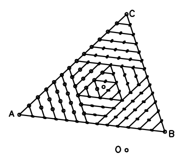
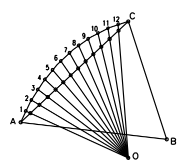
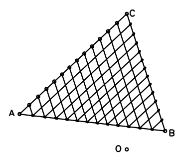
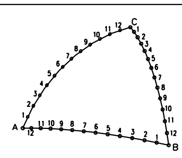
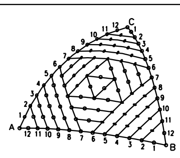
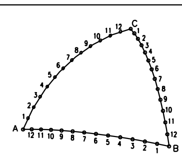
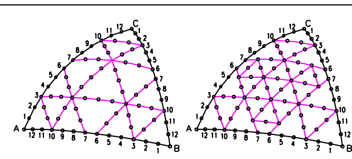
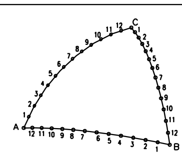
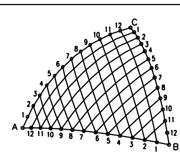
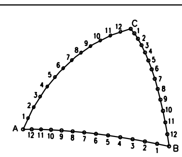
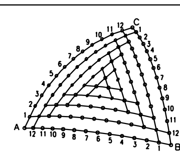
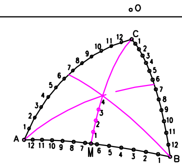
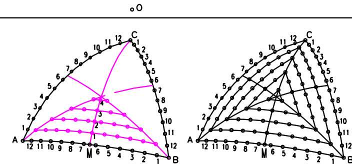


Fig.3 Paterns to the spherical wrapping up of the polyedral surface.

Table 1 The used tesselation methods

Tessellating operation	Division	Subdivision	Recommendations for use
Edge halving			It gives a good accuracy and uniformity of sphere approximation
Three-sided subdivision of the polyhedron face			It gives a good accuracy of sphere approximation. It is easy to compute.
Division of the edge of the spherical triangle and tripetal subdivision of the polyhedron face			It gives a good accuracy and uniformity of sphere approximation It is easy to compute.
Division of the edge of the spherical triangle and two-sided subdivision of the polyhedron face			Approximation accuracy and uniformity are low, but it enables the tessellation of caps without margin adjustment.
Tripetal subdivision of the spherical triangle			It gives a good accuracy and uniformity. of sphere approximation. It has a good aesthetic look.
Quasihalving			
Two-sided subdivision of the spherical triangle			Approximation accuracy and uniformity are low, but it enables the tessellation of caps without margin adjustment.
Concentric subdivision of the spherical triangle			It gives a very good approximation accuracy, but a lower uniformity. It has a good aesthetic look.
Division of the edge and the median arc of the spherical triangle and concentric subdivision of this triangle			It gives the best approximation accuracy and uniformity and is suitable for engineering purposes.

