

# Treatment of the Nonlinear Problems Through the Generalization of a Theorem from Parameterized Nonlinear Continuous Functions

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*Abstract:* - This paper focuses on  $\bar{g}$ -calculus for modified  $\bar{g}$ -functions as  $f_{\bar{g}}$ -function and relations through some interesting pseudo-functional equations as their solutions and also expression by some parameterized nonlinear continuous functions  $t_{\bar{g}}$ . The main problem we are dealing with in this research paper is generalizing the relations formulated by Rybárik's theorem for pseudo-functional equations and complementing it with some other treated cases expressed by some parameterized nonlinear continuous functions. We have proven in detail all the cases of Rybárik's theorem. Still, in this paper, we will generalize them and some exceptional cases related to the conditions that satisfy the set of parameters or functions  $(\alpha, \lambda, f, h, \bar{g})$  that participate in the relevant pseudo-nonlinear relations of the generalized theorem from parameterized nonlinear continuous functions. Furthermore, some exceptional cases for each relation are presented as main results that will connect this generalization theorem with Rybárik's theorem and lead us to new results. We have built a system of important pseudo-nonlinear relations for representation the  $f_{\bar{g}}$ -calculus,  $t_{\bar{g}}$ -calculus by the parameterized nonlinear continuous functions  $t_{\bar{g}}$ . Also, we are again in cases of classical analysis for the particular case of generator  $\bar{g}$  as  $\bar{g}$ -identity.

*Key-Words:* - Pseudo-Arithmetic Operation, Generator,  $\bar{g}$ -calculus,  $\bar{g}$ -functions,  $\bar{g}$ -transform, Modified Function, Pseudo-Functional Equation, Pseudo-Nonlinear Problem Generalized Theorem, Parameterized Nonlinear Continuous Function

Received: July 11, 2023. Revised: December 18, 2023. Accepted: February 4, 2024. Published: April 2, 2024.

## 1 Introduction

This field of Pseudo-Analysis is studied by many authors, [1], [2], [3], [4], [5], [6], [7], and the axiomatic concepts of pseudo-arithmetic operations (also their extension problem) are treated and supported by  $\bar{g}$ -generators:  $\{\bar{\oplus}, \bar{\odot}, \bar{\ominus}, \bar{\oslash}\} = \{\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}}, \bar{\ominus}_{\bar{g}}, \bar{\oslash}_{\bar{g}}\}$ . The concept of pseudo-arithmetical operations  $\{\oplus, \odot, \ominus, \oslash\}$  as a system generated by the generator  $g$ , was first introduced on the interval  $[0, +\infty]$  and then to the whole extended real line  $\bar{R} = [-\infty, +\infty]$ , [2], [4], [5], [7], [8], [9], [10], [11], and the generator  $\bar{g}$  is extended in  $\bar{R}$ , [12], [13].

The role of the extended pseudo-arithmetic operations is shown directly by taking the Rybárik, [2], rational functions and Pap  $\bar{g}$ -calculus, [1], [3]. The consistent system of  $\{\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}}, \bar{\ominus}_{\bar{g}}, \bar{\oslash}_{\bar{g}}\}$  is a valuable tool in treating [2] nonlinear problems and some elementary  $\bar{g}$ -functions ( $f_{\bar{g}}$ ) corresponding to the function  $f$  as modified functions by  $\bar{g}$ -transform are derived as

solutions of some functional equations using the results of Aczél, [14]. We are generalizing the relations formulated by Rybárik's theorem for pseudo-functional equations and complementing it with some other treated cases expressed by some parameterized nonlinear continuous functions  $t_{\bar{g}}$ , [15], [16]. This paper stops at  $\bar{g}$ -calculus for these modified  $\bar{g}$ -functions and their relations through interesting functional equations, [2] and represent the system of important pseudo-nonlinear relations for the  $f_{\bar{g}}$ -calculus,  $t_{\bar{g}}$ -calculus by functions  $t_{\bar{g}}$ .

The extended forms of  $\bar{g}$ -calculus, [1], [3], [5], [6], [8], [9], [10], [12], are present in four relations listed below:

$$\bar{\oplus}_{\bar{g}}(x, y) = x \bar{\oplus}_{\bar{g}} y = \bar{g}^{-1}(\bar{g}(x) + \bar{g}(y));$$

$$\bar{\odot}_{\bar{g}}(x, y) = x \bar{\odot}_{\bar{g}} y = \bar{g}^{-1}(\bar{g}(x) \cdot \bar{g}(y));$$

$$\bar{\ominus}_{\bar{g}}(x, y) = x \bar{\ominus}_{\bar{g}} y = \bar{g}^{-1}(\bar{g}(x) - \bar{g}(y));$$

$$\bar{\oslash}_{\bar{g}}(x, y) = x \bar{\oslash}_{\bar{g}} y = \bar{g}^{-1}(\bar{g}(x)/\bar{g}(y)).$$

## 2 Problem Formulation

### 2.1 Study Method and Mathematical Instrument

We recall some concepts of Pseudo-Analysis, the properties of the system pseudo-operations and their generators, the process of modifying function by  $\bar{g}$ -transforms, interweaving  $\bar{g}$ -functions in some interesting pseudo-functional equations, [2], [10], [16] and other related results. Our study method consists of the systematization of the theoretical material explicitly treated in the field of Generated Pseudo-Analysis, and through a detailed mathematical analysis, we describe the generalization of some relations featured by pseudo-functional equations treated by Rybárik, [2] and presented by Theorem 4 in, [2]. The generalisation process is related not only to the five cases treated by Rybárik in, [2], verified by us previously in, [8], but also to three other new cases generalized in this paper and expressed by some parameterized nonlinear continuous functions.

Rybárik's theorem is proved in detail for all the cases by theorem 2.3 in, [8]. Still, in this paper, we will generalize them and some exceptional cases related to the conditions that satisfy the set of our parameters or functions  $(\alpha, \lambda, f, h, \bar{g})$  that participate in the eight relevant relations of the generalized theorem 3.1 in this paper at the problem solution session. Furthermore, some exceptional cases for each relation will connect this generalization theorem with the very interesting Theorem 4 of Rybárik and lead us to some new results summarized as particular cases (C1 ÷ C8) at the main result session.

The mathematical instrument provides us with the algorithms for all the cases treated in the paper. This instrument leads us to extensions of pseudo-functional equations and exceptional cases, showing once again the role of the  $\bar{g}$ -generator in the processes of transformation and generalization. Moreover, the diversity of the generator's nature in this generalization process leads to a concrete link between Pseudo-Analysis and Classical Analysis.

This treatment again highlights the connection of Pseudo-Analysis, as a field of our study, with Pseudo-Linear Algebra, Information Theory, Elementary Algebra and other areas of pure mathematics.

#### 2.1.1 A Definition and Some Relations for $\bar{g}$ -function

**Definition 2.2.1.** Let  $f$  be a function on  $]a, b[ \subseteq ]-\infty, +\infty[$ , and the function  $\bar{g}$  be a generator of the consistent system of pseudo-arithmetical operations  $\{\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}}, \bar{\ominus}_{\bar{g}}, \bar{\oslash}_{\bar{g}}\}$ , [2], [15], [16].

The function  $f_{\bar{g}}$  given by  $f_{\bar{g}}(x) = \bar{g}^{-1}(f(\bar{g}(x)))$  for every  $x \in (\bar{g}^{-1}(a), \bar{g}^{-1}(b))$  is said to be  $\bar{g}$ -function corresponding to the function  $f$ .

The parameterized nonlinear continuous functions  $\mathbf{t}$ ,  $\mathbf{t}: \bar{R} \times \bar{R} \rightarrow \bar{R}$ ,  $a_1, a_2, a_3, a_4 \in R$  are defined in the form below:

$$\begin{aligned} \mathbf{t}^{(+, \cdot)(a_1, a_2, a_3, a_4)}(x, y) &= \\ &= a_1 \cdot x \cdot y + a_2 \cdot x + a_3 \cdot y + a_4. \end{aligned}$$

By the definition for the composition of the two our functions  $\mathbf{t}$  and  $f$  we get [15]:

$$\begin{aligned} h(x, y) &= (\mathbf{t} \circ f)(x, y) = \mathbf{t}(f(x), f(y)) = \\ &= \mathbf{t}^{(+, \cdot)(a_1, a_2, a_3, a_4)}(f(x), f(y)). \end{aligned}$$

Also, by the definition of the function  $\mathbf{t}$  we have the form:

$$\begin{aligned} \mathbf{t}^{(+, \cdot)(a_1, a_2, a_3, a_4)}(f(x), f(y)) &= \\ &= a_1 \cdot f(x) \cdot f(y) + a_2 \cdot f(x) + a_3 \cdot f(y) + a_4. \end{aligned}$$

When  $\bar{g}$ -transform is applied for  $h = \mathbf{t} \circ f$ , the definition of  $\bar{g}$ -function brings us the equation of  $h_{\bar{g}}$  as:

$$\begin{aligned} h_{\bar{g}}(x, y) &= \left( \mathbf{t}^{(+, \cdot)(a_1, a_2, a_3, a_4)}(f(x), f(y)) \right)_{\bar{g}} = \\ &= t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})}(\bar{g}^{-1}(a_1), \bar{g}^{-1}(a_2), \bar{g}^{-1}(a_3), \bar{g}^{-1}(a_4))(f_{\bar{g}}(x), f_{\bar{g}}(y)). \end{aligned}$$

For the special cases when  $f(x) = \bar{g}(x)$ ;  $f(y) = \bar{g}(y)$ , we have:

$$\bar{g}^{-1}(\bar{g}(x)) = \bar{g}_{\bar{g}}(x) = x_{\bar{g}} = x, \text{ and}$$

$$\bar{g}^{-1}(\bar{g}(y)) = \bar{g}_{\bar{g}}(y) = y_{\bar{g}} = y.$$

So,  $\mathbf{t}_{\bar{g}}$  can be written in the form:

$$\begin{aligned} \mathbf{t}^{(+, \cdot)(a_1, a_2, a_3, a_4)}(\bar{g}(x), \bar{g}(y)) &= \\ &= a_1 \cdot \bar{g}(x) \cdot \bar{g}(y) + a_2 \cdot \bar{g}(x) + a_3 \cdot \bar{g}(y) + a_4, \end{aligned}$$

$$\begin{aligned} \mathbf{t}_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})}(\bar{g}^{-1}(a_1), \bar{g}^{-1}(a_2), \bar{g}^{-1}(a_3), \bar{g}^{-1}(a_4))(x, y) &= \\ &= (\bar{g}^{-1}(a_1) \bar{\odot}_{\bar{g}} x \bar{\odot}_{\bar{g}} y) \bar{\oplus}_{\bar{g}} (\bar{g}^{-1}(a_2) \bar{\odot}_{\bar{g}} x) \bar{\oplus}_{\bar{g}} \\ &\quad \bar{\oplus}_{\bar{g}} (\bar{g}^{-1}(a_3) \bar{\odot}_{\bar{g}} y) \bar{\oplus}_{\bar{g}} (\bar{g}^{-1}(a_4)). \end{aligned}$$

For  $\bar{g}$ -normed and  $a_1 = a_2 = a_3 = a_4 = 1$ , the form of  $\mathbf{t}$  and  $\mathbf{t}_{\bar{g}}$  is respectively:

$$\mathbf{t}^{(+, \cdot)(1, 1, 1, 1)}(x, y) = x \cdot y + x + y + 1,$$

$$\begin{aligned} \mathbf{t}_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})(1, 1, 1, 1)}(x, y) &= \\ &= (x \bar{\odot}_{\bar{g}} y) \bar{\oplus}_{\bar{g}} (x) \bar{\oplus}_{\bar{g}} (y) \bar{\oplus}_{\bar{g}} (1). \end{aligned}$$

