Managing the Existing in Real Life Indeterminancy

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Abstract: - The present paper deals with the reasons and processes that led from fuzziness to neutrosophy. Starting from definitions and examples of fuzzy sets and logic and of intuitionistic fuzzy sets, it proceeds to the description of the existing in real life indeterminancy and defines the connected to it concept of neutrosophic set, introduced by Smarandache in 1995. The basic operations between neutrosophic sets are presented and the paper closes with the concept of neutrosophic topological space, which generalizes the notions of fuzzy and intuitionistic fuzzy topological space. It is shown that the classical concepts of convergence and continuity in topological spaces, of compact topological space and of Hausdorff topological space can be extended to neutrosophic topological spaces.

Key-Words: - Fuzzy set (FS), fuzzy logic (FL), intuitionistic fuzzy set (IFS), indeterminancy, neutrosophic set (NS), neutrosophic topological space (NTS).

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1 Introduction

The frequently appearing in real life *uncertainty* is due to the shortage of knowledge regarding some situation. Roughly speaking, the amount of the existing uncertainty is equal to the difference of the amount of the necessary knowledge needed for interpreting or determining the evolution of a situation, minus the already existing knowledge about this situation. In other words, uncertainty represents the total amount of potential information in the situation, which implies that a reduction of uncertainty due to a new evidence (e.g. receipt of a message) indicates a gain of an equal amount of information. This is the reason for which the measures of uncertainty have been also adopted as measures of information [1, Chapter 5]).

The existing uncertainty is due to various reasons, like randomness, imprecise or incomplete data, vague information, etc. As a result, different taxonomies of the uncertainty have been proposed. Primarily, however, one may distinguish between the uncertainty due to *randomness* and the uncertainty due to *imprecision*. The former is related to well-defined events whose outcomes cannot be predicted in advance, like the turning of a coin, the throwing of a die, etc. On the other hand, the latter occurs when the events are well defined, but the possible outcomes cannot be expressed in a crisp form; e.g. “the weather tomorrow will be rather rainy”.

The uncertain problems need imprecise methods that could deal with different types of uncertainties to increase the understanding of the outcomes. *Probability* theory has been proved sufficient for managing the cases of uncertainty due to randomness. Starting from the Zadeh’s *fuzzy set (FS)* [2] and *fuzzy logic* [3], however, several other theories developed during the last 60 years for managing equally well the other types of uncertainty [4].

FSs describe effectively the uncertainty due to *vagueness*, which is created when one is unable to clearly differentiate between two classes, such as “a person of average height” and “a tall person”. Atanassov introduced in 1986 the concept of *intuitionistic FS (IFS)* [5] by adding to the Zadeh’s *membership degree* for each element of the set of the discourse the degree of *non-membership*.

Smarandanche, motivated by the many existing in real life neutral situations - like <friend, neutral, enemy>, <positive, zero, negative>, <small, medium, high>, <male, transgender, female>, <win, draw, defeat>, etc. - introduced in 1995 the concept of *neutosophic set (NS)* [6] by adding a third *degree* of *indeterminancy* between the degrees of membership and non-membership.

The present paper aims at reviewing the process that led from fuzziness to neutrosophy and at discussing the future perspectives of the corresponding theories. The rest of the paper is formulated as follows: Section 2 contains basic definitions and examples of FS and IFS. Section 3 is devoted to NSs and the definition of inteterminancy. Section 3 describes the concept of *neutrosophic topological space (NTS)* and the article closes with the final conclusions and some hints for future research presented in section 5.

2 Preliminaries

The electrical engineer Lofti Aliasker Zadeh, Professor of Computer Science at the University of Berkeley, introduced in 1965 the concept of FS [2] as follows:

Definition 1: Let U be the universal set of the discourse, then a FS A in U is defined with the help of its *membership function* m: $U \rightarrow [0,1]$ as the set of the ordered pairs

$$A = \{(x, m(x)): x \in U\} \quad (1)$$

The real number m(x) is called the membership degree of *x* in A**.** The greater is m(*x*), the more *x* satisfies the characteristic property of A. Many authors, for reasons of simplicity, identify a fuzzy set with its membership function.

