On Representations of Reciprocals as a Sum of Three Distinct Unit Fractions

POTŮČEK, R.

Department of Mathematics and Physics, University of Defence,
Kounicova 65, 662 10 Brno,
CZECH REPUBLIC

ORCID iD: 0000-0003-4385-691X

Abstract: We study representations of reciprocals of positive integers in the form $1/n = 1/a + 1/b + 1/c$, where $n < a < b < c$. We derive explicit bounds for the variables and present two transformations reducing the problem to a two-variable setting and to a divisor-based parametrization. While the latter provides structural insight via the factorization of m^2, it does not capture all admissible solutions. Therefore, we propose a direct enumeration algorithm that examines $O(n^2 \log n)$ candidate pairs for each fixed n and guarantees complete results. Numerical computations for $1 \leq n \leq 100$ illustrate the irregular behavior of the number of representations and its dependence on the arithmetic structure of n.

Key-Words: reciprocal, Egyptian fraction, Erdős–Straus conjecture, Diophantine equation, divisor function, computer algebra system Maple

Received: March 21, 2026. Revised: June 15, 2026. Accepted: August 9, 2026. Published: November 2, 2026.

"The Pythagoreans called the number 3 (the triad) the only number that equals the sum of all terms below it, and the only number whose sum with those below equals the product of them and itself."

— Pythagorean View

1 Introduction

Egyptian fractions, representations of fractions as sums of distinct unit fractions, have intrigued mathematicians for centuries. Classical results and general surveys can be found, for example, in [1], [2], [3], and [4]. More recent developments concerning representations of unit fractions as sums of two or more terms are discussed in [5], [6], [7], and [8]. From commonly available online encyclopedic sources, basic information about Egyptian fractions can be found in [9], [10]. Algorithmic and computational aspects have been investigated in [11] and [12]. While the decomposition of fractions into two unit fractions is well-studied, decompositions into three unit fractions introduce more complex Diophantine equations. In this paper we focus on representations of the form

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}, \tag{1}$$

with distinct denominators $a, b, c \in \mathbb{N}$, where $n < a < b < c$, and n is fixed. For example,

$$\frac{1}{9} = \frac{1}{10} + \frac{1}{100} + \frac{1}{900}.$$

The problem considered here belongs to the broader study of Egyptian fractions, which also includes the famous *Erdős–Straus conjecture*, an unproven statement in number theory dating back to 1948. Basic information about this conjecture can be found on the website [13]. Major articles discussing this hypothesis include, for example, the articles [14], [15] and [16].

A discussion of the Erdős–Straus conjecture and the solution of Diophantine equations can be found in [17]. A recent popular account devoted to this issue is the contribution [18].

This conjecture states that, for every integer n that is greater than or equal to 2, there exist positive integers x, y, z for which

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

In other words, the number $4/n$ can be written as a sum of three positive unit fractions. For example,

$$\frac{4}{7} = \frac{1}{2} + \frac{1}{14} + \frac{1}{28}.$$

The goal of this paper, which is a natural continuation of the author's previous works [19] and [20], is to investigate structural properties of equation (1), derive bounds for the variables, and study computational aspects of enumerating its solutions.

The main contribution of this paper is a complete characterization of solutions together with an efficient computational algorithm and a clarification of the limitations of divisor-based approaches.

2 Basic Transformations of the Equation

The equation (1) under consideration is Diophantine in nature, seeking integer solutions. We begin by transforming the equation into forms amenable to analysis. Starting with the equation (1) we multiply through by $nabc$ to obtain

$$abc = n(ab + ac + bc), \qquad n < a < b < c. \quad (2)$$

Solving equation (2) for c gives

$$c = \frac{nab}{ab - n(a+b)}.$$

Hence the denominator must satisfy

$$ab - n(a+b) > 0.$$

Proposition 1. *Introducing the substitutions*

$$a = n + x, \quad b = n + y, \quad c = n + z,$$

the equation (1) *leads to the symmetric Diophantine equation*

$$xyz = n^2(x + y + z + 2n). \quad (3)$$

Proof. Using the above substitutions, we rewrite the left-hand side of equation (2) to the form

$$\begin{aligned} abc &= (n+x)(n+y)(n+z) \\ &= n^3 + n^2(x+y+z) + n(xy+xz+yz) + xyz. \end{aligned}$$

The sum $ab + ac + bc$ then will be of the form

$$\begin{aligned} (n+x)(n+y) + (n+x)(n+z) + (n+y)(n+z) \\ = 3n^2 + 2n(x+y+z) + (xy+xz+yz). \end{aligned}$$

After multiplying by n we obtain the right side of equation (2) of the form

$$\begin{aligned} n(ab+ac+bc) \\ = 3n^3 + 2n^2(x+y+z) + n(xy+xz+yz). \end{aligned}$$