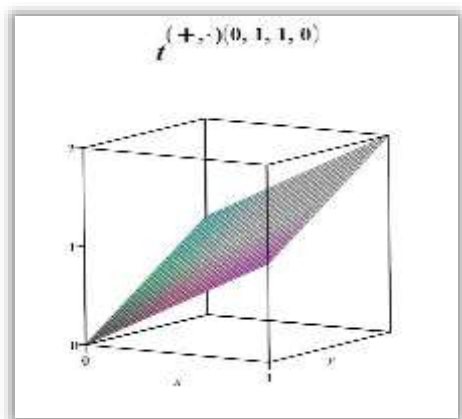
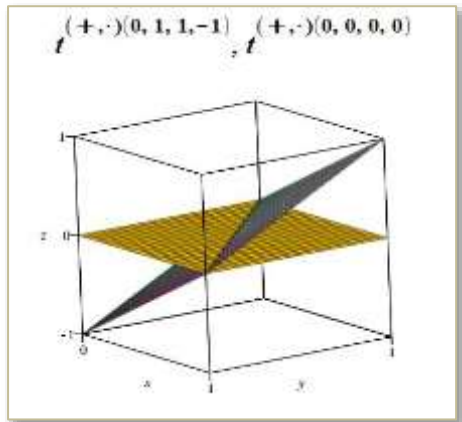
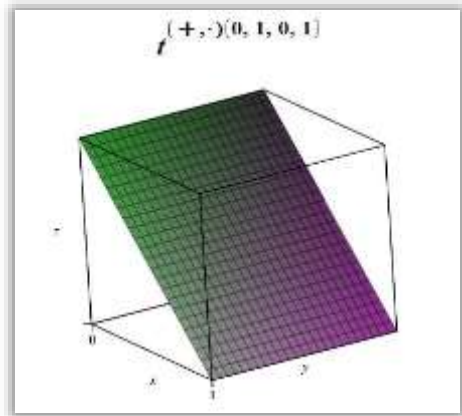
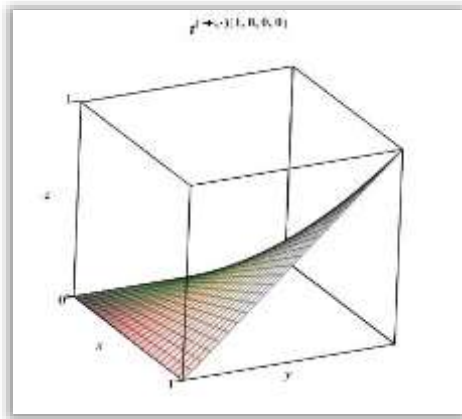


Fig. 2.2.1. Some parameterized nonlinear continuous functions  $t$ , for  $\bar{g}$  - normed

**Theorem 2.2.2.** Let  $\bar{g}$  be a generator of the consistent system of pseudo-arithmetical operations  $\{\oplus, \odot, \ominus, \oslash\} = \{\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}}, \bar{\ominus}_{\bar{g}}, \bar{\oslash}_{\bar{g}}\}$ . Let  $f$  and  $h$  be continuous functions on  $]a, b[ \subseteq ]-\infty, +\infty[$  and  $\alpha, \lambda \in ]-\infty, +\infty[$  are constants [2], [8]. Then for every  $x \in ]\bar{g}^{-1}(a), \bar{g}^{-1}(b)[$  we have:

1.  $(\alpha \cdot f)_{\bar{g}} = \bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} f_{\bar{g}} = \bar{\odot}_{\bar{g}}(\bar{g}^{-1}(\alpha), f_{\bar{g}})$
2.  $(f + h)_{\bar{g}} = f_{\bar{g}} \bar{\oplus}_{\bar{g}} h_{\bar{g}} = \bar{\oplus}_{\bar{g}}(f_{\bar{g}}, h_{\bar{g}})$
3.  $(f - h)_{\bar{g}} = f_{\bar{g}} \bar{\ominus}_{\bar{g}} h_{\bar{g}} = \bar{\ominus}_{\bar{g}}(f_{\bar{g}}, h_{\bar{g}})$
4.  $(f \cdot h)_{\bar{g}} = f_{\bar{g}} \bar{\odot}_{\bar{g}} h_{\bar{g}} = \bar{\odot}_{\bar{g}}(f_{\bar{g}}, h_{\bar{g}})$
5.  $\left(\frac{f}{h}\right)_{\bar{g}} = f_{\bar{g}} \bar{\oslash}_{\bar{g}} h_{\bar{g}} = \bar{\oslash}_{\bar{g}}(f_{\bar{g}}, h_{\bar{g}})$ , with the conditions for values of function  $h$ : for each value of  $x \in ]\bar{g}^{-1}(a), \bar{g}^{-1}(b)[$ ,  $h(x) \neq 0$ .

By the definition of the  $\bar{g}$  - calculus, the modified function  $f_{\bar{g}}$  and the parameterized nonlinear continuous functions  $t, t_{\bar{g}}$ , step by step, we take the pseudo-nonlinear relations for each case 1 ÷ 5 of the theorem.

$$\begin{aligned}
 1. (\alpha \cdot f)_{\bar{g}} &= \left( t^{(+, \cdot)}(0, \alpha, 0, 0)(f(x), f(y)) \right)_{\bar{g}} = \\
 &= \left( t^{(+, \cdot)}(0, 0, 0, \alpha)(f(x), f(y)) \cdot t^{(+, \cdot)}(0, 1, 0, 0)(f(x), f(y)) \right)_{\bar{g}} = \\
 &= \bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} f_{\bar{g}} = \bar{\odot}_{\bar{g}}(\bar{g}^{-1}(\alpha), f_{\bar{g}}) = \\
 &= t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})}(\bar{g}^{-1}(0), \bar{g}^{-1}(\alpha), \bar{g}^{-1}(0), \bar{g}^{-1}(0))(f_{\bar{g}}(x), f_{\bar{g}}(y)) = \\
 &= t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})}(\bar{g}^{-1}(0), \bar{g}^{-1}(0), \bar{g}^{-1}(0), \bar{g}^{-1}(\alpha))(f_{\bar{g}}(x), f_{\bar{g}}(y)) \bar{\odot}_{\bar{g}} \\
 &\quad \bar{\odot}_{\bar{g}} t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})}(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0))(f_{\bar{g}}(x), f_{\bar{g}}(y)) \\
 2. (f + h)_{\bar{g}} &= \\
 &= \left( t^{(+, \cdot)}(0, 1, 0, 0)(f(x), f(y)) + t^{(+, \cdot)}(0, 1, 0, 0)(h(x), h(y)) \right)_{\bar{g}} = \\
 &= f_{\bar{g}} \bar{\oplus}_{\bar{g}} h_{\bar{g}} = \bar{\oplus}_{\bar{g}}(f_{\bar{g}}, h_{\bar{g}}) = \\
 &= t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})}(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0))(f_{\bar{g}}(x), f_{\bar{g}}(y)) \bar{\oplus}_{\bar{g}} \\
 &\quad \bar{\oplus}_{\bar{g}} t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})}(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0))(h_{\bar{g}}(x), h_{\bar{g}}(y)). \\
 3. (f - h)_{\bar{g}} &= \\
 &= \left( t^{(+, \cdot)}(0, 1, 0, 0)(f(x), f(y)) - t^{(+, \cdot)}(0, 1, 0, 0)(h(x), h(y)) \right)_{\bar{g}} = \\
 &= f_{\bar{g}} \bar{\ominus}_{\bar{g}} h_{\bar{g}} = \bar{\ominus}_{\bar{g}}(f_{\bar{g}}, h_{\bar{g}}) =
 \end{aligned}$$

$$= t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})}(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0)) (f_{\bar{g}}(x), f_{\bar{g}}(y)) \bar{\ominus}_{\bar{g}}$$

$$\bar{\ominus}_{\bar{g}} t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})}(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0)) (h_{\bar{g}}(x), h_{\bar{g}}(y)).$$

$$4. (f \cdot h)_{\bar{g}} =$$

$$= \left( t^{(+, \cdot)(0, 1, 0, 0)}(f(x), f(y)) \cdot t^{(+, \cdot)(0, 1, 0, 0)}(h(x), h(y)) \right)_{\bar{g}} =$$

$$= f_{\bar{g}} \bar{\ominus}_{\bar{g}} h_{\bar{g}} = \bar{\ominus}_{\bar{g}} (f_{\bar{g}}, h_{\bar{g}}) =$$

$$= t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})}(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0)) (f_{\bar{g}}(x), f_{\bar{g}}(y)) \bar{\ominus}_{\bar{g}}$$

$$\bar{\ominus}_{\bar{g}} t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})}(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0)) (h_{\bar{g}}(x), h_{\bar{g}}(y)).$$

$$5. \left( \frac{f}{h} \right)_{\bar{g}} = \left( \frac{t^{(+, \cdot)(0, 1, 0, 0)}(f(x), f(y))}{t^{(+, \cdot)(0, 1, 0, 0)}(h(x), h(y))} \right)_{\bar{g}} =$$

$$= f_{\bar{g}} \bar{\ominus}_{\bar{g}} h_{\bar{g}} = \bar{\ominus}_{\bar{g}} (f_{\bar{g}}, h_{\bar{g}}) =$$

$$= t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})}(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0)) (f_{\bar{g}}(x), f_{\bar{g}}(y)) \bar{\ominus}_{\bar{g}}$$

$$\bar{\ominus}_{\bar{g}} t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})}(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0)) (h_{\bar{g}}(x), h_{\bar{g}}(y)),$$

with the conditions for values of function  $h$ : for each value of  $x \in ]\bar{g}^{-1}(a), \bar{g}^{-1}(b)[$ ,  $h(x) \neq 0$ .

By the definition of the  $\bar{g}$ -calculus and  $f_{\bar{g}}$  as a  $\bar{g}$ -function corresponding to the function  $f$ , also, the parameterized nonlinear continuous functions  $t, t_{\bar{g}}$ , all these relations of Theorem 4 of Rybárik are proven in, [2] (case 2) and [8], (case 1, 3, 4, 5).

### 3 Problem Solution

In this session, we are treating the nonlinear problems through the generalization of Theorem 4 of Rybárik presented in [2] and its proof for each of the eight generalized cases and complementing the process with some other treated cases expressed by some parameterized nonlinear continuous functions  $t, t_{\bar{g}}$ , [15], [16]. Considering the conclusions from this generalization, we will also parallelize the connections with the treatments in Theorem 4 of Rybárik, with the values of the parameters or the conditions dictated for the functions.

#### 3.1 The Generalized Theorem for Some Pseudo-Functional Equations

**Theorem 3.1.** Let  $\bar{g}$  be a generator of the consistent system of four pseudo-arithmetical operations. Let  $f$  and  $h$  be continuous functions

on interval  $]a, b[ \subseteq ]-\infty, +\infty[$  and  $\alpha, \lambda \in ]-\infty, +\infty[$  are constants [2].

