Properties of Abiyev’s Triangle

POLADOVA VUSALA NADIR
Ministry of Science and Education Republic of Azerbaijan Institute of Radiation Problems
Address: 9 B.Vahabzade str., Baku, AZERBAIJAN, AZ1143

Abstract: The procedure for writing Abiyev’s triangle using symmetric graphs has been explained. The Fibonacci and Lucas sequences are formed, respectively, by adding the numbers from the lines of the triangles on the left and right sides of this triangle. Symmetrical diagonal elements of these triangles generate Pascal's and analogous triangles, and vice versa. The diagonal numbers of these triangles make up the elements of the left and right triangles of Abiyev’s triangle. The numbers in the lines of these ultimate triangles represent the coefficients of the polynomial terms corresponding to the binomial $a^n \pm b^n = f_n[(a+b), ab]$. Mathematical conversions are considerably simpler when polynomials are used in the calculations. Examples of this have been provided. Using this method, we obtain a simple solution to one of the Ramanujan’s problems.

Keywords: Lucas, Fibonacci, Abiyev, sequences, polynomial, coefficient, triangle

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1. Introduction

It is known that Fibonacci numbers arise as sums of diagonals in Pascal’s triangle.

$$F_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-1-k}{k} = \frac{(n-1-k)!}{k!(n-1-2k)!} \quad (1)$$

for all $n \geq 0$.

$$F_{7+1} = F_8 = \sum_{k=0}^{3} \binom{n-k}{k} =$$
$$= \binom{7}{0} + \binom{6}{1} + \binom{5}{2} + \binom{4}{3} = 1+6+10+4=21$$

This is the sum of the numbers in the 8[th] line on the left side of Abiyev’s triangle [1, 2]. This leads us to the conclusion that the sum of the diagonal numbers in Pascal's triangle forms the spectrum of Fibonacci numbers. The objective of this paper is to calculate the spectrum of Fibonacci and Lucas numbers using the $a^n \pm b^n = f_n[(a+b), ab]$ equation and to demonstrate the application of the Abiyev Triangle in solving algebraic equations [4, 5, 6].

2.1. Writing a triangle

First, let's form the middle column using natural numbers, starting from 0 (Table 1). Then, we write 0 in front of even numbers and 1 in front of odd numbers on the left side of the middle column. On the right side of this middle column, we write 2 in front of the even numbers, and in front of the odd numbers, we write odd numbers themselves to form the 1[st] column on the left and right. Next, we write 1;1 in front of the numbers 0;1 on the left side of these columns and 1;1 in front of the numbers 2;3 on the right side. We use mutually symmetrical columns to generate the second column on the left and right sides of this table. By repeating this rule, we also make the 3[rd], 4[th],... columns. As you can see, the number of columns is equal to $\frac{n+1}{2}$ for odd n's and $\frac{n+2}{2}$ for even n's (Table 1).

If we add the numbers in the lines of this table, we get the Fibonacci sequence on the left, and the Lucas sequence on the right:

$$F_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-1-k}{k}; \; L_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{k} \binom{n-1-k}{k-1}$$

Examples:

$$F_7 = \sum_{k=0}^{3} \binom{n-k}{k} = \binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3} =$$
$$= 1+5+6+1=13 \quad (2)$$

$$L_6 = \sum_{k=0}^{3} \frac{6}{k} \binom{5-k}{k-1} = \frac{6}{0} \binom{5}{-1} + \frac{6}{1} \binom{4}{0} +$$

$$+\frac{6}{2}\binom{3}{1}+\frac{6}{3}\binom{2}{2}=1+6+9+2=18 \qquad (3)$$

2.2. Spectrum of Fibonacci and Lucas sequences

Note that Abiyev proposed to write the function $f_n[(a+b),ab]$ in the form of a binomial $a^n \pm b^n$ [3]:

$$f_n[(a+b),ab]=$$
$$=c_0(a+b)^n-c_1(a+b)^{n-2}ab+$$
$$+c_2(a+b)^{n-4}(ab)^2+\ldots+c_n(ab)^n$$

Given the roots of the golden ratio equation $\left[a=\frac{1}{2}(1+\sqrt{5});\ b=\frac{1}{2}(1-\sqrt{5})\right]$, substituting the values of their sum ($a+b=1$) and product ($ab=-1$) into the above function we obtain:

$$f_n=c_0+c_1+c_2+...+c_n$$

The Fibonacci number spectrum corresponds to the coefficients of a^n-b^n, the numbers (in orange) on the left side of Abiyev's triangle, while the Lucas numbers spectrum corresponds to coefficients of a^n+b^n, the numbers (in green) on the right side of the triangle (Table 1).