A crisp subset A of U is a fuzzy set in U with membership function taking the values *m*(*x*)=1, if *x* belongs to A, and 0 otherwise. The classical operations on crisp sets (intersection, union, complement, etc.) are generalized in a natural way to FSs [1, Chapter 2].

The infinite-valued on the interval [0,1] FL is defined with the help of the concept of FS [3]. Through FL, the fuzzy terminology is translated by algorithmic procedures into numerical values, operations are performed upon those values and the outcomes are returned into natural language statements in a reliable manner [7]. An important advantage of FL is that its rules are set in natural language with the help of linguistic, and therefore fuzzy, variables [8]. For general facts on FSs and FL we refer to the books [1] and [9, Chapters 4-7].

A difficulty appears, however, in FSs with the definition of the membership function, which is not unique, depending on the subjective perceptions of the user. In fact, the way of perceiving a concept (e.g. "tall") is different from person to person, depending on the "signals" that each one receives from the real world about it. Thus, the only restriction about the definition of the membership function is to be compatible with common logic; otherwise the FS does not give a reliable representation of the corresponding real situation.

For a more accurate quantification of the uncertainty Kassimir Atanassov, Professor of Mathematics at the Bulgarian Academy of Sciences, introduced in 1986 the concept of IFS [5] as follows:

Definition 2: An IFS A in the universe U is defined with the help of its membership function m: $U \rightarrow [0,1]$ and its *non-membership function* n: $U \rightarrow [0,1]$ as the set of the ordered triples

$$A = \{(x, m(x), n(x)): x \in U, 0 \leq m(x) + n(x) \leq 1\} \quad (2)$$

One can write m(x) + n(x) + h(x) = 1, where h(x) is called the *hesitation* or *uncertainty degree* of x. When h(x) = 0, then the corresponding IFS is reduced to an ordinary FS. The characterization intuitionistic is due to the fact that an IFS contains the intuitionistic idea, as it incorporates the degree of hesitation.

For example, if A is the IFS of the clever students of a class and (x, 0.6, 0.2) ∈ A, then there is a 60% probability for the student x to be characterized as clever, a 20% probability to be characterized as not clever, and a 20% hesitation to be characterized as either clever or not. Most notions and operations concerning FS can be extended to IFS, which simulate successfully the existing imprecision in human thinking [10].

3. Neutrosophic Sets

3.1 Basic Concepts

The Romanian writer and mathematician Florentin Smarandache, Professor at the branch of Gallup of the New Mexico University, introduced in 1995 the degree of *indeterminancy/neutrality* of the elements of the universal set U in a subset of U and defined the concept of NS [6] as follows:

Definition 3: A *single valued NS (SVNS)* A in U is of the form

$$A = \{(x,T(x),I(x),F(x)): x \in U, T(x),I(x),F(x) \in [0,1], 0 \leq T(x)+I(x)+F(x) \leq 3\} \quad (3)$$

In (3) T(x), I(x), F(x) are the degrees of *truth*, *indeterminancy and falsity* of x in A respectively, called the *neutrosophic components* of x. For simplicity, we write A<T, I, F>.

The etymology of the term "neutrosophy" comes from the adjective "neutral΄ and the Greek word "sophia" (wisdom) and, according to Smarandanche who introduced it, means "the knowledge of neutral thought".

For example, let U be the set of the players of a football team and let A be the SVNS of the good players of U. Then each player x of U is characterized by a *neutrosophic triplet* (t, i, f) with respect to A, with t, i, f in [0, 1]. For instance, x(0.6, 0.2, 0.4) ∈ A means that there is a 60% probability for x to be a good player, a 20% probability to be unknown if x is a good player and a 40% probability for x to not be a good player. In particular, x(0,1,0)∈ A means that we do not know absolutely nothing about x's affiliation with A.

Indeterminancy is understood to be in general everything which is between the opposites of truth and falsity [11].