Then, for every  $x \in ](a), \bar{g}^{-1}(b)[$ , we have the functional relations:

$$1. (\alpha + f)_{\bar{g}} = \bar{g}^{-1}(\alpha) \bar{\oplus}_{\bar{g}} f_{\bar{g}} = \bar{\oplus}_{\bar{g}} (\bar{g}^{-1}(\alpha), f_{\bar{g}}).$$

$$2. (\alpha \cdot f)_{\bar{g}} = \bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} f_{\bar{g}} = \bar{\odot}_{\bar{g}} (\bar{g}^{-1}(\alpha), f_{\bar{g}}).$$

$$3. (\alpha \cdot f + \lambda \cdot h)_{\bar{g}} =$$

$$= (\bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} f_{\bar{g}}) \bar{\oplus}_{\bar{g}} (\bar{g}^{-1}(\lambda) \bar{\odot}_{\bar{g}} h_{\bar{g}}) =$$

$$= (\alpha \cdot f)_{\bar{g}} \bar{\oplus}_{\bar{g}} (\lambda \cdot h)_{\bar{g}}.$$

$$4. [\alpha \cdot (f + h)]_{\bar{g}} = \bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} (f_{\bar{g}} \bar{\oplus}_{\bar{g}} h_{\bar{g}}) =$$

$$= \bar{\odot}_{\bar{g}} (\bar{g}^{-1}(\alpha), (f_{\bar{g}} \bar{\oplus}_{\bar{g}} h_{\bar{g}})).$$

$$5. (\alpha \cdot f - \lambda \cdot h)_{\bar{g}} =$$

$$= (\bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} f_{\bar{g}}) \bar{\ominus}_{\bar{g}} (\bar{g}^{-1}(\lambda) \bar{\odot}_{\bar{g}} h_{\bar{g}}).$$

$$6. [\alpha \cdot (f - h)]_{\bar{g}} =$$

$$= \bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} (f_{\bar{g}} \bar{\ominus}_{\bar{g}} h_{\bar{g}}) =$$

$$= \bar{\odot}_{\bar{g}} (\bar{g}^{-1}(\alpha), (f_{\bar{g}} \bar{\ominus}_{\bar{g}} h_{\bar{g}})).$$

$$7. \left( \frac{\alpha f}{\lambda h} \right)_{\bar{g}} = (\bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} f_{\bar{g}}) \bar{\odot}_{\bar{g}} (\bar{g}^{-1}(\lambda) \bar{\odot}_{\bar{g}} h_{\bar{g}})$$

for conditions  $\alpha \neq \lambda \neq 1, \lambda \neq 0$ .

$$8. [(\alpha \cdot f) \cdot (\lambda \cdot h)]_{\bar{g}} = (\alpha \cdot f)_{\bar{g}} \bar{\odot}_{\bar{g}} (\lambda \cdot h)_{\bar{g}} =$$

$$= (\bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} \bar{g}^{-1}(\lambda)) \bar{\odot}_{\bar{g}} (f_{\bar{g}}(x) \bar{\odot}_{\bar{g}} h_{\bar{g}}(x)).$$

Proof: By the definition of the  $\bar{g}$ -calculus, the modified function  $f_{\bar{g}}$  and the parameterized nonlinear continuous functions  $t, t_{\bar{g}}$ , step by step, we take the relations for each case 1 ÷ 8 of the theorem.

$$1. (\alpha + f)_{\bar{g}} = (\alpha + f(x))_{\bar{g}} =$$

$$= \left( t^{(+, \cdot)(0, 0, 0, \infty)}(f(x), f(y)) + \right)_{\bar{g}} =$$

$$= \left( t^{(+, \cdot)(0, 1, 0, 0)}(f(x), f(y)) \right)_{\bar{g}} =$$

$$= \left( t^{(+, \cdot)(0, 1, 0, \infty)}(f(x), f(y)) \right)_{\bar{g}} =$$

$$= \bar{g}^{-1}(\alpha + f(\bar{g}(x))) =$$

$$= \bar{g}^{-1} \left\{ \bar{g}(\bar{g}^{-1}(\alpha)) + \bar{g} \left( \bar{g}^{-1}(f(\bar{g}(x))) \right) \right\} =$$

$$= \bar{g}^{-1}(\alpha) \bar{\oplus}_{\bar{g}} f_{\bar{g}}(x) =$$

$$\begin{aligned}
&= \bar{g}^{-1}(\alpha) \bar{\oplus}_{\bar{g}} f_{\bar{g}}(x) = \bar{\oplus}_{\bar{g}} \left( \bar{g}^{-1}(\alpha), f_{\bar{g}} \right) = \\
&= t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})}(\bar{g}^{-1}(0), \bar{g}^{-1}(\alpha), \bar{g}^{-1}(0), \bar{g}^{-1}(0)) \left( f_{\bar{g}}(x), f_{\bar{g}}(y) \right) \bar{\oplus}_{\bar{g}} \\
&\bar{\oplus}_{\bar{g}} t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})}(\bar{g}^{-1}(0), \bar{g}^{-1}(\alpha), \bar{g}^{-1}(0), \bar{g}^{-1}(0)) \left( h_{\bar{g}}(x), h_{\bar{g}}(y) \right) = \\
&= t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})}(\bar{g}^{-1}(0), \bar{g}^{-1}(\alpha), \bar{g}^{-1}(0), \bar{g}^{-1}(\alpha)) \left( f_{\bar{g}}(x), f_{\bar{g}}(y) \right).
\end{aligned}$$

This pseudo-linear relation shows that the  $\bar{g}$ -transform of the addition of a constant with a function is equal to the pseudo-addition of a pseudo-constant with a  $\bar{g}$ -function.

$$2. (\alpha \cdot f)_{\bar{g}} = \bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} f_{\bar{g}} = \bar{\odot}_{\bar{g}} \left( \bar{g}^{-1}(\alpha), f_{\bar{g}} \right).$$

This pseudo-linear relation is proved in detail in, [10] and session 2.2., the Theorem 2.2.2., case 1

$$\begin{aligned}
3. (\alpha \cdot f + \lambda \cdot h)_{\bar{g}} &= \left( \alpha \cdot f(x) + \lambda \cdot h(x) \right)_{\bar{g}} = \\
&= \bar{g}^{-1} \left( \alpha \cdot f(\bar{g}(x)) + \lambda \cdot h(\bar{g}(x)) \right) = \\
&= \left( t^{(+, \cdot)}(0, \alpha, 0, 0) (f(x), f(y)) + \right. \\
&\quad \left. + t^{(+, \cdot)}(0, \lambda, 0, 0) (h(x), h(y)) \right)_{\bar{g}} = \\
&= \bar{g}^{-1} \left( \bar{g}(\bar{g}^{-1}(\alpha)) \cdot \bar{g} \left( \bar{g}^{-1} \left( f(\bar{g}(x)) \right) \right) + \right. \\
&\quad \left. + \bar{g}(\bar{g}^{-1}(\lambda)) \cdot \bar{g} \left( \bar{g}^{-1} \left( h(\bar{g}(x)) \right) \right) \right) = \\
&= \bar{g}^{-1} \left\{ \bar{g} \left( \frac{\bar{g}^{-1} \left( \bar{g}(\bar{g}^{-1}(\alpha)) \cdot \bar{g} \left( \bar{g}^{-1} \left( f(\bar{g}(x)) \right) \right) \right)}{\bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} f_{\bar{g}}(x)} \right) + \right. \\
&\quad \left. + \bar{g} \left( \frac{\bar{g}^{-1} \left( \bar{g}(\bar{g}^{-1}(\lambda)) \cdot \bar{g} \left( \bar{g}^{-1} \left( h(\bar{g}(x)) \right) \right) \right)}{\bar{g}^{-1}(\lambda) \bar{\odot}_{\bar{g}} h_{\bar{g}}(x)} \right) \right\} = \\
&= \bar{g}^{-1} \left\{ \bar{g} \left( \bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} f_{\bar{g}}(x) \right) + \right. \\
&\quad \left. + \bar{g} \left( \bar{g}^{-1}(\lambda) \bar{\odot}_{\bar{g}} h_{\bar{g}}(x) \right) \right\} = \\
&= \left( \bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} f_{\bar{g}}(x) \right) \bar{\oplus}_{\bar{g}} \left( \bar{g}^{-1}(\lambda) \bar{\odot}_{\bar{g}} h_{\bar{g}}(x) \right) = \\
&= (\alpha \cdot f)_{\bar{g}} \bar{\oplus}_{\bar{g}} (\lambda \cdot h)_{\bar{g}} = \\
&= t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})}(\bar{g}^{-1}(0), \bar{g}^{-1}(\alpha), \bar{g}^{-1}(0), \bar{g}^{-1}(0)) \left( f_{\bar{g}}(x), f_{\bar{g}}(y) \right) \bar{\oplus}_{\bar{g}} \\
&\bar{\oplus}_{\bar{g}} t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})}(\bar{g}^{-1}(0), \bar{g}^{-1}(\lambda), \bar{g}^{-1}(0), \bar{g}^{-1}(0)) \left( f_{\bar{g}}(x), f_{\bar{g}}(y) \right).
\end{aligned}$$

This pseudo-nonlinear relation shows that the  $\bar{g}$ -transform of addition of two results functions (each function is a monomial as the multiplication

of a constant with a function) is equal to the pseudo-addition of two modified functions (each modified function is a pseudo-multiplication of pseudo-constant with a  $\bar{g}$ -function).

$$\begin{aligned}
4. [\alpha \cdot (f + h)]_{\bar{g}} &= \left[ \alpha \cdot (f(x) + h(x)) \right]_{\bar{g}} = \\
&= \bar{g}^{-1} \left( \alpha \cdot (f(\bar{g}(x)) + h(\bar{g}(x))) \right) = \\
&= \left( t^{(+, \cdot)}(0, 0, 0, \alpha) (f(x), f(y)) \cdot \right. \\
&\quad \left. + t^{(+, \cdot)}(0, 1, 0, 0) (f(x), f(y)) + \right. \\
&\quad \left. + t^{(+, \cdot)}(0, 1, 0, 0) (h(x), h(y)) \right)_{\bar{g}} = \\
&= \left( t^{(+, \cdot)}(0, 0, 0, \alpha) (f(x), f(y)) \cdot \right. \\
&\quad \left. + t^{(+, \cdot)}(0, 1, 0, 0) (f(x), f(y)) + \right. \\
&\quad \left. + t^{(+, \cdot)}(0, 1, 0, 0) (h(x), h(y)) \right)_{\bar{g}} = \\
&= \bar{g}^{-1} \left\{ \bar{g}(\bar{g}^{-1}(\alpha)) \cdot \bar{g} \left( \bar{g}^{-1} \left( \bar{g}(\bar{g}^{-1}(f(\bar{g}(x)))) + \right. \right. \right. \\
&\quad \left. \left. + \bar{g}(\bar{g}^{-1}(h(\bar{g}(x)))) \right) \right) \right\} = \\
&= \bar{g}^{-1} \left( \bar{g}(\bar{g}^{-1}(\alpha)) \cdot \bar{g} (f_{\bar{g}}(x) \bar{\oplus}_{\bar{g}} h_{\bar{g}}(x)) \right) = \\
&= \bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} (f_{\bar{g}}(x) \bar{\oplus}_{\bar{g}} h_{\bar{g}}(x)) = \\
&= \bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} (f_{\bar{g}} \bar{\oplus}_{\bar{g}} h_{\bar{g}}) = \\
&= \bar{\odot}_{\bar{g}} \left( \bar{g}^{-1}(\alpha), (f_{\bar{g}} \bar{\oplus}_{\bar{g}} h_{\bar{g}}) \right) = \\
&= t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})}(\bar{g}^{-1}(0), \bar{g}^{-1}(0), \bar{g}^{-1}(0), \bar{g}^{-1}(\alpha)) \left( f_{\bar{g}}(x), f_{\bar{g}}(y) \right) \bar{\odot}_{\bar{g}} \\
&\bar{\oplus}_{\bar{g}} t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})}(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0)) \left( f_{\bar{g}}(x), f_{\bar{g}}(y) \right) \bar{\oplus}_{\bar{g}} \\
&\bar{\oplus}_{\bar{g}} t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})}(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0)) \left( f_{\bar{g}}(x), f_{\bar{g}}(y) \right).
\end{aligned}$$