Examples:

$$a^n-b^n=a^8-b^8=(a-b)x(x^6-6x^4y+$$
$$+10x^2y^2-4y^2);$$

$a=3$; $b=2$ for $a+b=x=5$; $ab=y=6$.

$$3^8-2^8=$$
$$=1\cdot 5(5^6-6\cdot 5^4\cdot 6+10\cdot 5^2\cdot 6^2-$$
$$-4\cdot 6^3)=5(24625-23364)=$$
$$=5\cdot 1261=6305$$

Note that the binomial $a^n \pm b^n$ is easily calculated for conjugate complex numbers a and b:

$$a=c+id,\ b=c-id,\ a+b=2c,$$
$$ab=c^2+d^2.$$

2.3. Examples of the application of this formula.

Example 1.

$$\begin{cases} x+y=1 \\ x^2+y^2=2 \end{cases} \Longrightarrow xy=\frac{-1}{2}$$

$$x^{11}+y^{11}=?$$

$$\frac{x^{11}+y^{11}}{x+y}=(x+y)^{10}-11(x+y)^8xy+$$
$$+44(x+y)^6(xy)^2-77(x+y)^4(xy)^3+$$
$$+55(x+y)^2(xy)^4-11(xy)^5=$$
$$=1^{10}-11\cdot 1^8\cdot\left(\frac{-1}{2}\right)+44\cdot 1^6\cdot\left(\frac{-1}{2}\right)^2-$$
$$-77\cdot 1^4\cdot\left(\frac{-1}{2}\right)^3+55\cdot 1^2\cdot\left(\frac{-1}{2}\right)^4-$$
$$-11\cdot\left(\frac{-1}{2}\right)^5=1+11\cdot\frac{1}{2}+44\cdot\frac{1}{4}+77\cdot\frac{1}{8}+$$
$$+55\cdot\frac{1}{16}+11\cdot\frac{1}{32}=$$
$$=\frac{32+11\cdot 16+44\cdot 8+77\cdot 4+55\cdot 2+11}{32}=$$
$$\frac{32+176+352+308+110+11}{32}=\frac{989}{32}=30.90625$$

Example 2.

$$\begin{cases} x+x^{-1}=\sqrt{13} \\ x^5-x^{-5}=? \end{cases}$$

$$x^2+x^{-2}=(\sqrt{13})^2-2=11$$

$$(x-x^{-1})^2=x^2+x^{-2}-2=9;$$
$$x-x^{-1}=3$$

$$\frac{x^5-x^{-5}}{x-x^{-1}}=(x+x^{-1})^4-$$
$$-3(x+x^{-1})^2(x\cdot x^{-1})+(x\cdot x^{-1})^2=$$
$$=(\sqrt{13})^4-3(\sqrt{13})^2+1=169-39+1=131$$

$$x^5-x^{-5}=(x-x^{-1})\cdot 131=3\cdot 131=393$$

Example 3.

$$\begin{cases} x+y=10 \\ x^5+y^5=20000 \end{cases}$$
$$x=?,\ y=?$$

$$\frac{x^5+y^5}{x+y}=(x+y)^4-5(x+y)^2xy+5(xy)^2$$

$$x^5+y^5=10(10^4-5\cdot 10^2xy+5(xy)^2)=$$
$$=10[10000-500xy+5(xy)^2]=20000$$

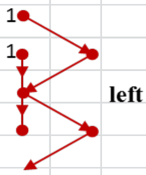
$$5(xy)^2-500xy+8000=0$$
$$(xy)^2-100xy+1600=0$$
$$(xy)=50\pm\sqrt{2500-1600}=50\pm30$$
$$(xy)_1=20; \qquad (xy)_2=80$$
$$\begin{cases}x+y=10\\ xy=20\end{cases} \qquad \begin{cases}x+y=10\\ xy=80\end{cases}$$
$$t^2-10t+20=0; \qquad u^2-10u+80=0$$
$$t=5\pm\sqrt{25-20}=5\pm\sqrt{5}; \quad u=5\pm\sqrt{-55}$$
$$(x_1;y_1)=(5-\sqrt{5};5+\sqrt{5})$$
$$(x_2;y_2)=(5+\sqrt{5};5-\sqrt{5})$$
$$(x_3;y_3)=(5-i\sqrt{5};5+i\sqrt{5})$$
$$(x_4;y_4)=(5+i\sqrt{5};5-i\sqrt{5})$$