Substituting into equation (2), we obtain the equation

$$\begin{aligned} n^3 + n^2(x+y+z) + n(xy+xz+yz) + xyz \\ = 3n^3 + 2n^2(x+y+z) + n(xy+xz+yz). \end{aligned}$$

After isolating the product xyz, we obtain the equation

$$xyz = 2n^3 + n^2(x+y+z),$$

which is clearly equivalent to the Diophantine equation (3). □

The equation (1) can also be transformed into another algebraic relation. The following proposition establishes this relation:

Proposition 2. *Let*

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}, \qquad n < a < b < c,$$

where $a, b, c, n \in \mathbb{N}$. *Define*

$$x = \frac{ab}{n}, \qquad y = \frac{ac}{n}, \qquad z = \frac{bc}{n}.$$

Then

$$xyz = n(x+y+z)^2.$$

Proof. From

$$abc = n(ab + ac + bc)$$

and the definitions of x, y, z, we have

$$ab = nx, \qquad ac = ny, \qquad bc = nz,$$

and hence

$$ab + ac + bc = n(x + y + z).$$

Therefore

$$abc = n^2(x + y + z).$$

Moreover,

$$(abc)^2 = (ab)(ac)(bc) = n^3xyz.$$

Substituting the former relation into the latter gives

$$n^4(x+y+z)^2 = n^3xyz,$$

and thus

$$xyz = n(x+y+z)^2.$$

□

Remark 1. *The transformation above is one-way in general. Although* a, b, c, n *are integers, the quantities* x, y, z *need not be integers. Consequently, the transformed equation is used only as an auxiliary relation and not as a complete parametrization of the integer solutions of equation* (1).

3 Bounds for Variables

In this section, we derive basic bounds for the variables appearing in equation (1). These bounds play a crucial role in reducing the search space and form the basis for the computational approach developed later.

Lemma 1. *If*

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}, \qquad n < a < b < c,$$

then

$$n < a < 3n.$$

Proof. Since $a < b < c$, we have

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} < \frac{1}{a} + \frac{1}{a} + \frac{1}{a} = \frac{3}{a},$$

hence

$$a < 3n.$$

On the other hand, $a > n$ is immediate, since otherwise

$$\frac{1}{a} \geq \frac{1}{n}.$$

□

Lemma 2. *Let*

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}, \qquad n < a < b < c,$$

and, for a fixed admissible value of a, define

$$m = \frac{an}{a-n}.$$

Then

$$\max\{a, m\} < b < 2m.$$

Consequently, the integer variable b satisfies

$$\max\{a+1, \lfloor m \rfloor + 1\} \leq b \leq \lceil 2m \rceil - 1.$$

Proof. From

$$\frac{1}{n} - \frac{1}{a} = \frac{a-n}{an} = \frac{1}{m},$$

the original equation can be rewritten as

$$\frac{1}{m} = \frac{1}{b} + \frac{1}{c}.$$

Since $b < c$, we have

$$\frac{1}{b} + \frac{1}{c} > \frac{1}{b},$$

and hence

$$\frac{1}{m} > \frac{1}{b},$$

which gives

$$b > m.$$

Moreover, since $c > b$,

$$\frac{1}{b} + \frac{1}{c} < \frac{2}{b}.$$

Therefore

$$\frac{1}{m} < \frac{2}{b},$$

and consequently

$$b < 2m.$$

Together with the condition $b > a$, this yields

$$\max\{a, m\} < b < 2m.$$

The corresponding integer bounds follow immediately. □

Proposition 3. *Let $n < a < b < c$ and let*

$$m = \frac{an}{a-n}.$$

For fixed n and a, every admissible value of b uniquely determines c by

$$c = \frac{abn}{ab - an - bn}.$$

Moreover, if

$$\max\{a, m\} < b < 2m,$$

then

$$ab - an - bn > 0$$

and the resulting value of c satisfies

$$c > b.$$

Hence, for such a and b, the triple (a, b, c) is a solution if and only if

$$(ab - an - bn) \mid abn.$$

Proof. Starting from

$$\frac{1}{n} - \frac{1}{a} = \frac{1}{b} + \frac{1}{c},$$

we obtain

$$\frac{a-n}{an} = \frac{b+c}{bc}.$$

After clearing denominators,

$$abn + acn = abc,$$

and therefore

$$c(ab - an - bn) = abn.$$

Thus

$$c = \frac{abn}{ab - an - bn}.$$

Since

$$b > m = \frac{an}{a-n},$$

we have

$$b(a-n) > an,$$

and hence

$$ab - an - bn = b(a-n) - an > 0.$$

Furthermore,

$$c > b$$

is equivalent to

$$\frac{abn}{ab-an-bn} > b,$$

which, since the denominator is positive, is equivalent to

$$ab - an - bn < an.$$

This is equivalent to

$$b(a-n) < 2an,$$

or

$$b < 2m.$$

Thus the inequalities

$$\max\{a, m\} < b < 2m$$

guarantee both a positive denominator and the required ordering $c > b$.