This pseudo-nonlinear relation shows that the  $\bar{g}$ -transform of the multiplication of a constant with a sum function (addition of two functions  $f, h$ ) is equal to the pseudo-multiplication of a pseudo-constant with the pseudo-addition of two modified functions ( $f_{\bar{g}}, h_{\bar{g}}$ ).

$$\begin{aligned}
5. (\alpha \cdot f - \lambda \cdot h)_{\bar{g}} &= \left( \alpha \cdot f(x) - \lambda \cdot h(x) \right)_{\bar{g}} = \\
&= \bar{g}^{-1} \left( \alpha \cdot f(\bar{g}(x)) - \lambda \cdot h(\bar{g}(x)) \right) = \\
&= \left( t^{(+, \cdot)}(0, \alpha, 0, 0) (f(x), f(y)) - \right. \\
&\quad \left. - t^{(+, \cdot)}(0, \lambda, 0, 0) (h(x), h(y)) \right)_{\bar{g}} =
\end{aligned}$$

$$\begin{aligned}
&= \bar{g}^{-1} \left( \frac{\bar{g}(\bar{g}^{-1}(\alpha)) \cdot \bar{g}(\bar{g}^{-1}(f(\bar{g}(x))))}{-\bar{g}(\bar{g}^{-1}(\lambda)) \cdot \bar{g}(\bar{g}^{-1}(h(\bar{g}(x))))} \right) = \\
&= \bar{g}^{-1} \left\{ \frac{\bar{g} \left( \frac{\bar{g}(\bar{g}^{-1}(\alpha)) \cdot \bar{g}(\bar{g}^{-1}(f(\bar{g}(x))))}{\bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} f_{\bar{g}}(x)} \right)}{-\bar{g} \left( \frac{\bar{g}(\bar{g}^{-1}(\lambda)) \cdot \bar{g}(\bar{g}^{-1}(h(\bar{g}(x))))}{\bar{g}^{-1}(\lambda) \bar{\odot}_{\bar{g}} h_{\bar{g}}(x)} \right)} \right\} = \\
&= \bar{g}^{-1} \left\{ \frac{\bar{g}(\bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} f_{\bar{g}}(x))}{-\bar{g}(\bar{g}^{-1}(\lambda) \bar{\odot}_{\bar{g}} h_{\bar{g}}(x))} \right\} = \\
&= (\bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} f_{\bar{g}}(x)) \bar{\ominus}_{\bar{g}} (\bar{g}^{-1}(\lambda) \bar{\odot}_{\bar{g}} h_{\bar{g}}(x)) = \\
&= (\alpha \cdot f)_{\bar{g}} \bar{\ominus}_{\bar{g}} (\lambda \cdot h)_{\bar{g}} = \\
&= t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(\alpha), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (f_{\bar{g}}(x), f_{\bar{g}}(y)) \bar{\ominus}_{\bar{g}} \\
&\bar{\ominus}_{\bar{g}} t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(\lambda), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (f_{\bar{g}}(x), f_{\bar{g}}(y)).
\end{aligned}$$

This pseudo-nonlinear relation shows that the  $\bar{g}$  – transform of subtraction of two functions (each function is a monomial as the multiplication of a constant with a function) is equal to the pseudo-subtraction of two modified functions (each modified function is the pseudo-multiplication of pseudo-constant with a  $\bar{g}$  – function).

$$\begin{aligned}
6. [\alpha \cdot (f - h)]_{\bar{g}} &= [\alpha \cdot (f(x) - h(x))]_{\bar{g}} = \\
&= \bar{g}^{-1} (\alpha \cdot (f(\bar{g}(x)) - h(\bar{g}(x)))) = \\
&= \left( \begin{array}{c} \mathbf{t}^{(+, \cdot)(0,0,0,\alpha)}(f(x), f(y)) \cdot \\ \mathbf{t}^{(+, \cdot)(0,1,0,0)}(f(x), f(y)) - \\ -\mathbf{t}^{(+, \cdot)(0,1,0,0)}(h(x), h(y)) \end{array} \right)_{\bar{g}} = \\
&= \bar{g}^{-1} \left\{ \frac{\bar{g}(\bar{g}^{-1}(\alpha)) \cdot \bar{g} \left( \frac{\bar{g}(\bar{g}^{-1}(f(\bar{g}(x))))}{-\bar{g}(\bar{g}^{-1}(h(\bar{g}(x))))} \right)}{\bar{f}_{\bar{g}}(x) \bar{\ominus}_{\bar{g}} h_{\bar{g}}(x)} \right\} = \\
&= \bar{g}^{-1} (\bar{g}(\bar{g}^{-1}(\alpha)) \cdot \bar{g} (f_{\bar{g}}(x) \bar{\ominus}_{\bar{g}} h_{\bar{g}}(x))) = \\
&= \bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} (f_{\bar{g}}(x) \bar{\ominus}_{\bar{g}} h_{\bar{g}}(x)) = \\
&= \bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} (f_{\bar{g}} \bar{\ominus}_{\bar{g}} h_{\bar{g}}) =
\end{aligned}$$

$$\begin{aligned}
&= \bar{\odot}_{\bar{g}} (\bar{g}^{-1}(\alpha), (f_{\bar{g}} \bar{\ominus}_{\bar{g}} h_{\bar{g}})) = \\
&= t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(0), \bar{g}^{-1}(0), \bar{g}^{-1}(\alpha))} (f_{\bar{g}}(x), f_{\bar{g}}(y)) \bar{\odot}_{\bar{g}} \\
&\bar{\odot}_{\bar{g}} \left( \frac{t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (f_{\bar{g}}(x), f_{\bar{g}}(y)) \bar{\ominus}_{\bar{g}}}{\bar{\ominus}_{\bar{g}} t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (f_{\bar{g}}(x), f_{\bar{g}}(y))} \right)
\end{aligned}$$

This pseudo-nonlinear relation shows that the  $\bar{g}$  – transform of the multiplication of a constant with the subtraction of two functions ( $f, h$ ) is equal to the pseudo-multiplication of a pseudo-constant with the pseudo-subtraction of two modified functions ( $f_{\bar{g}}, h_{\bar{g}}$ ).

$$\begin{aligned}
7. \left( \frac{\alpha \cdot f}{\lambda \cdot h} \right)_{\bar{g}} &= \bar{g}^{-1} \left( \frac{\alpha \cdot f(\bar{g}(x))}{\lambda \cdot h(\bar{g}(x))} \right) = \\
&= \bar{g}^{-1} \left\{ \frac{\bar{g}(\bar{g}^{-1}(\alpha \cdot f(\bar{g}(x))))}{\bar{g}(\bar{g}^{-1}(\lambda \cdot h(\bar{g}(x))))} \right\} = \\
&= \left( \frac{\mathbf{t}^{(+, \cdot)(0,\alpha,0,0)}(f(x), f(y))}{\mathbf{t}^{(+, \cdot)(0,\lambda,0,0)}(h(x), h(y))} \right)_{\bar{g}} = \\
&= \bar{g}^{-1} \left\{ \frac{\bar{g} \left( \frac{\bar{g}(\bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} f_{\bar{g}}(x))}{\bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} f_{\bar{g}}(x)} \right)}{\bar{g} \left( \frac{\bar{g}(\bar{g}^{-1}(\lambda) \bar{\odot}_{\bar{g}} h_{\bar{g}}(x))}{\bar{g}^{-1}(\lambda) \bar{\odot}_{\bar{g}} h_{\bar{g}}(x)} \right)} \right\} = \\
&= \bar{g}^{-1} \left\{ \frac{\bar{g} \left( \frac{\bar{g}(\bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} f_{\bar{g}}(x))}{\bar{g}(\bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} f_{\bar{g}}(x))} \right)}{\bar{g} \left( \frac{\bar{g}(\bar{g}^{-1}(\lambda) \bar{\odot}_{\bar{g}} h_{\bar{g}}(x))}{\bar{g}(\bar{g}^{-1}(\lambda) \bar{\odot}_{\bar{g}} h_{\bar{g}}(x))} \right)} \right\} = \\
&= \bar{g}^{-1} \left( \frac{\bar{g}(\bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} f_{\bar{g}}(x))}{\bar{g}(\bar{g}^{-1}(\lambda) \bar{\odot}_{\bar{g}} h_{\bar{g}}(x))} \right) = \\
&= (\bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} f_{\bar{g}}(x)) \bar{\odot}_{\bar{g}} (\bar{g}^{-1}(\lambda) \bar{\odot}_{\bar{g}} h_{\bar{g}}(x)) = \\
&= (\bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} f_{\bar{g}}) \bar{\odot}_{\bar{g}} (\bar{g}^{-1}(\lambda) \bar{\odot}_{\bar{g}} h_{\bar{g}}) = \\
&= t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(\alpha), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (f_{\bar{g}}(x), f_{\bar{g}}(y)) \bar{\odot}_{\bar{g}} \\
&\bar{\odot}_{\bar{g}} t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(\lambda), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (h_{\bar{g}}(x), h_{\bar{g}}(y)),
\end{aligned}$$

when the conditions  $\alpha \neq \lambda \neq 1$  and  $\lambda \neq 0$  are met.

This pseudo-nonlinear relation shows that the  $\bar{g}$  – transform of the division of two functions (each factor is a multiplication of a constant with a function) is equal to the pseudo-division of two modified functions (each modified function is a pseudo-multiplication of a pseudo-constant with a  $\bar{g}$  – function).

8. We prove this point in two ways, according to the associative and the commutative properties of multiplication and pseudo-multiplication operation and apply points 1, 4 of the theorem 2.2.2. The equivalence of verifying the two paths and their same results will lead us to the complete verification of point 8.

First, we transformed  $(\alpha \cdot f) \cdot (\lambda \cdot h)$  without changing the present form and applying point 1 of the theorem 2.2.2. for the two linear expressions  $\alpha \cdot f$  and  $\lambda \cdot h$  inside the brackets. Further, use point four again from this theorem for pseudo-multiplication of the two result pseudo-linear functional expressions.

