Heat Distribution on Thin Plates with Point Sources

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Abstract: Steady heat distribution problems with point sources are considered. The problems are governed by Poisson equations. Such Poisson equations, generally, may not be solved analytically. To solve these equations, numerical methods called Dual Reciprocity Methods (DRM) are employed. The DRM is applied to solve problems involving steady heat distribution on a thin plate with point sources. Variations in the sources position are considered. The sources are located along a diagonal line and a line of symmetry of the plate. Applying these methods, numerical solutions of the problems are presented and discussed. Sources located near the center of the plates result in higher value of total temperature. Moreover, total temperature of the plate, with sources placed along the diagonal line, is higher than that resulted from sources at the line of symmetry.

Key–Words: Heat distribution, Point source, Dual reciprocity method.

1 Introduction

Heat distribution in media is one of Physics problems that is modelled mathematically. One of the purpose of modelling the problems mathematically is to study and solve the problem. A number of researchers have derived and study mathematical model of heat distribution problems. Such researchers are Haberman [3], Persson [4], and Tsai and Eagar [7].

There are, generally, two methods to solve mathematical models, analytically and numerically. For analytical method, the problems that can be solved using this method are limited, mostly simple problems. For more general problems, the method used are numerical methods. A class of the numerical methods that may be applied to solve heat distribution problems are Boundary Element Methods (BEM). These methods have been employed to solve various problems such as infiltration problems [2, 6], crack problems [5], and stress intensity [1]. In this paper, we employ a BEM which is known as Dual Reciprocity Method (DRM) to solve the problems.

2 Problem Formulation

We consider a homogeneous isotropic square thin plate. The temperature of the sides of the square is maintained at 0. Here, dimension of the temperature is omitted. The plate with the boundary conditions is illustrated in Figure 1.

There are point sources in the square thin plate

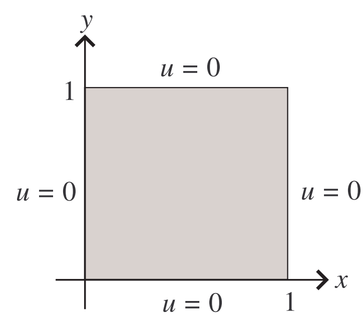

Figure 1: A thin plate with temperature of all the sides are maintained at 0.

with temperature maintained at 100. Given this situation, it is required to determine distribution of temperature on the thin plate.

3 Basic Equations

Problems involving two dimensional heat conduction is governed by

$$c\rho\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + Q(x, y), \qquad (1)$$

where u is the temperature, c is specific heat capacity, ρ is the density of the material, K_0 is the thermal conductivity, and Q is the source.

For steady heat conduction problems on thin plates with n point heat sources, Equation (1) can be written as

$$\frac{K_0}{c\rho}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]+\sum_{i=1}^{n} Q_i\delta(x,y;x_i,y_i)=0, \quad (2)$$

where (x_i, y_i) is point of source i, $i = 1, 2, \cdots, n$, Q_i is the source at point (x_i, y_i), and $\delta(x, y; x_i, y_i)$ is the Dirac delta function with source at (x_i, y_i). In this study $K_0/c\rho$ is set to be 1. The governing equation (2) is defined over a region Ω and its boundary, denoted by Γ. The region Ω is defined as

$$\{(x,y) : 0 < x < 1 \text{ and } 0 < y < 1\}. \quad (3)$$

Equation (2), which is defined over $\Omega \cup \Gamma$, may be solved numerically using a DRBEM. Following the procedure of the DRBEM, an integral equation for solution of Equation (2) is

$$\begin{aligned}\lambda(a,b)u(a,b) &= \int\limits_{\Gamma}\left[u(x,y)\frac{\partial}{\partial n}\mathcal{U}(x,y;a,b)\right.\\ &\quad \left.-\mathcal{U}(x,y;a,b)\frac{\partial}{\partial n}u(x,y)\right]ds\\ &\quad -\sum_{i=1}^{n}\mathcal{U}(x,y;a,b)\times\\ &\qquad Q_i\delta(x,y;x_i,y_i)dxdy, \quad (4)\end{aligned}$$