Consequently, the only remaining condition for c to be an integer is

$$ab - an - bn \mid abn.$$

□

Proposition 4. *Under the assumptions of the preceding lemma, the variable b satisfies the sharper upper bound*

$$b \le B(a,n),$$

where

$$B(a,n) = \\ = \frac{(2a+1)n - a + \sqrt{(4a^2+1)n^2 - 2an + a^2}}{2(a-n)}.$$

Equivalently, in terms of

$$m = \frac{an}{a-n},$$

we have

$$B(a,n) = m - \frac{1}{2} + \frac{1}{2}\sqrt{4m^2+1}.$$

In particular,

$$2m - \frac{1}{2} < B(a,n) < 2m.$$

Proof. For fixed n and a, the relation

$$\frac{1}{m} = \frac{1}{b} + \frac{1}{c}$$

holds. Since $c > b$ and b, c are integers,

$$c \ge b + 1.$$

Therefore

$$\frac{1}{m} = \frac{1}{b} + \frac{1}{c} \le \frac{1}{b} + \frac{1}{b+1} = \frac{2b+1}{b(b+1)}.$$

The boundary value is obtained when $c = b + 1$, that is,

$$\frac{1}{m} = \frac{2b+1}{b(b+1)}.$$

After rearrangement,

$$b^2 + (1-2m)b - m = 0.$$

The positive root is

$$B = m - \frac{1}{2} + \frac{1}{2}\sqrt{4m^2+1}.$$

Substituting

$$m = \frac{an}{a-n}$$

gives the stated expression for $B(a,n)$.

Finally,

$$2m < \sqrt{4m^2+1} < 2m+1,$$

and hence

$$2m - \frac{1}{2} < B < 2m.$$

□

Remark 2. *The simple bound*

$$b < 2m$$

and the sharper bound

$$b \le B(a,n) < 2m$$

are asymptotically equivalent, while the latter may reduce the integer search interval by at most one additional value of b. The simple bound is therefore preferable for the structural analysis, whereas the sharper bound may be used as an optional refinement in the computational implementation.

Example 1. *Consider the case*

$$n = 12, \qquad a = 28.$$

Then

$$m = \frac{an}{a-n} = \frac{28 \cdot 12}{28-12} = 21\,.$$

The bounds established above give

$$\max\{28, 21\} < b < 42\,,$$

and hence

$$29 \le b \le 41\,.$$

For each such b, the remaining denominator is uniquely determined by

$$c = \frac{abn}{ab - an - bn} = \frac{336b}{16b - 336}\,.$$

The divisibility condition

$$(16b - 336) \mid 336b$$

then determines the admissible values of b.

In this case the only admissible value is

$$b = 30\,,$$

which gives

$$c = \frac{336 \cdot 30}{17 \cdot 30 - 336} = 70\,.$$

Thus the unique representation corresponding to $n = 12$ and $a = 28$ is

$$\frac{1}{12} = \frac{1}{28} + \frac{1}{30} + \frac{1}{70}\,.$$

Remark 3. *For the same example,*

$$m^2 = 21^2 = 441$$

has the positive divisors

$$1, 3, 7, 9, 21, 49, 63, 147, 441\,.$$

If m is an integer, the factorization

$$(b - m)(c - m) = m^2$$

shows that a factorization

$$441 = uv$$

formally gives

$$b = 21 + u\,, \qquad c = 21 + v\,.$$

However, not every such factorization corresponds to a solution satisfying

$$28 < b < c\,.$$

For example, the factor $u = 1, 3, 7$ gives $b \le 28$, while $u = 21$ gives $b = c = 42$. The admissible factor is

$$u = 9\,,$$

which yields

$$b = 30\,, \qquad c = 70\,.$$

Thus only one of the formally available factorizations gives a representation with $a < b < c$.

This example illustrates why the divisor function $\tau(m^2)$ cannot by itself provide the exact number of representations for a fixed a.

4 Another Transformation of the Equation

We now rewrite equation (1) in a form that reveals a connection with the divisor structure of integers. Subtracting $1/a$ from both sides of equation (1), we obtain

$$\frac{a-n}{an} = \frac{1}{b} + \frac{1}{c}\,. \tag{4}$$

For a fixed pair (n, a), where $a > n$, define

$$m = \frac{an}{a-n}\,. \tag{5}$$

Equation (4) can then be written as

$$\frac{1}{m} = \frac{1}{b} + \frac{1}{c}\,.$$

Clearing denominators gives

$$bc = m(b + c)\,,$$

and hence

$$(b - m)(c - m) = m^2. \tag{6}$$

Since $1/m = 1/b + 1/c$ and $b, c > 0$, we have $b > m$ and $c > m$. Thus both factors on the left-hand side of equation (6) are positive.