8.1. This pseudo-multiplication case is also expressed by the parameterized nonlinear continuous functions  $t_{\bar{g}}$ .

$$\begin{aligned}
& [(\alpha \cdot f) \cdot (\lambda \cdot h)]_{\bar{g}} = [(\alpha \cdot f(x)) \cdot (\lambda \cdot h(x))]_{\bar{g}} = \\
& = \bar{g}^{-1} \left[ (\alpha \cdot f(\bar{g}(x))) \cdot (\lambda \cdot h(\bar{g}(x))) \right] = \\
& = \left( \begin{array}{c} \mathbf{t}^{(+, \cdot)(0, \alpha, 0, 0)}(f(x), f(y)) \cdot \\ \cdot \mathbf{t}^{(+, \cdot)(0, \lambda, 0, 0)}(h(x), h(y)) \end{array} \right)_{\bar{g}} = \\
& = \bar{g}^{-1} \left[ \left( \begin{array}{c} \bar{g}(\bar{g}^{-1}(\alpha)) \cdot \bar{g}(\bar{g}^{-1}(f(\bar{g}(x)))) \\ \cdot \left( \bar{g}(\bar{g}^{-1}(\lambda)) \cdot \bar{g}(\bar{g}^{-1}(h(\bar{g}(x)))) \right) \end{array} \right) \right] = \\
& = \bar{g}^{-1} \left\{ \begin{array}{l} \bar{g} \left( \frac{\bar{g}^{-1}(\bar{g}(\bar{g}^{-1}(\alpha)) \cdot \bar{g}(\bar{g}^{-1}(f(\bar{g}(x)))))}{\bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} f_{\bar{g}}(x)} \right) \\ \cdot \bar{g} \left( \frac{\bar{g}^{-1}(\bar{g}(\bar{g}^{-1}(\lambda)) \cdot \bar{g}(\bar{g}^{-1}(h(\bar{g}(x)))))}{\bar{g}^{-1}(\lambda) \bar{\odot}_{\bar{g}} h_{\bar{g}}(x)} \right) \end{array} \right\} = \\
& = \bar{g}^{-1} \left\{ \begin{array}{l} \bar{g}(\bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} f_{\bar{g}}(x)) \bar{\odot}_{\bar{g}} \\ \bar{\odot}_{\bar{g}} \bar{g}(\bar{g}^{-1}(\lambda) \bar{\odot}_{\bar{g}} h_{\bar{g}}(x)) \end{array} \right\} = \\
& = (\bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} f_{\bar{g}}(x)) \bar{\odot}_{\bar{g}} (\bar{g}^{-1}(\lambda) \bar{\odot}_{\bar{g}} h_{\bar{g}}(x)) = \\
& = (\alpha \cdot f)_{\bar{g}} \bar{\odot}_{\bar{g}} (\lambda \cdot h)_{\bar{g}} =
\end{aligned}$$

$$\begin{aligned}
& = t_{\bar{g}}^{\left( \bar{\oplus}_{\bar{g}, \bar{\odot}_{\bar{g}}} \right) (\bar{g}^{-1}(0), \bar{g}^{-1}(\alpha), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (f_{\bar{g}}(x), f_{\bar{g}}(y)) \bar{\odot}_{\bar{g}} \\
& \bar{\odot}_{\bar{g}} t_{\bar{g}}^{\left( \bar{\oplus}_{\bar{g}, \bar{\odot}_{\bar{g}}} \right) (\bar{g}^{-1}(0), \bar{g}^{-1}(\lambda), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (h_{\bar{g}}(x), h_{\bar{g}}(y)).
\end{aligned}$$

Secondly, according to the associative and the commutative properties of multiplication and pseudo-multiplication operation, we can transform  $(\alpha \cdot \lambda) \cdot (f \cdot h)$  by applying point 4 of the theorem 2.2.2. for the multiplication of constant  $c$  ( $c = \alpha \cdot \lambda$ ) with a product function (as multiplication of two functions  $f \cdot h$ ) inside the brackets. Further, use point one again from this theorem for pseudo-multiplication of the result pseudo-nonlinear functional expressions and pseudo-coefficients before them as a Constant Pseudo-Factor.

8.2. As in the case 8.1., the pseudo-multiplication is also expressed by the parameterized nonlinear continuous functions  $t_{\bar{g}}$ .

$$\begin{aligned}
& [(\alpha \cdot f) \cdot (\lambda \cdot h)]_{\bar{g}} = [(\alpha \cdot \lambda) \cdot (f(x) \cdot h(x))]_{\bar{g}} = \\
& = \bar{g}^{-1} \left[ (\alpha \cdot \lambda) \cdot (f(\bar{g}(x)) \cdot h(\bar{g}(x))) \right] = \\
& = \left( \begin{array}{c} \mathbf{t}^{(+, \cdot)(0, 0, 0, \alpha)}(f(x), f(y)) \cdot \mathbf{t}^{(+, \cdot)(0, 0, 0, \lambda)}(h(x), h(y)) \\ \cdot \left( \mathbf{t}^{(+, \cdot)(0, 1, 0, 0)}(f(x), f(y)) \cdot \mathbf{t}^{(+, \cdot)(0, 1, 0, 0)}(h(x), h(y)) \right) \end{array} \right)_{\bar{g}} \\
& = \left( \begin{array}{c} \mathbf{t}^{(+, \cdot)(0, 0, 0, \alpha \cdot \lambda)}(f(x), f(y)) \cdot \\ \cdot \left( \mathbf{t}^{(+, \cdot)(0, 1, 0, 0)}(f(x), f(y)) \cdot \mathbf{t}^{(+, \cdot)(0, 1, 0, 0)}(h(x), h(y)) \right) \end{array} \right)_{\bar{g}} = \\
& = \left( \begin{array}{c} \mathbf{t}^{(+, \cdot)(0, 0, 0, \alpha \cdot \lambda)}(h(x), h(y)) \cdot \\ \cdot \left( \mathbf{t}^{(+, \cdot)(0, 1, 0, 0)}(f(x), f(y)) \cdot \mathbf{t}^{(+, \cdot)(0, 1, 0, 0)}(h(x), h(y)) \right) \end{array} \right)_{\bar{g}} = \\
& = \bar{g}^{-1} \left[ \begin{array}{c} \bar{g}(\bar{g}^{-1}(\alpha \cdot \lambda)) \cdot \\ \bar{g} \left( \bar{g}^{-1} \left( \frac{\bar{g} \left( \bar{g}^{-1} (f(\bar{g}(x))) \right)}{f_{\bar{g}}(x)} \right) \right) \\ \cdot \bar{g} \left( \bar{g}^{-1} \left( \frac{\bar{g} \left( \bar{g}^{-1} (h(\bar{g}(x))) \right)}{h_{\bar{g}}(x)} \right) \right) \end{array} \right] = \\
& = \bar{g}^{-1} \left[ \begin{array}{c} \bar{g}(\bar{g}^{-1}(\alpha \cdot \lambda)) \cdot \\ \bar{g} \left( \bar{g}^{-1} \left( \frac{\bar{g} \left( \bar{g}^{-1} (f_{\bar{g}}(x)) \right) \cdot \bar{g} \left( \bar{g}^{-1} (h_{\bar{g}}(x)) \right)}{f_{\bar{g}}(x) \bar{\odot}_{\bar{g}} h_{\bar{g}}(x)} \right) \right) \end{array} \right] = \\
& = \bar{g}^{-1} \left[ \bar{g}(\bar{g}^{-1}(\alpha \cdot \lambda)) \cdot \bar{g} \left( f_{\bar{g}}(x) \bar{\odot}_{\bar{g}} h_{\bar{g}}(x) \right) \right] =
\end{aligned}$$

$$\begin{aligned}
&= \bar{g}^{-1} \left[ \bar{g} \left( \frac{\bar{g}^{-1}(\bar{g}(\bar{g}^{-1}(\alpha)) \cdot \bar{g}(\bar{g}^{-1}(\lambda)))}{\bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} \bar{g}^{-1}(\lambda)} \right) \cdot \bar{g}(f_{\bar{g}}(x) \bar{\odot}_{\bar{g}} h_{\bar{g}}(x)) \right] = \\
&= \bar{g}^{-1} \left[ \bar{g}(\bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} \bar{g}^{-1}(\lambda)) \cdot \bar{g}(f_{\bar{g}}(x) \bar{\odot}_{\bar{g}} h_{\bar{g}}(x)) \right] = \\
&= (\bar{g}^{-1}(\alpha) \bar{\odot}_{\bar{g}} \bar{g}^{-1}(\lambda)) \bar{\odot}_{\bar{g}} (f_{\bar{g}}(x) \bar{\odot}_{\bar{g}} h_{\bar{g}}(x)) = \\
&= (\alpha \cdot f)_{\bar{g}} \bar{\odot}_{\bar{g}} (\lambda \cdot h)_{\bar{g}} = \\
&= \left( \begin{array}{c} t_{\bar{g}}^{(\oplus_{\bar{g}}, \bar{\odot}_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(0), \bar{g}^{-1}(0), \bar{g}^{-1}(\alpha))} (f_{\bar{g}}(x), f_{\bar{g}}(y)) \bar{\odot}_{\bar{g}} \\ \bar{\odot}_{\bar{g}} t_{\bar{g}}^{(\oplus_{\bar{g}}, \bar{\odot}_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(0), \bar{g}^{-1}(0), \bar{g}^{-1}(\lambda))} (h_{\bar{g}}(x), h_{\bar{g}}(y)) \end{array} \right) \bar{\odot}_{\bar{g}} \\
&\bar{\odot}_{\bar{g}} \left( \begin{array}{c} t_{\bar{g}}^{(\oplus_{\bar{g}}, \bar{\odot}_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (f_{\bar{g}}(x), f_{\bar{g}}(y)) \bar{\odot}_{\bar{g}} \\ \bar{\odot}_{\bar{g}} t_{\bar{g}}^{(\oplus_{\bar{g}}, \bar{\odot}_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (f_{\bar{g}}(x), f_{\bar{g}}(y)) \end{array} \right) \\
&= \\
&= \left( t_{\bar{g}}^{(\oplus_{\bar{g}}, \bar{\odot}_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(0), \bar{g}^{-1}(0), \bar{g}^{-1}(\alpha \cdot \lambda))} (f_{\bar{g}}(x), f_{\bar{g}}(y)) \right) \bar{\odot}_{\bar{g}} \\
&\bar{\odot}_{\bar{g}} \left( \begin{array}{c} t_{\bar{g}}^{(\oplus_{\bar{g}}, \bar{\odot}_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (f_{\bar{g}}(x), f_{\bar{g}}(y)) \bar{\odot}_{\bar{g}} \\ \bar{\odot}_{\bar{g}} t_{\bar{g}}^{(\oplus_{\bar{g}}, \bar{\odot}_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (f_{\bar{g}}(x), f_{\bar{g}}(y)) \end{array} \right) = \\
&= \left( t_{\bar{g}}^{(\oplus_{\bar{g}}, \bar{\odot}_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(0), \bar{g}^{-1}(0), \bar{g}^{-1}(\alpha \cdot \lambda))} (h_{\bar{g}}(x), h_{\bar{g}}(y)) \right) \bar{\odot}_{\bar{g}} \\
&\bar{\odot}_{\bar{g}} \left( \begin{array}{c} t_{\bar{g}}^{(\oplus_{\bar{g}}, \bar{\odot}_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (f_{\bar{g}}(x), f_{\bar{g}}(y)) \bar{\odot}_{\bar{g}} \\ \bar{\odot}_{\bar{g}} t_{\bar{g}}^{(\oplus_{\bar{g}}, \bar{\odot}_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (f_{\bar{g}}(x), f_{\bar{g}}(y)) \end{array} \right) \\
&= \\
&= t_{\bar{g}}^{(\oplus_{\bar{g}}, \bar{\odot}_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(\alpha), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (f_{\bar{g}}(x), f_{\bar{g}}(y)) \bar{\odot}_{\bar{g}} \\
&\bar{\odot}_{\bar{g}} t_{\bar{g}}^{(\oplus_{\bar{g}}, \bar{\odot}_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(\lambda), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (h_{\bar{g}}(x), h_{\bar{g}}(y)).
\end{aligned}$$

### 3.2 Main Results

For relations of theorem 4, we have considered specific values of constants  $\alpha$ ,  $\lambda$  or condition for functions  $f$ ,  $h$  and generator  $\bar{g}$ , summarizing some exceptional cases (C1 ÷ C8) addressed below. These cases bring us back to the conditions of the Theorem 4 in, [2].