which can be written as

$$\begin{aligned}\lambda(a,b)u(a,b) &= \int\limits_{\Gamma}\left[u(x,y)\frac{\partial}{\partial n}\mathcal{U}(x,y;a,b)\right.\\ &\quad \left.-\mathcal{U}(x,y;a,b)\frac{\partial}{\partial n}u(x,y)\right]ds\\ &\quad -\sum_{i=1}^{n}Q_i\mathcal{U}(x_i,y_i;a,b), \quad (5)\end{aligned}$$

where

$$\mathcal{U}(x,y;a,b)=\frac{1}{2\pi}\ln\sqrt{(x-a)^2+(y-b)^2},$$

is the fundamental solution of Laplace equation, and

$$\lambda(a,b)=\begin{cases}1, & \text{if } (a,b)\in\Omega\\ \frac{1}{2} & \text{if } (a,b) \text{ on smooth part of } \Gamma\end{cases}.$$

Implementing DRBEM needs a number of elements and interior collocation points. The boundary Γ is discretized into a number of elements or segments joined end to end. Let K be the number of segments and L be the number of interior collocation points. Segments $\Gamma^{(k)}$, $k = 1, 2, \cdots, K$ are the elements or segments such that $\Gamma \approx \bigcup_{k=1}^{K}\Gamma^{(k)}$. For $k = 1, 2, \cdots, K$, point $(a^{(k)}, b^{(k)})$ is the mid point of segment $C^{(k)}$, which is the collocation points at $C^{(k)}$. Points $(a^{(l)}, b^{(l)})$, $l = K + 1, K + 2, \cdots, K + L$, are the interior collocation points. Using these elements and collocation points, Equation (5) may be approximated by

$$\begin{aligned}\lambda(a,b)u(a,b) &= \sum_{k=1}^{K}\left[\mathbf{G}_2^{(k)}(a,b)u^{(k)}\right.\\ &\quad \left.-\mathbf{G}_1^{(k)}(a,b)u_n^{(k)}\right]\\ &\quad +\sum_{i=1}^{n}Q_i\mathcal{U}(x_i,y_i;a,b). \quad (6)\end{aligned}$$

By substituting (a, b) with $(a^{(m)}, b^{(m)})$, $m = 1, 2, \cdots, K + L$ we may obtain a system of linear algebraic equations

$$\begin{aligned}\lambda^{(m)}u^{(m)} &= \sum_{k=1}^{K}\left[\mathbf{G}_2^{(k)}(a^{(m)},b^{(m)})u^{(k)}\right.\\ &\quad \left.-\mathbf{G}_1^{(k)}(a^{(m)},b^{(m)})u_n^{(k)}\right]\\ &\quad +\sum_{i=1}^{n}Q_i\mathcal{U}(x_i,y_i;a^{(m)},b^{(m)}),\\ &\quad m=1,2,\cdots,K+L, \quad (7)\end{aligned}$$

where $\lambda^{(m)} = \lambda(a^{(m)}, b^{(m)})$, $u^{(m)}$ and $u_n^{(m)}$ are respectively the values of u and $\partial u/\partial n$ at $(a^{(m)}, b^{(m)})$,

$$\mathbf{G}_1^{(k)}(a^{(m)},b^{(m)})=\int\limits_{\Gamma^{(k)}}\mathcal{U}(x,y;a^{(m)},b^{(m)})ds,$$

and

$$\mathbf{G}_2^{(k)}(a^{(m)},b^{(m)})=\int\limits_{\Gamma^{(k)}}\frac{\partial}{\partial n}\left[\mathcal{U}(x,y;a^{(m)},b^{(m)})\right]ds.$$

Solving the system of linear algebraic equation (7), we obtain the values of u and $\partial u/\partial n$ at collocation points. Value of u at any point $(a, b) \in \Omega \cup \Gamma$, may be computed using Equation (6).