The identity (6) holds whether or not m is an integer. However, a parametrization in terms of integer divisors of m^2 is available only when $m \in \mathbb{N}$. Indeed, writing

$$m = n + \frac{n^2}{a-n}\,,$$

we see that

$$m \in \mathbb{N} \quad \Longleftrightarrow \quad (a-n) \mid n^2.$$

If m is not an integer, equation (6) may still have solutions with integer denominators b and c, but the integer-divisor parametrization described below does not apply.

Proposition 5. *Let $n, a, b, c \in \mathbb{N}$ satisfy*

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}, \qquad n < a < b < c,$$

and let $m = an/(a-n)$. Then

$$\frac{1}{m} = \frac{1}{b} + \frac{1}{c} \qquad \textit{and} \qquad (b-m)(c-m) = m^2.$$

If, in addition, $m \in \mathbb{N}$, then

$$u = b - m, \qquad v = c - m$$

are positive integers satisfying $uv = m^2$, and

$$b = m + u, \qquad c = m + v.$$

Conversely, a factorization $m^2 = uv$ into positive integers gives a solution with $a < b < c$ precisely when

$$\max\{0, a-m\} < u < m, \qquad v = \frac{m^2}{u}.$$

Proof. The first identity follows directly from (4) and the definition of m. Multiplying $1/m = 1/b + 1/c$ by mbc gives

$$bc = m(b+c),$$

which is equivalent to (6).

If m is an integer, then $u = b - m$ and $v = c - m$ are positive integers, and equation (6) yields $uv = m^2$. Conversely, let u be a positive divisor of m^2 and put $v = m^2/u$. The inequalities

$$\max\{0, a-m\} < u < m$$

ensure that $a < m + u$ and $u < v$. Consequently, $b = m + u$ and $c = m + v$ satisfy $a < b < c$. Moreover, $uv = m^2$ implies

$$(b-m)(c-m) = m^2,$$

which is equivalent to $1/m = 1/b + 1/c$ and therefore to the original equation. □

The proposition gives an exact divisor-based characterization for a fixed admissible value of a when m is an integer. In this case, the relevant solutions correspond precisely to the divisors u of m^2 satisfying

$$\max\{0, a-m\} < u < m.$$

The ordering conditions are essential: not every factorization of m^2 yields denominators satisfying $a < b < c$.

When m is not an integer, this divisor-based characterization is unavailable. Therefore, the parametrization above describes only a subset of all possible solutions of equation (1). A complete enumeration must also account for admissible pairs (a, b) for which m is not an integer.

5 A Divisor-Based Approach and Its Limitations

The identity

$$(b-m)(c-m) = m^2,$$

derived in the preceding section, reveals a connection between the original equation and the divisor structure of integers. However, its use for enumerating solutions requires some care.

Recall that

$$m = \frac{an}{a-n} = n + \frac{n^2}{a-n}.$$

Thus, for integers $a > n$ and $n \geq 1$,

$$m \in \mathbb{N} \quad \Longleftrightarrow \quad (a-n) \mid n^2.$$

Only in this case can the identity

$$(b-m)(c-m) = m^2$$

be parametrized directly by positive integer divisors of m^2. If $m \notin \mathbb{N}$, the identity remains valid, but this integer-divisor parametrization is not applicable.

Even when $m \in \mathbb{N}$, the number of divisors of m^2 does not by itself give the exact number of admissible representations for a fixed a. As shown in the preceding section, only divisors $u \mid m^2$ satisfying

$$\max\{0, a-m\} < u < m$$

yield denominators

$$b = m + u, \qquad c = m + \frac{m^2}{u}$$

with $a < b < c$. The ordering conditions must therefore be taken into account when counting solutions.

The restriction to integer values of m also excludes valid representations. For example, consider $n = 3$ and $a = 5$. In this case,

$$m = \frac{an}{a-n} = \frac{15}{2},$$

which is not an integer. Nevertheless, the following four representations satisfy the required ordering of denominators:

$$\frac{1}{3} = \frac{1}{5} + \frac{1}{8} + \frac{1}{120},$$
$$\frac{1}{3} = \frac{1}{5} + \frac{1}{9} + \frac{1}{45},$$
$$\frac{1}{3} = \frac{1}{5} + \frac{1}{10} + \frac{1}{30},$$
$$\frac{1}{3} = \frac{1}{5} + \frac{1}{12} + \frac{1}{20}.$$

These solutions cannot be obtained from a factorization of m^2 into positive integer divisors using the parametrization described above.