C1. For modification by  $\bar{g}$  – transform by addition of a constant with a function, if we replace

the value of the parameter  $\alpha=1$  for each generator  $\bar{g}$  or  $\bar{g}$  – normed in case (1) also, expressed by the parameterized nonlinear continuous functions  $t_{\bar{g}}$ , we get the pseudo-linear relations:

$$\begin{aligned}
(\alpha + f)_{\bar{g}} &= (1 + f)_{\bar{g}} = \\
&= \bar{g}^{-1}(1) \oplus f_{\bar{g}} \xrightarrow{(\bar{g}\text{-normed})} 1 \bar{\oplus}_{\bar{g}} f_{\bar{g}}. \\
(1 + f)_{\bar{g}} &= \bar{g}^{-1}(1) \oplus f_{\bar{g}} \xrightarrow{(\bar{g}\text{-normed})} 1 \bar{\oplus}_{\bar{g}} f_{\bar{g}}. \\
(1 + f)_{\bar{g}} &= \left( \mathbf{t}^{(+, \cdot)(0,1,0,1)}(f(x), f(y)) \right)_{\bar{g}} = \\
&= t_{\bar{g}}^{(\oplus_{\bar{g}}, \bar{\odot}_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(1))} (f_{\bar{g}}(x), f_{\bar{g}}(y)) = \\
&= \bar{g}^{-1}(1) \oplus f_{\bar{g}}. \\
(1 + f)_{\bar{g}} &= \\
&= \left( \begin{array}{c} \mathbf{t}^{(+, \cdot)(0,0,0,1)}(f(x), f(y)) \\ + \mathbf{t}^{(+, \cdot)(0,1,0,0)}(f(x), f(y)) \end{array} \right)_{\bar{g}} \xrightarrow{(\bar{g}\text{-normed})} \\
&\xrightarrow{(\bar{g}\text{-normed})} t_{\bar{g}}^{(\oplus_{\bar{g}}, \bar{\odot}_{\bar{g}})(0,0,0,1)} (f_{\bar{g}}(x), f_{\bar{g}}(y)) \bar{\oplus}_{\bar{g}} \\
&\bar{\oplus}_{\bar{g}} t_{\bar{g}}^{(\oplus_{\bar{g}}, \bar{\odot}_{\bar{g}})(0,1,0,0)} (f_{\bar{g}}(x), f_{\bar{g}}(y)) = 1 \bar{\oplus}_{\bar{g}} f_{\bar{g}}. \\
&\left( \mathbf{t}^{(+, \cdot)(0,1,0,1)}(f(x), f(y)) \right)_{\bar{g}} \xrightarrow{(\bar{g}\text{-normed})} \\
&\xrightarrow{(\bar{g}\text{-normed})} t_{\bar{g}}^{(\oplus_{\bar{g}}, \bar{\odot}_{\bar{g}})(0,1,0,1)} (f_{\bar{g}}(x), f_{\bar{g}}(y)).
\end{aligned}$$

C2. If we replace the value of the parameter  $\alpha=1$  for each generator  $\bar{g}$  or  $\bar{g}$  – normed in case (2) for  $\bar{g}$  – transform of multiplication of a constant with a function also, expressed by the parameterized nonlinear continuous functions  $t_{\bar{g}}$ , we get the pseudo-linear relations:

$$\begin{aligned}
(\alpha \cdot f)_{\bar{g}} &= \bar{g}^{-1}(1) \bar{\odot}_{\bar{g}} f_{\bar{g}} \xrightarrow{(\bar{g}\text{-normed})} 1 \bar{\odot}_{\bar{g}} f_{\bar{g}}; \\
(\alpha \cdot f)_{\bar{g}} &= (1 \cdot f)_{\bar{g}} = f_{\bar{g}} = \bar{g}^{-1}(1) \bar{\odot}_{\bar{g}} f_{\bar{g}}; \\
f_{\bar{g}} &= \bar{g}^{-1}(1) \bar{\odot}_{\bar{g}} f_{\bar{g}} \xrightarrow{(\bar{g}\text{-normed})} 1 \bar{\odot}_{\bar{g}} f_{\bar{g}}. \\
(1 \cdot f)_{\bar{g}} &= f_{\bar{g}} = \left( \mathbf{t}^{(+, \cdot)(0,1,0,0)}(f(x), f(y)) \right)_{\bar{g}} = \\
&= t_{\bar{g}}^{(\oplus_{\bar{g}}, \bar{\odot}_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (f_{\bar{g}}(x), f_{\bar{g}}(y)) = \\
&= \bar{g}^{-1}(1) \bar{\odot}_{\bar{g}} f_{\bar{g}}. \\
&\left( \mathbf{t}^{(+, \cdot)(0,1,0,0)}(f(x), f(y)) \right)_{\bar{g}} = \\
&= t_{\bar{g}}^{(\oplus_{\bar{g}}, \bar{\odot}_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (f_{\bar{g}}(x), f_{\bar{g}}(y)).
\end{aligned}$$



$$\begin{aligned} & \left( \mathbf{t}^{(+, \cdot)(0,1,0,0)}(f(x), f(y)) \right)_{\bar{g}} \xrightarrow{(\bar{g}\text{-normed})} \\ & \xrightarrow{(\bar{g}\text{-normed})} t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})(0,1,0,0)}(f_{\bar{g}}(x), f_{\bar{g}}(y)). \end{aligned}$$

C3. If we replace the value of the parameter  $\alpha$ ,  $\lambda$  and function  $f$ , also  $\bar{g}$ -normed in case (3), for the  $\bar{g}$ -transform of the addition of two functions also, expressed by the parameterized nonlinear continuous functions  $t_{\bar{g}}$ , we get some unique pseudo-nonlinear relations.

3.1. If we replace the value of the parameters  $\alpha = \lambda = 1$  in case (3), we get the relation for  $\bar{g}$ -modified sum function:

$$\begin{aligned} (f + h)_{\bar{g}} &= f_{\bar{g}} \bar{\oplus}_{\bar{g}} h_{\bar{g}} = \bar{\oplus}_{\bar{g}}(f_{\bar{g}}, h_{\bar{g}}). \\ & \left( \mathbf{t}^{(+, \cdot)(0,1,0,0)}(f(x), f(y)) + \right. \\ & \left. + \mathbf{t}^{(+, \cdot)(0,1,0,0)}(h(x), h(y)) \right)_{\bar{g}} \xrightarrow{(\bar{g}\text{-normed})} \\ & \xrightarrow{(\bar{g}\text{-normed})} t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})(0,1,0,0)}(f_{\bar{g}}(x), f_{\bar{g}}(y)) \bar{\oplus}_{\bar{g}} \\ & \bar{\oplus}_{\bar{g}} t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})(0,1,0,0)}(h_{\bar{g}}(x), h_{\bar{g}}(y)). \end{aligned}$$

We quickly see that we are in the conditions of Rybárik's theorem treated in, [2], and this is proved for each case (theorem 2.3) in, [10].

3.2. For the value of the parameters  $\alpha = \lambda = 1$  and the function  $f(x) = c = 1$  (i.e. as a constant function with value 1), the generator  $\bar{g}$  as  $\bar{g}$ -normed, we get the pseudo-linear relations:

$$(1 + f)_{\bar{g}} = \bar{g}^{-1}(1) \bar{\oplus}_{\bar{g}} f_{\bar{g}} \xrightarrow{(\bar{g}\text{-normed})} 1 \bar{\oplus}_{\bar{g}} f_{\bar{g}}.$$

This pseudo-linear relation is also the particular case C1.

C4. If we replace the value of the parameter  $\alpha$  and function  $f$ , also  $\bar{g}$ -normed in case (4), we get some unique relations.

4.1. If we replace the value of the parameter  $\alpha = 1$  and  $\bar{g}$ -normed in case (4) for  $\bar{g}$ -transform for the addition of two functions, we get a pseudo-nonlinear functional equation as a relation:

$$(f + h)_{\bar{g}} = f_{\bar{g}} \bar{\oplus}_{\bar{g}} h_{\bar{g}} = \bar{\oplus}_{\bar{g}}(f_{\bar{g}}, h_{\bar{g}}).$$

This pseudo-nonlinear relation is also the particular case C3.1. We quickly see that we are in the conditions of Rybárik's theorem, case 2 in, [2].

4.2. For the value of the parameters  $\alpha = 1$  and the function  $f(x) = c = 1$  (i.e. as a constant function with value 1), the generator  $\bar{g}$  as  $\bar{g}$ -normed in case (4) also, expressed by the parameterized nonlinear continuous functions  $t_{\bar{g}}$ , we get the pseudo-linear relations:

$$(1 + h)_{\bar{g}} = \bar{g}^{-1}(1) \bar{\oplus}_{\bar{g}} h_{\bar{g}} \xrightarrow{(\bar{g}\text{-normed})} 1 \bar{\oplus}_{\bar{g}} h_{\bar{g}}.$$

$$\begin{aligned} & \left( \mathbf{t}^{(+, \cdot)(0,0,0,1)}(h(x), h(y)) + \right. \\ & \left. + \mathbf{t}^{(+, \cdot)(0,1,0,0)}(h(x), h(y)) \right)_{\bar{g}} \xrightarrow{(\bar{g}\text{-normed})} \\ & \xrightarrow{(\bar{g}\text{-normed})} t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})(0,0,0,1)}(h_{\bar{g}}(x), h_{\bar{g}}(y)) \bar{\oplus}_{\bar{g}} \\ & \bar{\oplus}_{\bar{g}} t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})(0,1,0,0)}(h_{\bar{g}}(x), h_{\bar{g}}(y)). \\ & \left( \mathbf{t}^{(+, \cdot)(0,1,0,1)}(h(x), h(y)) \right)_{\bar{g}} \xrightarrow{(\bar{g}\text{-normed})} \end{aligned}$$

$$\xrightarrow{(\bar{g}\text{-normed})} t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})(0,1,0,1)}(h_{\bar{g}}(x), h_{\bar{g}}(y)).$$

This pseudo-linear relation is also the particular case C3.2.