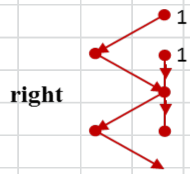
Conclusion

The diagonal elements in the left side of the Abiyev Triangle (red cells) constitute the Pascal Triangle and vice versa. On the other hand, the diagonal elements in the right side of the Abiyev Triangle (blue cells) constitute the Pascal Triangle analogue and vice versa (Table 2, 3).

Table 1.

Abiyev's triangle





0											0	0	2											2
1											1	1	1											1
1										1	0	2	2	1										3
2										1	1	3	3	1										4
3									1	2	0	4	2	4	1									7
5									1	3	1	5	5	5	1									11
8								1	4	3	0	6	2	9	6	1								18
13								1	5	6	1	7	7	14	7	1								29
21							1	6	10	4	0	8	2	16	20	8	1							47
34							1	7	15	10	1	9	9	30	27	9	1							76
55						1	8	21	20	5	0	10	2	25	50	35	10	1						123
89						1	9	28	35	15	1	11	11	55	77	44	11	1						199
144					1	10	36	56	35	6	0	12	2	36	105	112	54	12	1					322
233					1	11	45	84	70	21	1	13	13	91	182	156	65	13	1					521
377				1	12	55	120	126	56	7	0	14	2	49	196	294	210	77	14	1				843
610				1	13	66	165	210	126	28	1	15	15	140	378	450	275	90	15	1				1364
987			1	14	78	220	330	252	84	8	0	16	2	64	336	672	660	352	104	16	1			2207
1597			1	15	91	286	495	462	210	36	1	17	17	204	714	1122	935	442	119	17	1			3571
2584		1	16	105	364	715	792	462	120	9	0	18	2	81	540	1386	1782	1287	546	135	18	1		5778
4181		1	17	120	455	1001	1287	924	330	45	1	19	19	285	1254	2508	2717	1729	665	152	19	1		9349
6765	1	18	136	560	1365	2002	1716	792	165	10	0	20	2	100	825	2640	4290	4004	2275	800	170	20	1	15127

Table 2.

Pascal's triangle

													1														2^{0}
												1		1													2^{1}
											1		2		1												2^{2}
										1		3		3		1											2^{3}
									1		4		6		4		1										2^{4}
								1		5		10		10		5		1									2^{5}
							1		6		15		20		15		6		1								2^{6}
						1		7		21		35		35		21		7		1							2^{7}
					1		8		28		56		70		56		28		8		1						2^{8}
				1		9		36		84		126		126		84		36		9		1					2^{9}
			1		10		45		120		210		252		210		120		45		10		1				2^{10}
		1		11		55		165		330		462		462		330		165		55		11		1			2^{11}
	1		12		66		220		495		792		924		792		495		220		66		12		1		2^{12}
1		13		78		286		715		1287		1716		1716		1287		715		286		78		13		1	2^{13}

Table 3.

Analogous Pascal's triangle

														1		2															3
													1		3		2														6
												1		4		5		2													12
											1		5		9		7		2												24
										1		6		14		16		9		2											48
									1		7		20		30		25		11		2										96
								1		8		27		50		55		36		13		2									192
							1		9		35		77		105		91		49		15		2								384
						1		10		44		112		182		196		140		64		17		2							768
					1		11		54		156		294		378		336		204		81		19		2						1536
				1		12		65		210		450		672		714		540		285		100		21		2					3072
			1		13		77		275		660		1122		1386		1254		825		385		121		23		2				6144
		1		14		90		352		935		1782		2508		2640		2079		1210		506		144		25		2			12288
	1		15		104		442		1287		2717		4290		5148		4719		3289		1716		650		169		27		2		24576
1		16		119		546		1729		4004		7007		9438		9867		8008		5005		2366		819		196		29		2	49152

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