4 Results and Discussion

The method described in Section 3 is applied to solve problems involving steady heat distribution on thin plates described in Figure 1. The number of point sources in this study are two, Q_1 and Q_2. The values of the two sources are $Q_1 = Q_2 = 100$. We consider two classes of problems. The first class of problems are problems with sources at a diagonal. The other class of problems are problems with sources at the line $y = 0.5$. The position of sources are summarized in Table 1.

Table 1: List of the sources positions.

Source position	Source Points		
Along diagonal	(0.1,0.1) and (0.9,0.9)	(0.25,0.25) and (0.75,0.75)	(0.4,0.4) and (0.6,0.6)
Notation	D1	D2	D3
Along $y = 0.5$	(0.1,0.5) and (0.9,0.5)	(0.25,0.5) and (0.75,0.5)	(0.4,0.5) and (0.6,0.5)
Notation	L1	L2	L3

To apply the method presented in Section 3, the numbers of elements and interior collocation points used are 200 and 225 respectively. Some of the results obtained using the method with these elements and interior collocation points are presented in Figure 2 - Figure 4, and Table 2.

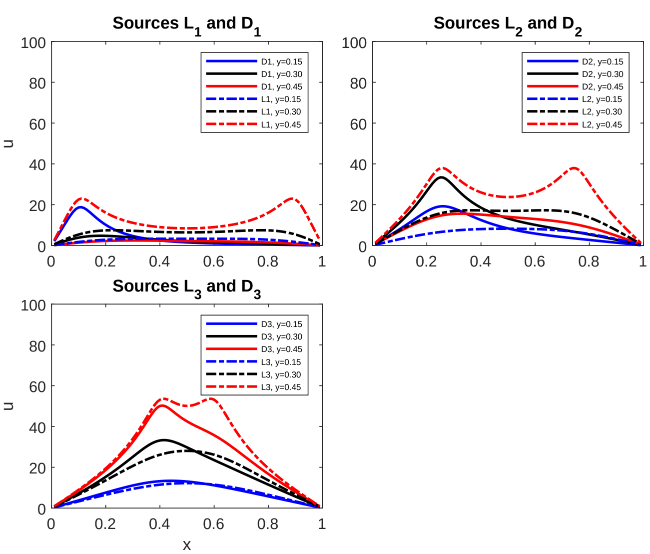

Figure 2: Graphs of u along x-axis at selected values of y.

Figure 2 shows numerical values of u along selected lines. The selected lines are $y = 0.15$, $y = 0.30$, and $y = 0.45$. It can be seen that as the sources go nearer the middle of the plates, temperature of the plates gets higher. It can also be seen in the figure that

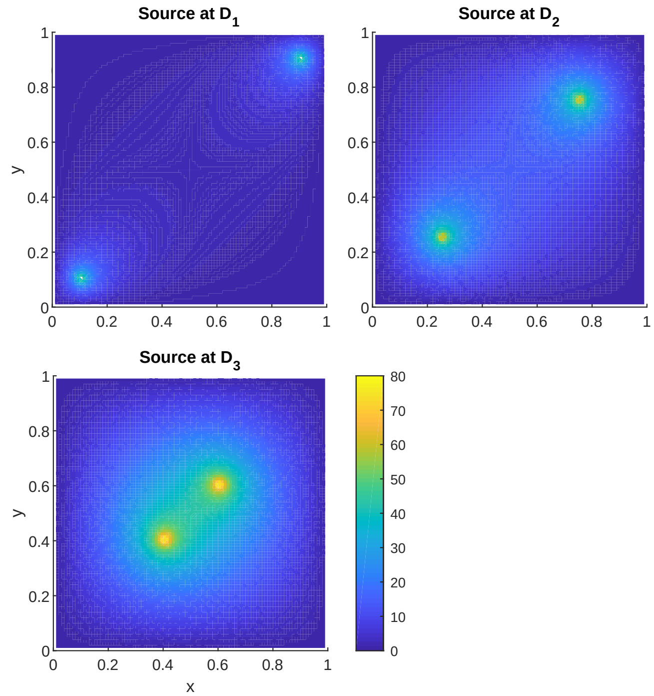

Figure 3: Surfaceplot of u for problems with sources at a diagonal.