C5 & C6. If we replace the value of the parameters  $\alpha = \lambda = 1$  in case (5) or  $\alpha = 1$  in case (6) also, expressed by the parameterized nonlinear continuous functions  $t_{\bar{g}}$ , we get the pseudo-nonlinear relation:

$$(f - h)_{\bar{g}} = f_{\bar{g}} \bar{\ominus}_{\bar{g}} h_{\bar{g}} = \bar{\ominus}_{\bar{g}}(f_{\bar{g}}, h_{\bar{g}}).$$

$$\left( \mathbf{t}^{(+, \cdot)(0,1,0,0)}(f(x), f(y)) - \right. \\ \left. - \mathbf{t}^{(+, \cdot)(0,1,0,0)}(h(x), h(y)) \right)_{\bar{g}} \xrightarrow{(\bar{g}\text{-normed})}$$

$$\xrightarrow{(\bar{g}\text{-normed})} t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})(0,1,0,0)}(f_{\bar{g}}(x), f_{\bar{g}}(y)) \bar{\ominus}_{\bar{g}}$$

$$\bar{\ominus}_{\bar{g}} t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})(0,1,0,0)}(h_{\bar{g}}(x), h_{\bar{g}}(y)).$$

C6. If we replace the value of the parameter  $\alpha = 1$  and the constant function  $f(x) = c = 1$ , as the generator as  $\bar{g}$ -normed in case (6) for  $\bar{g}$ -transform of the subtraction of a constant with a function also, expressed by the parameterized nonlinear continuous functions  $t_{\bar{g}}$ , we get the two pseudo-linear relations below:

$$(1 - h)_{\bar{g}} = \bar{g}^{-1}(1) \bar{\ominus}_{\bar{g}} h_{\bar{g}} \xrightarrow{(\bar{g}\text{-normed})} 1 \bar{\ominus}_{\bar{g}} h_{\bar{g}}.$$

$$\left( \mathbf{t}^{(+, \cdot)(0,0,0,1)}(h(x), h(y)) - \right. \\ \left. - \mathbf{t}^{(+, \cdot)(0,1,0,0)}(h(x), h(y)) \right)_{\bar{g}} \xrightarrow{(\bar{g}\text{-normed})}$$

$$\xrightarrow{(\bar{g}\text{-normed})} \left( t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})(0,0,0,1)}(h_{\bar{g}}(x), h_{\bar{g}}(y)) \bar{\ominus}_{\bar{g}} \right) \\ \bar{\ominus}_{\bar{g}} t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})(0,1,0,0)}(h_{\bar{g}}(x), h_{\bar{g}}(y)).$$

$$\left( \mathbf{t}^{(+, \cdot)(0,1,0,1)}(h(x), h(y)) \right)_{\bar{g}} \xrightarrow{(\bar{g}\text{-normed})}$$

$$\xrightarrow{(\bar{g}\text{-normed})} t_{\bar{g}}^{(\bar{\oplus}_{\bar{g}}, \bar{\odot}_{\bar{g}})(0,1,0,1)}(h_{\bar{g}}(x), h_{\bar{g}}(y)).$$

C7. Considering the values of the parameters  $\alpha$ ,  $\lambda$  or conditions for the functions  $f, h$  and generator  $\bar{g}$

( $\bar{g}$  – normed), we get some exceptional cases for this relation.

7.1. For two cases of the values of the parameters  $\alpha = \lambda \neq 0$  or  $\alpha = \lambda = 1$  and the generator as  $\bar{g}$  – normed also, expressed by the parameterized nonlinear continuous functions  $t_{\bar{g}}$ , we get a pseudo-nonlinear relation:

$$\begin{aligned} \left(\frac{f}{h}\right)_{\bar{g}} &= f_{\bar{g}} \bar{\mathcal{O}}_{\bar{g}} h_{\bar{g}}. \\ \left(\frac{\mathbf{t}^{(+,\cdot)(0,1,0,0)}(f(x), f(y))}{\mathbf{t}^{(+,\cdot)(0,1,0,0)}(h(x), h(y))}\right)_{\bar{g}} &= \\ &= \left(\frac{t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(0,1,0,0)}(f_{\bar{g}}(x), f_{\bar{g}}(y)) \bar{\mathcal{O}}_{\bar{g}}}{\bar{\mathcal{O}}_{\bar{g}} t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(0,1,0,0)}(h_{\bar{g}}(x), h_{\bar{g}}(y))}\right)_{\bar{g}}. \end{aligned}$$

7.2. Considering the values of the parameters  $\alpha = \lambda \neq 0$  or  $\alpha = \lambda = 1$ ,  $f(x) = c$  (constant function) and  $\bar{g}$  – normed for the generator  $\bar{g}$  also, expressed by the parameterized nonlinear continuous functions  $t_{\bar{g}}$ , we get the pseudo-nonlinear relation:

$$\begin{aligned} \left(\frac{c}{h}\right)_{\bar{g}} &= \bar{g}^{-1}(c) \bar{\mathcal{O}}_{\bar{g}} h_{\bar{g}}. \\ \left(\frac{\mathbf{t}^{(+,\cdot)(0,0,0,c)}(f(x), f(y))}{\mathbf{t}^{(+,\cdot)(0,1,0,0)}(h(x), h(y))}\right)_{\bar{g}} &= \\ &= t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(0), \bar{g}^{-1}(0), \bar{g}^{-1}(c))}(f_{\bar{g}}(x), f_{\bar{g}}(y)) \bar{\mathcal{O}}_{\bar{g}} \\ &\bar{\mathcal{O}}_{\bar{g}} t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0))}(h_{\bar{g}}(x), h_{\bar{g}}(y)). \end{aligned}$$

7.2.1. In the particular case of constant function  $f$  as  $f(x) = c = 1$  and the condition for values of function  $h$ , such that  $h(x) \neq 0$  for every value of  $x$  that  $x \in ]\bar{g}^{-1}(a), \bar{g}^{-1}(b)[$  also, expressed by the parameterized nonlinear continuous functions  $t_{\bar{g}}$ , we have this pseudo-nonlinear relation:

$$\begin{aligned} \left(\frac{1}{h}\right)_{\bar{g}} &= \bar{g}^{-1}(1) \bar{\mathcal{O}}_{\bar{g}} h_{\bar{g}} \xrightarrow{(\bar{g}\text{-normed})} 1 \bar{\mathcal{O}}_{\bar{g}} h_{\bar{g}}. \\ \left(\frac{\mathbf{t}^{(+,\cdot)(0,0,0,1)}(f(x), f(y))}{\mathbf{t}^{(+,\cdot)(0,1,0,0)}(h(x), h(y))}\right)_{\bar{g}} &= \\ &= \left(\frac{t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(0), \bar{g}^{-1}(0), \bar{g}^{-1}(1))}(f_{\bar{g}}(x), f_{\bar{g}}(y)) \bar{\mathcal{O}}_{\bar{g}}}{\bar{\mathcal{O}}_{\bar{g}} t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0))}(h_{\bar{g}}(x), h_{\bar{g}}(y))}\right)_{\bar{g}} \end{aligned}$$

$\xrightarrow{(\bar{g}\text{-normed})}$

$$\left(\frac{t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(0,0,0,1)}(f_{\bar{g}}(x), f_{\bar{g}}(y)) \bar{\mathcal{O}}_{\bar{g}}}{\bar{\mathcal{O}}_{\bar{g}} t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(0,1,0,0)}(h_{\bar{g}}(x), h_{\bar{g}}(y))}\right)_{\bar{g}}.$$

C8. If we take into consideration the values of the parameters  $\alpha = \lambda = 1$  or the condition  $\alpha \cdot \lambda = 1$  in case (8) and for the generator  $\bar{g}$  as  $\bar{g}$  – normed also, expressed by the parameterized nonlinear continuous functions  $t_{\bar{g}}$ , we get the pseudo-nonlinear relation for the  $\bar{g}$  – transform of the product function as below:

$$\begin{aligned} [(\alpha \cdot f) \cdot (\lambda \cdot h)]_{\bar{g}} &= [(\alpha \cdot \lambda) \cdot (f \cdot h)]_{\bar{g}} = \\ &= (f \cdot h)_{\bar{g}} = f_{\bar{g}} \bar{\mathcal{O}}_{\bar{g}} h_{\bar{g}}. \\ \left(\frac{\mathbf{t}^{(+,\cdot)(0,1,0,0)}(f(x), f(y)) \cdot}{\mathbf{t}^{(+,\cdot)(0,1,0,0)}(h(x), h(y))}\right)_{\bar{g}} &= \\ &= \left(\frac{t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(0,1,0,0)}(f_{\bar{g}}(x), f_{\bar{g}}(y)) \bar{\mathcal{O}}_{\bar{g}}}{\bar{\mathcal{O}}_{\bar{g}} t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(0,1,0,0)}(h_{\bar{g}}(x), h_{\bar{g}}(y))}\right)_{\bar{g}}. \end{aligned}$$

## 2.1 Results and Discussions

○ Reviewing all results according to C3.1, C4.1, C5, C6, C7.1, C8 as exceptional cases of the two treated theorems depending on the values of the parameters or conditions for the functions  $(\alpha, \lambda, f, h, \bar{g})$ , that participate in these relations and parallel with  $\bar{g}$  – calculus, [1], we obtain the four important pseudo-nonlinear relations for  $f_{\bar{g}}$  – calculus which correspond to points 2,3,4,5 of Rybárik’s theorem, [2]:

$$\begin{aligned} (f + h)_{\bar{g}} &= f_{\bar{g}} \bar{\oplus}_{\bar{g}} h_{\bar{g}} = \bar{\oplus}_{\bar{g}}(f_{\bar{g}}, h_{\bar{g}}); \\ (f - h)_{\bar{g}} &= f_{\bar{g}} \bar{\ominus}_{\bar{g}} h_{\bar{g}} = \bar{\ominus}_{\bar{g}}(f_{\bar{g}}, h_{\bar{g}}); \\ (f \cdot h)_{\bar{g}} &= f_{\bar{g}} \bar{\odot}_{\bar{g}} h_{\bar{g}} = \bar{\odot}_{\bar{g}}(f_{\bar{g}}, h_{\bar{g}}); \\ \left(\frac{f}{h}\right)_{\bar{g}} &= f_{\bar{g}} \bar{\oslash}_{\bar{g}} h_{\bar{g}} = \bar{\oslash}_{\bar{g}}(f_{\bar{g}}, h_{\bar{g}}). \end{aligned}$$

○ In the same way, we find the four important pseudo-nonlinear relations for  $f_{\bar{g}}$  – calculus which correspond to points 2, 3, 4, 5 of Rybárik’s theorem, [2], for each generator  $\bar{g}$  and the system of these relations is expressed by the parameterized nonlinear continuous functions  $t_{\bar{g}}$  as:

$$\begin{aligned} \left( \mathbf{t}^{(+,\cdot)(0,1,0,0)}(f(x), f(y)) + \right. & \left. + \mathbf{t}^{(+,\cdot)(0,1,0,0)}(h(x), h(y)) \right)_{\bar{g}} = \begin{pmatrix} t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (f_{\bar{g}}(x), f_{\bar{g}}(y)) \oplus_{\bar{g}} \\ \oplus_{\bar{g}} t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (h_{\bar{g}}(x), h_{\bar{g}}(y)) \end{pmatrix}; \\ \left( \mathbf{t}^{(+,\cdot)(0,1,0,0)}(f(x), f(y)) - \right. & \left. - \mathbf{t}^{(+,\cdot)(0,1,0,0)}(h(x), h(y)) \right)_{\bar{g}} = \begin{pmatrix} t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (f_{\bar{g}}(x), f_{\bar{g}}(y)) \ominus_{\bar{g}} \\ \ominus_{\bar{g}} t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (h_{\bar{g}}(x), h_{\bar{g}}(y)) \end{pmatrix}; \\ \left( \mathbf{t}^{(+,\cdot)(0,1,0,0)}(f(x), f(y)) \cdot \right. & \left. \cdot \mathbf{t}^{(+,\cdot)(0,1,0,0)}(h(x), h(y)) \right)_{\bar{g}} = \begin{pmatrix} t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (f_{\bar{g}}(x), f_{\bar{g}}(y)) \odot_{\bar{g}} \\ \odot_{\bar{g}} t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (h_{\bar{g}}(x), h_{\bar{g}}(y)) \end{pmatrix}; \\ \left( \frac{\mathbf{t}^{(+,\cdot)(0,1,0,0)}(f(x), f(y))}{\mathbf{t}^{(+,\cdot)(0,1,0,0)}(h(x), h(y))} \right)_{\bar{g}} & = \begin{pmatrix} t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (f_{\bar{g}}(x), f_{\bar{g}}(y)) \overline{\odot}_{\bar{g}} \\ \overline{\odot}_{\bar{g}} t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0))} (h_{\bar{g}}(x), h_{\bar{g}}(y)) \end{pmatrix}. \end{aligned}$$