Consequently, the divisor-based approach provides useful structural information, but it does not cover all solutions and cannot, by itself, be used for their exact enumeration. To obtain the complete set of representations, we use the direct enumeration method developed in the following sections. This method considers all admissible values of a and b and determines c from

$$c = \frac{abn}{ab - an - bn},$$

subject to the required integrality and ordering conditions.

6 The Main Result

The bounds derived in Section 3 reduce the original problem to a finite search over the variables a and b. For each admissible pair (a, b), the remaining denominator c is uniquely determined. The following theorem gives a complete characterization of all representations of equation (1).

Theorem 1. *Let $n \in \mathbb{N}$. The representations*

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}, \qquad n < a < b < c,$$

are in one-to-one correspondence with the integer pairs (a, b) satisfying

$$n + 1 \le a \le 3n - 1,$$

$$\max\{a, m\} < b < 2m, \qquad m = \frac{an}{a - n},$$

and

$$(ab - an - bn) \mid abn.$$

For every such pair, the corresponding denominator c is given uniquely by

$$c = \frac{abn}{ab - an - bn}.$$

Proof. Let (a, b, c) be a representation of equation (1). By the lemma in Section 3,

$$n < a < 3n,$$

and hence

$$n + 1 \le a \le 3n - 1.$$

For a fixed admissible value of a, define

$$m = \frac{an}{a - n}.$$

Subtracting $1/a$ from both sides of equation (1) gives

$$\frac{1}{m} = \frac{1}{b} + \frac{1}{c}.$$

Since $b < c$, the bounds established in Section 3 yield

$$\max\{a, m\} < b < 2m.$$

Multiplying equation (1) by $nabc$ gives

$$abc = n(ab + ac + bc),$$

which can be rearranged as

$$c(ab - an - bn) = abn.$$

Consequently,

$$c = \frac{abn}{ab - an - bn}.$$

The bounds on b ensure that the denominator is positive and that $c > b$. Since c is an integer, it follows that

$$(ab - an - bn) \mid abn.$$

Thus every representation determines a pair (a, b) satisfying all the stated conditions.

Conversely, let (a, b) be an integer pair satisfying these conditions. The divisibility condition ensures that the value

$$c = \frac{abn}{ab - an - bn}$$

is an integer. By the proposition in Section 3, the bounds on b guarantee that $c > b$ and that equation (1) holds. Since $a < b$, the resulting triple satisfies $n < a < b < c$ and is therefore an admissible representation.

Finally, c is uniquely determined by (a, b). Conversely, each representation determines exactly one such pair. The correspondence is therefore one-to-one. □

Corollary 1. Let $E(n)$ denote the number of representations of $1/n$ as a sum of three distinct unit fractions satisfying $n < a < b < c$. Then

$$E(n) = \#\left\{ \begin{array}{r} (a, b) \in \mathbb{N}^2 : \; n + 1 \le a \le 3n - 1, \\ \max\{a, m\} < b < 2m, \\ (ab - an - bn) \mid abn \end{array} \right\},$$

where $m = an/(a - n)$.

Proof. The result follows directly from the one-to-one correspondence established in the theorem. □

The theorem provides an exact characterization of all representations, including those for which m is not an integer. It therefore forms the mathematical basis for the complete enumeration algorithm described in the following section.

7 Computational Algorithms

The characterization established in Section 6 leads to a direct enumeration algorithm. For each fixed positive integer n, the algorithm examines all admissible values of a and b and determines c from

$$c = \frac{abn}{ab - an - bn}.$$

The value of c is accepted only if it is an integer and satisfies $c > b$. The bounds derived in Section 3 and refined in Section 6 reduce the search to a finite number of candidate pairs.

The following Maple procedure implements the direct enumeration. The bounds for b follow from $\max\{a, m\} < b < 2m$, where $m = an/(a-n)$. The procedure returns all representations for each n from 1 to N.

```
E3S := proc(N::posint)
local a,b,c,n,m,den,r,sol,bmin,bmax;
for n from 1 to N do
sol := [];
  for a from n+1 to 3*n-1 do
    m := a*n/(a-n);
    bmin := max(a+1,floor(m)+1);
    bmax := ceil(2*m)-1;
    for b from bmin to bmax do
      den := a*b-a*n-b*n;
      if den > 0 then
        c := iquo(a*b*n,den,'r');
        if r = 0 and c > b then
          sol := [op(sol),[a,b,c]];
        fi;
      fi;
    od;
  od;
print(n,nops(sol),sol);
od;
end proc:
E3S(100);
```