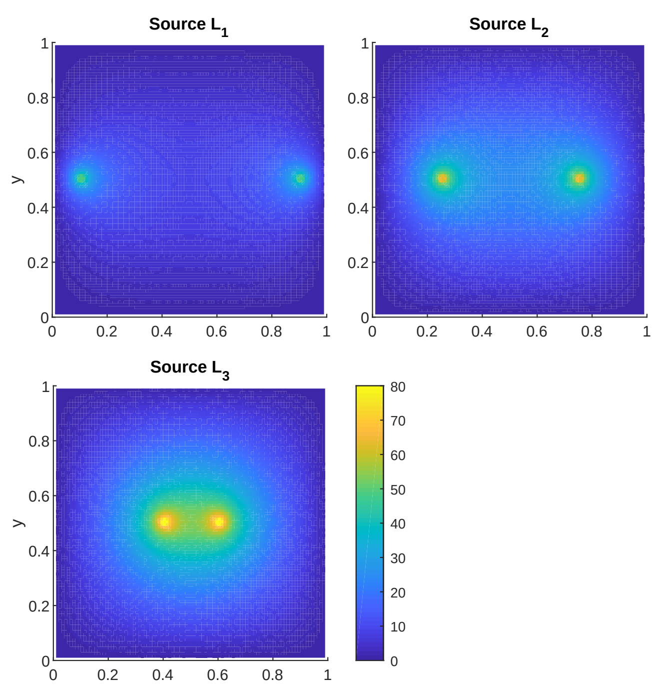

Figure 4: Surfaceplot of u for problems with sources at line $y = 0.5$.

the highest temperature occurs at the line $y = 0.45$ for sources at the line $y = 0.5$, especially when the sources are nearer to point (0.5,0.5).

The distribution of the temperature over the plates are shown in Figure 3 and Figure 4. The surface plots are generated by computing numerical values of 99×99 point equally spaced for each plate. Figure 3 shows surface plots of temperature over plates for sources at diagonal. From the surface plots, we can observe that the nearer the sources to point (0.5,0.5) the higher temperature distribution on the plates. Figure 4 presents surface plots of temperature with source at line $y = 0.5$. As before, the sources at (0.4,0.5) and (0.6,0.5), which are nearest to point (0.5,0.5), result in higher temperature over the plates.

Total temperature over the plates may be computed using formula

$$\int_0^1 \int_0^1 u(x,y)\, dxdy.$$

Since u may not be obtained analytically, the formula above may be approximated using the numerical values of u used to generate the surface plots. Using these values of u, the formula may be approximated by

$$0.01 \quad \times \quad \left[\sum_{i=1}^{99} \sum_{j=1}^{99} u(i,j) + \sum_{i=1}^{99} u(i,99) + \sum_{j=1}^{99} u(99,j) + u(99,99) \right], \qquad (8)$$

where $u(i,j)$ is the numerical value of u at point $(0.01 \times i, 0.01 \times j)$. The results obtained using Formula (8) are shown in Table 2.

Table 2: Total amount of temperature on the plates.

Source position	D1	D2	D3
Total temperature	261.86	906.14	1375.31
Source position	L1	L2	L3
Total temperature	581.26	1147.12	1423.48

From Table 2, the lowest value of total temperature occurs when the sources are placed at (0.25,0.25) and (0.75,0.75). The highest value of total temperature is on the plate with sources at (0.5,0.4) and (0.5,0.6). These results indicate that sources at a diagonal and near the boundary result in lower total temperature. On the other hand, sources at line $y = 0.5$ and near the centre of the plates result in higher total temperature on the plates.

5 Conclusion

The results described and presented in the Section 4 show that the sources of heat affect the distribution of temperature on the plates. The sources near the boundary, especially at a diagonal of the plates, result in lower value of total temperature of the plates. Higher value of total temperature of the plates occurs when sources are placed near the centre of the plates, especially along line $y = 0.5$ or $x = 0.5$.

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