- The system of four important pseudo-nonlinear relations for  $f_{\bar{g}}$  – calculus, expressed by  $t_{\bar{g}}$  –

*function* and with generator  $\bar{g}$  – *normed*, has the form as below:

$$\begin{aligned} \left( \mathbf{t}^{(+,\cdot)(0,1,0,0)}(f(x), f(y)) + \mathbf{t}^{(+,\cdot)(0,1,0,0)}(h(x), h(y)) \right)_{\bar{g}} & \xrightarrow{(\bar{g}\text{-normed})} \begin{pmatrix} t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(0,1,0,0)} (f_{\bar{g}}(x), f_{\bar{g}}(y)) \oplus_{\bar{g}} \\ \oplus_{\bar{g}} t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(0,1,0,0)} (h_{\bar{g}}(x), h_{\bar{g}}(y)) \end{pmatrix}; \\ \left( \mathbf{t}^{(+,\cdot)(0,1,0,0)}(f(x), f(y)) - \mathbf{t}^{(+,\cdot)(0,1,0,0)}(h(x), h(y)) \right)_{\bar{g}} & \xrightarrow{(\bar{g}\text{-normed})} \begin{pmatrix} t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(0,1,0,0)} (f_{\bar{g}}(x), f_{\bar{g}}(y)) \ominus_{\bar{g}} \\ \ominus_{\bar{g}} t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(0,1,0,0)} (h_{\bar{g}}(x), h_{\bar{g}}(y)) \end{pmatrix}; \\ \left( \mathbf{t}^{(+,\cdot)(0,1,0,0)}(f(x), f(y)) \cdot \mathbf{t}^{(+,\cdot)(0,1,0,0)}(h(x), h(y)) \right)_{\bar{g}} & \xrightarrow{(\bar{g}\text{-normed})} \begin{pmatrix} t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(0,1,0,0)} (f_{\bar{g}}(x), f_{\bar{g}}(y)) \odot_{\bar{g}} \\ \odot_{\bar{g}} t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(0,1,0,0)} (h_{\bar{g}}(x), h_{\bar{g}}(y)) \end{pmatrix}; \\ \left( \frac{\mathbf{t}^{(+,\cdot)(0,1,0,0)}(f(x), f(y))}{\mathbf{t}^{(+,\cdot)(0,1,0,0)}(h(x), h(y))} \right)_{\bar{g}} & \xrightarrow{(\bar{g}\text{-normed})} \begin{pmatrix} t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(0,1,0,0)} (f_{\bar{g}}(x), f_{\bar{g}}(y)) \overline{\odot}_{\bar{g}} \\ \overline{\odot}_{\bar{g}} t_{\bar{g}}^{(\oplus_{\bar{g}}, \ominus_{\bar{g}})(0,1,0,0)} (h_{\bar{g}}(x), h_{\bar{g}}(y)) \end{pmatrix}. \end{aligned}$$

- Looking at the above two forms of expression of the  $f_{\bar{g}}$  – calculus system expressed by the parameterized nonlinear continuous functions  $t_{\bar{g}}$  and according to the cases of the generator  $\bar{g}$ , we can say that we have built a system of  $t_{\bar{g}}$  – calculus.
- The exceptional cases C3.1, C4.1, C5, C6, C7.1, C8 bring us the displays of  $\bar{g}$  – transform of the sum, difference, product and quotient function  $(f + g; f - g; f \cdot h; \frac{f}{h})$  as exceptional cases of our generalized theorem 3.1., so the  $f_{\bar{g}}$  – calculus system itself.

- The values of parameters or conditions for functions  $(\alpha, \lambda, f, h, \bar{g})$  that participate in all relations and are applied to the point of the generalized theorem 3.1., bring us to the conditions of Rybárik’s theorem [2] treating some exceptional cases of it.
- For  $\bar{g}$  – function  $(f_{\bar{g}})$ , we get a representation as a pseudo-linear expression by C2, where the pseudo-constant is  $\bar{g}^{-1}(1)$  for each generator  $\bar{g}$  or 1 if  $\bar{g}$  – normed. Very interesting is the case of  $\bar{g}$  – identity  $(\bar{g} - id)$  that brings us to

classical analysis and  $f_{\bar{g}} = f$ . The pseudo-linear expression of  $f_{\bar{g}}$ :

$$f_{\bar{g}} = \begin{cases} \bar{g}^{-1}(1) \overline{\odot}_{\bar{g}} f_{\bar{g}} & \text{for each generator } \bar{g} \\ 1 \overline{\odot}_{\bar{g}} f_{\bar{g}} & \text{for } \bar{g} - \text{normed} \\ f & \text{for } \bar{g} - \text{id} \end{cases}$$

- We get a representation for  $\bar{g}$ -function  $(\alpha \cdot f_{\bar{g}})$ , as a pseudo-linear expression by C2,

$$(\alpha \cdot f)_{\bar{g}}(x) = \begin{cases} 1 \overline{\odot}_{\bar{g}} f_{\bar{g}}(x) & \text{for } \alpha = 1, \bar{g} - \text{normed} \\ \bar{g}^{-1}(1) \overline{\odot}_{\bar{g}} f_{\bar{g}} & \text{for each generator } \bar{g}, \alpha = 1 \\ f(x) & \text{for } \alpha = 1, \bar{g} - \text{id} \\ \alpha \cdot f(x) & \text{for each } \alpha, \bar{g} - \text{id} \\ \bar{g}^{-1}(\alpha) \odot f_{\bar{g}}(x) & \text{for other } \bar{g} \text{ and } \alpha \end{cases}.$$

$$\left( \mathbf{t}^{(+, \cdot)(0, \alpha, 0, 0)}(f(x), f(y)) \right)_{\bar{g}} = \begin{cases} t_{\bar{g}}^{(\overline{\oplus}_{\bar{g}}, \overline{\odot}_{\bar{g}})(0, 1, 0, 0)}(f_{\bar{g}}(x), f_{\bar{g}}(y)) & \text{for } \alpha = 1, \bar{g} - \text{normed} \\ t_{\bar{g}}^{(\overline{\oplus}_{\bar{g}}, \overline{\odot}_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(1), \bar{g}^{-1}(0), \bar{g}^{-1}(0))}(f_{\bar{g}}(x), f_{\bar{g}}(y)) & \text{for each } \bar{g}, \alpha = 1 \\ \mathbf{t}^{(+, \cdot)(0, 1, 0, 0)}(f(x), f(y)) & \text{for } \alpha = 1, \bar{g} - \text{id} \\ \mathbf{t}^{(+, \cdot)(0, \alpha, 0, 0)}(f(x), f(y)) & \text{for each } \alpha, \bar{g} - \text{id} \\ t_{\bar{g}}^{(\overline{\oplus}_{\bar{g}}, \overline{\odot}_{\bar{g}})(\bar{g}^{-1}(0), \bar{g}^{-1}(\alpha), \bar{g}^{-1}(0), \bar{g}^{-1}(0))}(f_{\bar{g}}(x), f_{\bar{g}}(y)) & \text{for other } \bar{g} \text{ and } \alpha \end{cases}.$$

- When the generator  $\bar{g}$  applies as  $\bar{g}$ -identity ( $\bar{g}(x) = x$ ) for all relations in theorem 3.1. the process of generalization associates with the Classical Analysis, showing the Generated Pseudo-Analysis as its generalization, [3], [9], [11], [12].
- These eight pseudo-linear or pseudo-nonlinear functional relations connect us to the four classes of functions that are their solutions, extending the study even further into their forms and properties, [2], [3], [10], [11], [15], ensuring further developments in these classes and the  $\bar{g}$ -integrals with three classes [1], [3], [4], [5], [6], [11], [16], [17], [18], [19], [20].
- We note that in this paper, we are limited by considering cases of two functions in linear or nonlinear relations during the process of modification and generalization by  $\bar{g}$ -transform, but further on, we will have more functions, precisely three more cases as  $(f^n)_{\bar{g}}; (\sum_{i=1}^n f)_{\bar{g}}; (\sum_{i=1}^n f_i)_{\bar{g}}$ .
- Attending the developments in Pseudo-Analysis and specifically in the processes of its generalizations [1], [2], [9], [11], [16], [17], [21], we plan to further study of  $\bar{g}$ -calculus,

where the pseudo-constant is  $\bar{g}^{-1}(\alpha)$  or  $\bar{g}^{-1}(1)$  when  $\alpha = 1$  for each generator  $\bar{g}$ , also  $\bar{g}^{-1}(\alpha) = 1$  if  $\bar{g}$ -normed or  $\bar{g}$ -id and  $\alpha = 1$ :

$f_{\bar{g}}$ -calculus and  $\bar{g}$ -derivative to contribute to the construction and completion of the Table of  $\bar{g}$ -Derivatives for  $\bar{g}$ -functions with general formulas.

- The studies in Pseudo-Analysis will be directed not only to nonlinearity problems but also to uncertainty and optimization, with many applications,

## 4 Conclusion

The main problem we are dealing with in this research paper is the treatment of nonlinear problems by the generalization of the relations formulated by Theorem 4 of Rybárik in, [2] as pseudo-linear or pseudo-nonlinear functional equations and its further completion with other treated cases. The parameterized nonlinear continuous functions  $t_{\bar{g}}$  during these generalization process, play a special role, providing us with important relationships and conclusions. This theorem 3.1., as a generalization of Theorem 4 of Rybárik presented in, [2], brings us back to the conditions of Rybárik's theorem as a particular case after some additional conditions. That follows from

the applications of some conditions for the parameters  $\alpha, \lambda$  or functions  $f, h$  and the generator  $\bar{g}$  over our theorem's eight pseudo-linear or pseudo-nonlinear relations.

The generators  $\bar{g}$  and specially  $\bar{g}$  – normed, the parameterized nonlinear continuous functions  $t_{\bar{g}}$  are critical to process of  $\bar{g}$  – transforming all the functions or expressions where they are combined according to pseudo-operations and, even more, for transforming the relations for each case of theorem 4 treated by Rybárik in, [2]. Furthermore, we emphasize that for the particular case of generator as  $\bar{g}$  – identity ( $\bar{g}(x) = x$ ), we are again in cases of Classical Analysis.

By this generalization are earned some pseudo-linear or pseudo-nonlinear functional equations, which are also related to the four classes of  $\bar{g}$  – functions treated as their solutions, [2], [3], [8], [10], [16], [18], [19].

The nonlinear problems are treated through the generalization theorem 3.1. from the parameterized nonlinear continuous functions  $t_{\bar{g}}$  along its eight exceptional cases. The study have led us to interesting and mutual conclusions about the cases treated in these two essential theorems taken into consideration also should open up horizons for us to deepen our studies for  $f_{\bar{g}}$  – calculus,  $t_{\bar{g}}$  – calculus,  $\bar{g}$  – derivative and  $\bar{g}$  – integrals of the second class with many applications in Generated Pseudo-Analysis, [3], [5], [6], [7], [8], [12], [15], [16], [17], [18], [19], [20].

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