The second procedure, denoted by `E3Sd`, implements the divisor-based parametrization discussed in Sections 4 and 5. It considers only those values of a for which

$$m = \frac{an}{a-n} \in \mathbb{N},$$

or, equivalently,

$$(a-n) \mid n^2.$$

For such values, the identity

$$(b-m)(c-m) = m^2$$

allows the denominators to be written as

$$b = m + u, \qquad c = m + \frac{m^2}{u},$$

where u is a positive integer divisor of m^2. The ordering conditions $a < b < c$ are equivalent to

$$\max\{0, a-m\} < u < m.$$

The procedure therefore enumerates only the representations satisfying these conditions. In general, it does not produce all representations of $1/n$.

```
with(NumberTheory):

E3Sd := proc(N::posint)
local a,m,n,u,b,c,sol;
for n from 1 to N do
sol := [];
  for a from n+1 to 3*n-1 do
    if (a*n) mod (a-n) = 0 then
      m := iquo(a*n,a-n);
      for u in Divisors(m^2) do
        if max(0,a-m)<u and u<m then
          b := m+u;
          c := m+iquo(m^2,u);
          sol := [op(sol),[a,b,c]];
        fi;
      od;
    fi;
  od;
print(n,nops(sol),sol);
od;
end proc:
E3Sd(100);
```

The two procedures serve different purposes. The direct algorithm `E3S` is intended to enumerate all representations satisfying the conditions of the main theorem, whereas `E3Sd` enumerates only the subclass for which m is an integer. Comparing their outputs illustrates the limitation of the divisor-based approach and provides a computational check of the structural results established in the preceding sections.

8 Numerical Results

The number of representations $E(n)$ of the reciprocal $1/n$ as a sum of three distinct unit fractions,

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c},$$

exhibits irregular behaviour, reflecting the arithmetic structure of n. Although larger values of $E(n)$ sometimes occur for integers with a richer divisor structure, the numerical data do not establish a simple relationship between $E(n)$ and the divisor function $\tau(n)$.

We computed the values of $E(n)$ together with $\tau(n)$ using the algorithm E3S for $1 \le n \le 100$. The results are summarized in Table 1.

n	1	2	3	4	5	6	7	8	9	10
τ	1	2	2	3	2	4	2	4	3	4
E	1	6	15	22	30	45	36	62	69	84
n	11	12	13	14	15	16	17	18	19	20
τ	2	6	2	4	4	5	2	6	2	6
E	56	142	53	124	178	118	67	191	74	274
n	21	22	23	24	25	26	27	28	29	30
τ	4	4	2	8	3	4	4	6	2	8
E	227	145	87	342	146	162	216	322	100	461
n	31	32	33	34	35	36	37	38	39	40
τ	2	6	4	4	4	9	2	4	4	8
E	84	257	304	199	435	508	79	204	360	580
n	41	42	43	44	45	46	47	48	49	50
τ	2	8	2	6	6	4	2	10	3	6
E	115	587	98	455	618	192	129	676	217	417
n	51	52	53	54	55	56	57	58	59	60
τ	4	6	2	8	4	8	4	4	2	12
E	369	449	119	573	543	759	367	240	166	1236
n	61	62	63	64	65	66	67	68	69	70
τ	2	4	6	7	4	8	2	6	4	8
E	102	261	857	428	568	717	115	537	460	1018
n	71	72	73	74	75	76	77	78	79	80
τ	2	12	2	4	6	6	4	8	2	10
E	155	1126	112	276	839	543	687	807	141	1148
n	81	82	83	84	85	86	87	88	89	90
τ	5	4	2	12	4	4	4	8	2	12
E	447	285	184	1633	581	276	483	945	184	1663
n	91	92	93	94	95	96	97	98	99	100
τ	4	6	4	4	4	12	2	6	6	9
E	628	540	454	298	715	1175	137	644	1094	1076

Table 1: The values of $\tau(n)$ and $E(n)$ (denoted here by τ and E) for $n = 1, \ldots, 100$
[source: own Maple computation]

These results indicate that the function $E(n)$ does not follow a simple monotonic or multiplicative pattern (see Figure 1).

It was noted above that the divisor-based algorithm E3Sd does not generate all solutions of equation (1). To illustrate this, Table 2 compares the values of $E(n)$ obtained by the direct enumeration algorithm (E3S) and the divisor-based algorithm (E3Sd) for $n = 1, \ldots, 10$.

Specifically, for the decomposition of the fraction $1/3$, the divisor-based algorithm E3Sd fails to detect four valid decompositions for which the parameter