In an IFS the inderterminancy is equal by default with the hesitancy, i.e. we have I(x)=1- T(x) – F(x). Also, in a FS is I(x)=0 and F(x) = 1 – T(x), whereas in a crisp set is T(x)=1 (or 0) and F(x)= 0 (or 1). In other words, crisp sets, FSs and IFSs are special cases of SVNSs.

When the sum T(x) + I(x) + F(x) of the neutrosophic components of x ∈ U in a SVNS A on U is <1, then it leaves room for incomplete information about x, when is equal to 1 for complete information and when is greater than 1 for *parasconsistent* (i.e. contradiction tolerant) information about x. A SVNS may contain simultaneously elements leaving room for all the previous types of information.

When T(x) + I(x) + F(x)<1, ∀ x ∈ U, then the corresponding SVNS is usually referred as *picture FS (PiFS)* [12]. In this case 1- T(x)-I(x)-F(x) is called the degree of *refusal membership* of x in A. The PiFSs based models are adequate in situations where we face human opinions involving answers of types yes, abstain, no and refusal. Voting is a representative example of such a situation.

The difference between the *general definition of a NS* and the previously given definition of a SVNS is that in the general definition T(x), I(x) and F(x) may take values in the non-standard unit interval]−0, 1+[(including values <0 or >1) [6]. This could happen in real world situations. For example, in a company with full-time work for its employees 40 hours per week an employee, upon his work, could belong by $\frac{40}{40}$=1 to the company (full-time job) or by $\frac{30}{40}$<1 (part-time job) or by $\frac{45}{40}$>1 (over-time job). Assume further that a full-time employee caused a damage to his job's equipment, the cost of which must be taken from his salary. Then, if the cost is equal to $\frac{50}{40}$ of his weekly salary, the employee belongs this week to the company by $-\frac{10}{40}$<0.

NSs, apart from vagueness, manage as well the cases of uncertainty due to *ambiguity* and *inconsistency*. In ambiguity the existing information leads to several interpretations by different observers. For example, the statement "Boy no girl" written as "Boy, no girl" means boy, but written as "Boy no, girl" means girl. On the other hand, inconsistency appears when two or more pieces of information cannot be true at the same time. As a result the obtainable in this case information is conflicted or undetermined. For example, "the probability for raining tomorrow is 80%, but this does not mean that the probability for not raining is 20%, because they might be hidden weather factors".

The difficulty, however, of defining properly the neutrosophic components of an object still exists, for the same reason as for the membership function of a FS described in the previous section. The same also happens with IFSs, and generally for any generalization of FSs involving membership functions. At any case, theories for managing the uncertainty related to FSs have been developed where the definition of a membership function is not necessary (*grey systems/numbers* [13]) or it is overpassed by using either a pair of sets which give the lower and the upper approximation of the original crisp set (*rough sets* [14]) or a suitable set of linguistic parameters (e.g. *soft sets* [15]). The present author has recently proposed a hybrid assessment model using soft sets and grey numbers as tools [16].

3.2 Operations on Neutrosophic Sets

The classical operations on crisp sets are generalized to NSs [17]. Here, for simplicity, we consider SVNSs and we define the subset and the complement of a SVNS, as well as the union and intersection of two SVNSs.

Definition 4: Let A<T_A, I_A, F_A> and B<T_B, I_B, F_B> be two SVNSs in the universe U. Then A is called a *subset* of B (A⊆B), if, and only if, $T_A(x) \leq T_B(x)$, $I_A(x) \leq I_B(x)$ and $F_A(x) \geq F_B(x)$, ∀x∈U. If we have simultaneously A⊆B and B⊆A, then A and B are *equal* (A=B) SVNSs.

Definition 5: The *complement* of a SVNS A<T_A, I_A, F_A> in U is the SVNS c(A)<F_A, 1-I_A, T_A> in U.

Definition 6: Let A<T_A, I_A, F_A> and B<T_B, I_B, F_B> be two SVNSs in the universe U. Then the *union* A∪B is the SVNS C<T_C, I_C, F_C> in U, with $T_C = \max\{T_A, T_B\}$, $I_C = \max\{I_A, I_B\}$ and $F_C = \min\{F_A, F_B\}$.