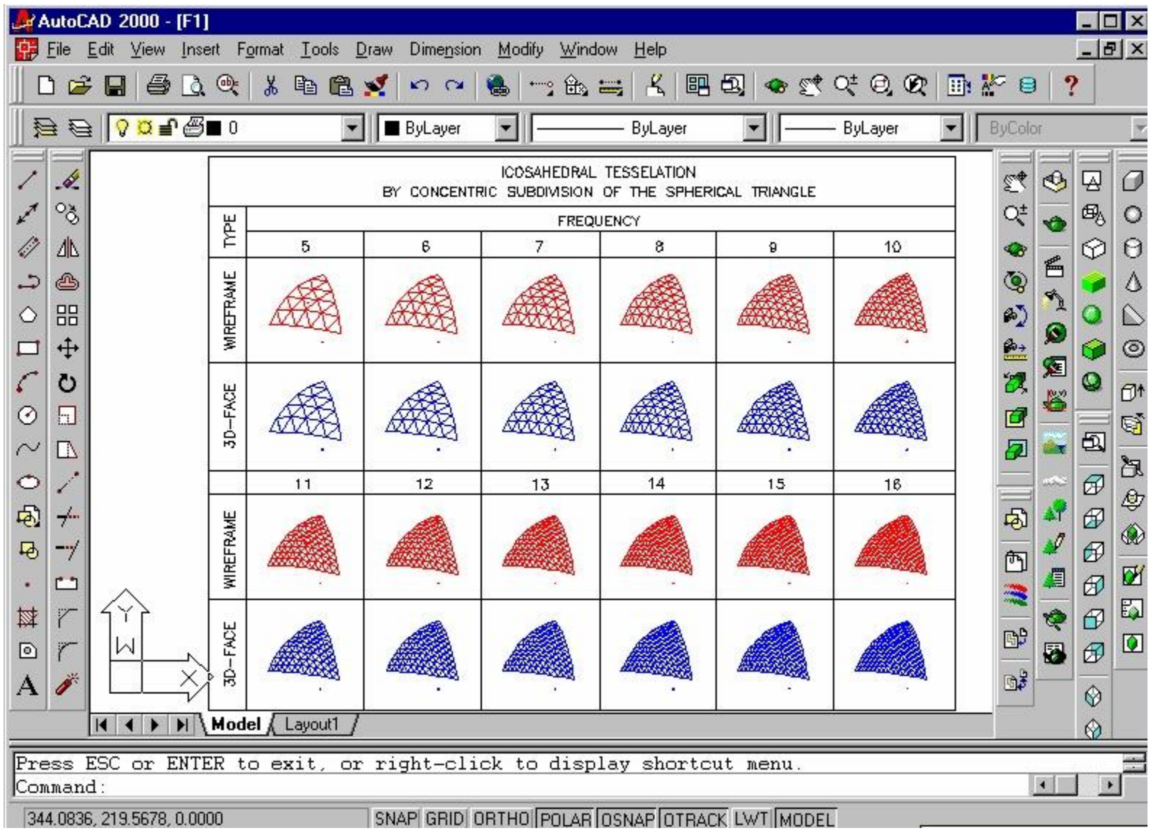


Fig.1 Library page.