$$m = \frac{an}{a-n}$$

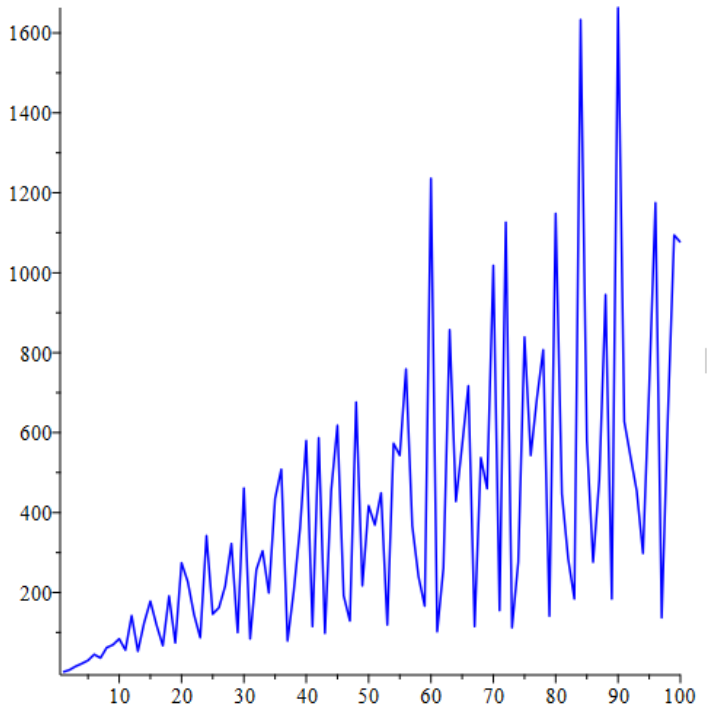


Figure 1: The line graph of values of $E(n)$ for $n = 1, \ldots, 100$
[source: own Maple computation]

n	1	2	3	4	5	6	7	8	9	10
E3S	1	6	15	22	30	45	36	62	69	84
E3Sd	1	6	11	17	17	41	14	41	41	59

Table 2: The numbers of decompositions $E(n)$ obtained by the E3S and E3Sd algorithms for $n = 1, \ldots, 10$
[source: own Maple computation]

is not an integer:

$$\frac{1}{3} = \frac{1}{5} + \frac{1}{8} + \frac{1}{120},$$
$$\frac{1}{3} = \frac{1}{5} + \frac{1}{9} + \frac{1}{45},$$
$$\frac{1}{3} = \frac{1}{5} + \frac{1}{10} + \frac{1}{30},$$
$$\frac{1}{3} = \frac{1}{5} + \frac{1}{12} + \frac{1}{20}.$$

In all four cases, the parameter m is not an integer:

$$m = \frac{an}{a-n} = \frac{5 \cdot 3}{5-3} = \frac{15}{2}.$$

Consequently, these representations cannot be obtained by the divisor-based algorithm E3Sd.

9 Discussion

The results obtained in this paper reveal a connection between representations of $1/n$ as a sum of three distinct unit fractions and the multiplicative structure of integers.

In particular, for solutions corresponding to an integer value of m, the transformation

$$(b-m)(c-m) = m^2$$

shows that their number is related to the divisor structure of m^2. This provides an explanation for some of the irregularities observed in the numerical values of $E(n)$.

The numerical results also reveal pronounced irregularities in the behaviour of $E(n)$. Relatively large values of $E(n)$ often occur for integers with a richer divisor structure, whereas primes and integers with relatively few divisors tend to yield fewer representations. This suggests that the arithmetic structure of n plays an important role in determining the number of its representations. However, a precise relationship between $E(n)$ and standard divisor functions has not yet been established.

Although the bounds $n < a < 3n$ significantly reduce the search space, the distribution of solutions remains highly non-uniform. A more detailed analytic description of the behaviour of $E(n)$ may therefore be possible, potentially involving divisor functions and their average orders.

The problem is related to classical questions concerning Egyptian fractions, as well as to the Erdős–Straus conjecture, in which similar Diophantine structures arise.

Further research may focus on:

- asymptotic estimates for $E(n)$,
- extensions to representations involving differences,
- generalizations to sums of more than three unit fractions.

The sequence of values $E(n)$ is included in the On-Line Encyclopedia of Integer Sequences (OEIS) as sequence A227610 [21], whose initial terms are

$$(1, 6, 15, 22, 30, 45, 36, 62, 69, 84, \ldots).$$

Conclusion

We have investigated representations of the form

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}, \quad n < a < b < c,$$

and derived a structural characterization of their solutions.

By combining algebraic transformations with explicit bounds on the variables, we reduced the problem to a finite two-variable search and developed an effective algorithm for the complete enumeration of solutions. We also examined a divisor-based parametrization, which provides an alternative approach for solutions corresponding to integer values of the auxiliary parameter, but does not cover all possible representations.

The computational results obtained for $n \leq 100$ illustrate the irregular behaviour of the number of representations $E(n)$ and provide further insight into its dependence on the arithmetic properties of n.

The results contribute to the study of Egyptian fractions and provide a basis for further investigations, particularly into the asymptotic behaviour of $E(n)$ and more general unit-fraction decompositions.

References:

[1] Sierpiński, W. Elementary Theory of Numbers, North-Holland Publishing Company, 1988. ISBN 978-0-444-86662-2.