Definition 7: Let A<T_A, I_A, F_A> and B<T_B, I_B, F_B> be two SVNSs in the universe U. Then the *intersection* A∩B is the SVNS C<T_C, I_C, F_C> in U, with T_C = min {T_A, T_B}, I_C = min {I_A, I_B} and F_C = max {F_A, F_B}.

It is straightforward to check that if A and B are crisp sets (FSs, IFSs) then the previous definitions are reduced to the corresponding definitions of crisp sets (FSs, IFSs).

Example 1: Let U= {x_1, x_2, x_3} be the universal set and let A={(0.3,0.3,0.6,x_1), (0.5,0.3,0.4,x_2), (0.7,0.2,0.5,x_3)}, B ={(0.6,0.1,0.2,x_1), (0.3,0.2,0.5, x_2), (0.3,0.1,0.6,x_3)}be two SVNSs in U. Then:

i) Neither A $\subseteq$ B, nor B $\subseteq$ A (definition 4)

ii) c(A)={(0.6,0.7,0.3,x_1),(0.4,0.7,0.5,x_2), (0.5,0.8, 0.7,x_3)} and c(B) ={(0.2,0.9,0.6,x_1), (0.5,0.8,0.3, x_2) (0.6,0.9,0.3..,x_3)} (definition 5)

iii) A∪B={(0.6,0.3,0.6,x_1), (0.5,0.3,0.5, x_2), (0.3,0.1,0.5,x_3)} (definition 6)

iv) A∩B={(0.3,0.1,0.2,x_1), (0.3,0.2,0.4, x_2), (0.7,0.2,0.6,x_3)} (definition 7)

4. Neutrosophic Topological Spaces

FSs, FL and the related to them theories for managing the uncertainty [4] have found many and important applications to almost all sectors of human activity. Fuzzy mathematics, however, has also enormously developed in theoretical level giving important insights even to traditional branches of pure mathematics, like Algebra, Geometry, Analysis, Topology, etc.

Topological spaces is the most general category of mathematical spaces, in which fundamental mathematical concepts like convergence, continuity, compactness, etc. are defined (e.g. see [18]). Metric spaces and manifolds are special forms of topological spaces satisfying some extra conditions.

It is recalled that the concept of a topological space is defined as follows:

Definition 8: A *topology* T on a non-empty set U is defined as a collection of subsets of U such that:

1. U and the empty set belong to T, and
2. The union and intersection of any two elements of T belong also to T.

Trivial examples are the *discrete topology* of all subsets of U and the *non-discrete topology* T= {U, ∅}. The *usual* topology on the set ***R*** of real numbers is defined as the set of all subsets A of ***R*** with the property that, for each a in A, there exists ε>0, such that (a-ε, a+ ε) $\subseteq$ A.

The elements of a topology T on U are called *open subsets* of U and their complements are called *closed subsets* of U. The pair (U, T) defines a *topological space (TS)* on U.

The concept of TS has been extended to *fuzzy TS* [19], to *intuitionistic fuzzy TS* [20], to *soft TC* [21], etc. Here we describe how one can extend the concept of TS to *neutrosophic TS* [22].

Definition 9: i) The *empty NS* $\emptyset_N$ on the universe U is defined to be $\emptyset_N$ = {(x,0,0,1): x ∈ U}.

ii) The *universal NS* U_N on U is defined to be U_N = {(x,1,1,0): x ∈ U}.

It is straightforward to check that for each NS A in U is A∪U_N = U_N, A∩U_N = A, A∪$\emptyset_N$ = A and A∩$\emptyset_N$ = $\emptyset_N$.

Definition 10: A *neutrosophic topology* T on a non-empty set U is defined as a collection of NSs on U such that:

1. U_N and $\emptyset_N$ belong to T, and
2. The union and intersection of any two elements of T belong also to T.

Trivial examples are the *discrete neutrosophic topology* of all NSs in U and the *non-discrete neutrosophic topology* T= {U_N, $\emptyset_N$}.

The elements of a neutrosophic topology T on U are called *open NSs* in U and their complements are called *closed NSs* in U. The pair (U, T) defines a *neutrosophic topological space (NTS)* on U.