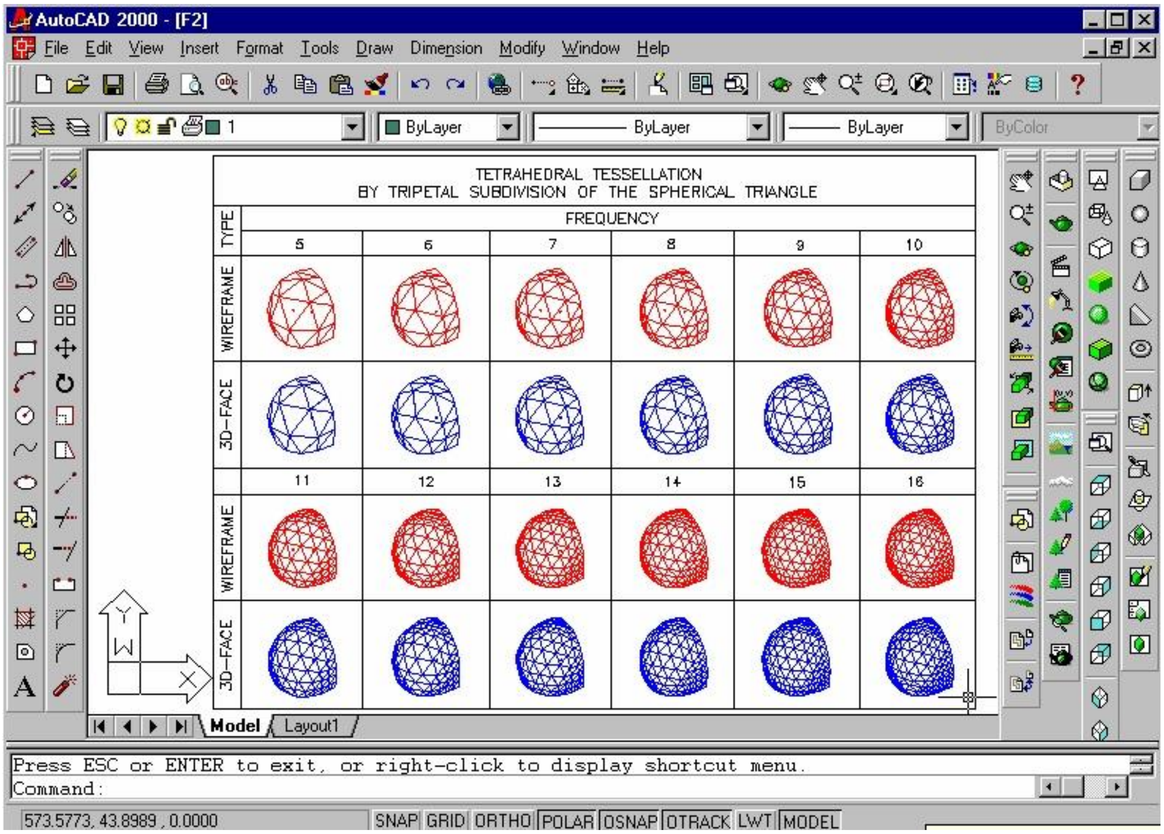


Fig.2 Library page.

The necessary time to construct the library pages for a tessellation method were less then those for the writing of a computer programme for the same method.
The drawings contained in the library are AutoCAD drawings, which may be directly used in applications.

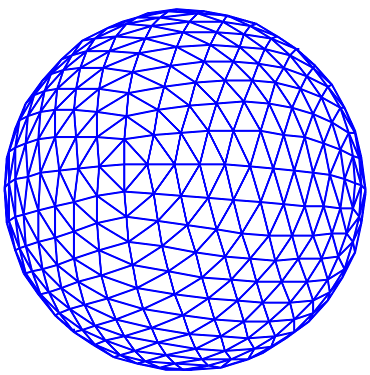


Fig.5 Patched polyedral surface obtained from an 11-frequency tesselated octahedral triangle using the concentric subdivision of the spherical triangle.

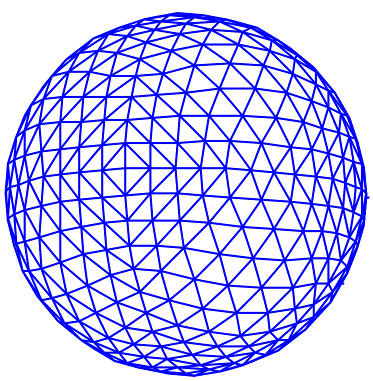


Fig.4 Example of a polyedral patch surface obtained from an 11-frequency tesselated octahedral triangle
using the tripetal subdivision of the spherical triangle.

3 Conclusions

Tessellations have practical applications in many realms, from art and architecture to science, technology, and production.
In design and architecture, tessellation refers to the paving of walls, floors, or other surfaces with a pattern of small tiles – tesserae - made of ceramics, glass, metal leaf, stone, or other materials. These tesserae normally are cut into geometric shapes that fit together perfectly in either simple or complex designs in a seemingly infinite pattern while providing continuous surface coverage. This is an ancient technique that you can see in buildings and wall murals in Greece, Italy, Turkey, India, and many other countries. Tessellations are particularly prominent in Islamic art, which forbids representational images of God; therefore, its designs favor abstract forms with mathematical underpinnings.
Tessellations appear in various scientific and engineering disciplines. Chemical discoveries show that certain carbon molecules take the shape of a truncated isocahedren. Geodesic domes are both theoretical 3D geometric constructs and built structures.

References:

[1] D. Anderson: *Constructing Goedesic Patch Sufraces.*1998, http://W3.one.net/˜monkey/geodesics/folding/
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[3] D. Cohn's. *AutoCAD R14 Essentials. A concise reference*. Addison Wesley Longman, 1999.