[2] Dickson, L. E. History of the Theory of Numbers, Vol. I, Chelsea Publishing, 1992. ISBN 978-0-8284-0086-2.

[3] Bello-Hernández M., Benito M., Fernández, E. On Egyptian fractions. arXiv:1010.2035v2, 24 pp., 2012. https://arxiv.org/abs/1010.2035, https://doi.org/10.48550/arXiv.1010.2035

[4] Butler, S., Erdős, P., Graham, R. Egyptian fractions with each denominator having three distinct prime divisors. Integers, 2015, Paper No. A51, 9 pp. https://www.math.ucsd.edu/~ronspubs/pre_tres_egyptian.pdf

[5] Clemente, F., Fortuny, J. M. Egyptian fractions and representation registers in the construction of the fraction concept. Eurasia Journal of Mathematics, Science and Technology Education, 2021, Vol. 17(7), 18 pp. https://doi.org/10.29333/ejmste/10992

[6] Xiao, X., Zhai, W. Error term of the mean value theorem for binary egyptian fractions. Open Mathematics, 2020, Vol. 18(1), p. 1250-1265. https://doi.org/10.1515/math-2020-0091

[7] Zhao, W., Lu, J., Wang, L. On the integral solutions of the egyptian fraction equation ap=1x+1y+1z. AIMS Mathematics, 2021, Vol. 6(5), p. 4930-4937. https://doi.org/10.3934/math.2021289

[8] Janthawee, M., Kanasri, N. R. On the SEL egyptian fraction expansion for real numbers. AIMS Mathematics, 2022, Vol. 7(8), p. 15094-15106. https://doi.org/10.3934/math.2022827

[9] Wikipedia contributors. Egyptian fractions. Wikipedia, The Free Encyclopedia, 2026. https://en.wikipedia.org/wiki/Egyptian_fraction

[10] Weisstein, Eric W. Egyptian Fraction. From MathWorld – A Wolfram Resource, 2026. https://mathworld.wolfram.com/EgyptianFraction.html

[11] Wang, Y. Review on greedy algorithm. Theoretical and Natural Science, 2023, Vol. 14(117), p. 233-239. https://www.researchgate.net/publication/376076453_Review_on_greedy_algorithm, https://doi.org/10.54254/2753-8818/14/20241041

[12] García, A. Greedy algorithms: a review and open problems. arXiv:2408.08935v2, 20 pp., 2024. https://arxiv.org/abs/2408.08935, https://doi.org/10.48550/arXiv.2408.08935

[13] Wikipedia contributors. Erdős–Straus conjecture. Wikipedia, The Free Encyclopedia, 2026. https://en.wikipedia.org/wiki/Erdos-Straus_conjecture

[14] Chun, J. H. Egyptian fractions, Sylvester's sequence, and the Erdős–Straus conjecture. 7 pp., 2011. https://studylib.net/doc/7971911/

[15] Elsholtz, Ch., Planitzer, S. The number of solutions of the Erdős-Straus Equation and sums of k unit fractions, arXiv:1805.02945, 21 pp., 2018. https://arxiv.org/abs/1805.02945

[16] Bradford, K. Elemental Patterns from the Erdős–Straus Conjecture. arXiv:2403.16047v1, 15 pp., 2024. https://arxiv.org/html/2403.16047v1

[17] Negash, D. J. Solutions to Diophantine Equation of Erdős–Straus Conjecture. arXiv:1812.05684v2, 13 pp., 2018. https://arxiv.org/abs/1812.05684

[18] Plevris, V. Erdős–Straus Conjecture: The Simple Fraction That Refuses to Be Proven. Medium Website – American online publishing platform, March 23, 2026. https://medium.com/@vplevris/erdos-straus-conjecture-the-simple-fraction-that-refuses-to-be-proven-f93cd6ee3208

[19] Potůček, R. On the Decompositions of a Reciprocal Into the Sum and the Difference of Two Reciprocals. International Journal of Mathematical and Computational Methods, 2025, Vol. 10, p. 231-239. ISSN 2367-895X. https://www.iaras.org/home/caijmcm/on-the-decompositions-of-a-reciprocal-into-the-sum-and-the-difference-of-two-reciprocals

[20] Potůček, R. Decompositions of $1/n$ into sums and differences of reciprocals of the forms $2/a$ and $3/b$. Mathematics in Education, Research and Applications, 2025, Vol. 11(1), p. 9-17. ISSN 2453-6881. https://doi.org/10.15414/meraa.2025.11.02.9-17, https://meraa.uniag.sk/docs/2025-01/potucek.pdf

[21] Sloane, N. J. A. The On-Line Encyclopedia of Integer Sequences. https://oeis.org/A227610, 2026.

Acknowledgement

The work presented in this paper has been supported by the Project for the Development of the Organization „DZRO Military autonomous and robotic systems“.