Example 2: Let U = {u} and let A = {(u ,0.5,0.5,0.4)}, B = {(u ,0.4,0.6,0.8)}, C = {(u ,0.5,0.6,0.4)}, D ={(u ,0.4,0.5,0.8)} be NSs in U. Then it is straightforward to check that the collection T = {$\emptyset_N$, U_N, A, B, C, D} is a neutrosophic topology on U.

We close by extending the concepts of *convergence*, *continuity*, *compact* TS and Hausdorf TS to NTSs.

Definition 11: Given two NSs A and B of the NTS (U, T), B is said to be a *neighbourhood* of A, if there exists an open NS O such that A $\subseteq$ O $\subset$ B. Further, we say that a sequence {A_n} of NSs of (U, T) *converges* to the NS A of (U, T), if there exists a positive integer m such that for each integer n≥m and each neighbourhood B of A we have that $A_n \subset$ B.

The following theorem generalizes the *Zadeh's extension principle* for FSs to NSs:

Theorem 1: Let U and V be two non-empty crisp sets and let g: U $\rightarrow$ V be a function. Then g can be extended to a function G mapping NSs in U to NSs sets in V.

Proof: Let $A<T_A, I_A, F_A>$ be a NS in U. Then its image G(A) is a NS B in V, whose neutrosophic components are defined as follows: Given v in V, consider the set $g^{-1}(v)=\{u \in U: g(u)=v\}$. If $g^{-1}(v)=\emptyset$, then $T_B(v)=0$, and if $g^{-1}(v)\neq\emptyset$, then $T_B(v)$ is equal to the maximal value of all $T_A(u)$ such that $u \in g^{-1}(v)$. Conversely, the inverse image $G^{-1}(B)$ is the NS A in U with truth membership function $T_A(u)=T_B(g(u))$, for each $u \in U$. In an analogous way one can determine the neutrosophic components I_B and F_B of B. -

Definition 12: Let (U,T) and (V,S) be two NTSs and let g be a function g: $U \to V$. According to Theorem 1, g can be extended to a function G which maps NSs of U to NSs of V. We say then that g is a *neutrosophicaly-continuous* function, if, and only if, the inverse image of each open NS of V through G is an open NS of U.

Definition 13: A family A = $\{A_i, i\in I\}$ of NSs of a NTS (U, T) is called a *cover* of U, if U = $\bigcup_{i\in I} A_i$. If the elements of A are open NSs, then A is called an *open cover* of U. Also, each NS subset of A which is also a cover of U is called a *sub-cover* of A. The NTS (U, T) is said to be *compact*, if every open cover of U contains a sub-cover with finitely many elements.

Definition 14: A NTS (U, T) is called a *T_1-NTS*, if, and only if, for each pair of elements u_1, u_2 of U, $u_1\neq u_2$, there exist at least two open NSs O_1 and O_2 such that $u_1\in O_1$, $u_2\notin O_1$ and $u_2\in O_2$, $u_1\notin O_2$.

Definition 15: A NTS (U, T) is called a *T_2-NTS*, if, and only if, for each pair of elements u_1, u_2 of U, $u_1\neq u_2$, there exist at least two open NSs O_1 and O_2 such that $u_1\in O_1$, $u_2\in O_2$ and $O_1\cap O_2 = \emptyset_N$.

A T_2-NTS is also called a *Hausdorff* or a *separable* NTS. Obviously a T_2-NTS is always a T_1-NTS.

4. Discussion and Conclusions

In this work the concept of NS in the universe U, introduced by Smarandanche in 1995, which describes the existing in real life indeterminancy, was studied. The basic operations between NSs were presented and the classical notion of TS was extended to NTS. It was further shown that convergence, continuity, compact TS and Hausdorff TS can be naturally extended to NTSs.

It looks in general that the combination of two or more of the alternative theories developed for tackling the existing in real life uncertainty, is a promising tool for obtaining better results in a variety of human activities characterized by uncertainty ([16], [23], etc.). Consequently this could be a fruitful area for future research.

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