

parameter that can act as the residual (a slack or a surplus) of the unaccounted activities.

We therefor assumed a solution consisting of three components of linear exponentiation and trigonometric and the residual with requisite simulation parameters that can be determined from time to time depending on consideration of some physical measurement in a real life situation . We hereby proposed a an interpolating function of the type given below

$$y(x) = a_0 + \alpha^x + a_1x + a_2 \sin(\beta x^2 + k) \quad (13)$$

The parameters can be selected carefully to complement or retract the response to the oscillatory nature of the dynamical system under consideration. This model can be made more complex to include special parameters to cover the level of variability and non-coherent behaviours that may vary speed and amplitude of oscillation. However the interpolating functions also have the tendency of being self-compensating depending on the value of the parameters.

3.1 The Exact Nonstandard scheme developed for Eq. (12) by ([6])

Mickens(1994) in his paper has developed an exact nonstandard scheme for the eq(12) given by

$$\left\{ \frac{y_{k+1} - 2y_k + y_{k-1}}{\varphi^2} \right\} = 2\varepsilon \left\{ \frac{y_k - \psi y_{k-1}}{\varphi} \right\} + \left\{ \frac{2(1-\psi)y_k + (x^2 + x^2 - 1)y_{k-1}}{\varphi^2} \right\} \quad (14)$$

with $\psi = \cosh \quad \varphi = 4\sin^2\left(\frac{h}{2}\right)$

In this work we will construct a new class of numerical schemes with the same qualitative properties as the corresponding second order initial value equation representing some special class of Harmonic Oscillator Differential equations Eq (12). We posit that this discrete model will also produce solutions and curves that behave like the exact Nonstandard schemes constructed by [6]. This work is based on a combination of both Interpolation method and Non-standard method as explained above.

This new discrete models are significant because of the combined methodology and its total congruency to the early works on the exact schemes for Oscillator equations. The schemes have been tested in a numerical experiment which produced some interesting 3D graphs..

4. Derivation of the Standard Finite Difference Scheme Using Eq(13)

Thus we can derive a discrete model based on equation (13) as given below:

$$y(x) = a_0 + \alpha^x + a_1x + a_2 \sin(\beta x^2 + k)$$

$$y' = \alpha^x \log \alpha + a_1 + a_2 2\beta x \cos(\beta x^2 + k) \quad (15)$$

$$y'' = a_2 [2\beta x (-2\beta x \sin(\beta x^2 + k) + 2\beta \cos(\beta x^2 + k))] + \alpha^x (\log \alpha)^2$$

$$y'' = a_2 [-(2\beta x)^2 \sin(\beta x^2 + k) + 2\beta \cos(\beta x^2 + k)] + \alpha^x (\log \alpha)^2 \quad (16)$$

$$y''' = a_2 [-(2\beta x)^3 \cos(\beta x^2 + k) + 4\beta x \sin(\beta x^2 + k)] + \alpha^x (\log \alpha)^3 \quad (17)$$

From (15), (16), (17)

$$a_1 = y' - \alpha^x \log \alpha - a_2 2\beta x \cos(\beta x^2 + k) \quad (18)$$

$$a_2 = \frac{y'' - \alpha^x (\log \alpha)^2}{[-(2\beta x)^2 \sin(\beta x^2 + k) + 2\beta \cos(\beta x^2 + k)]} \quad (19)$$

The discrete form

$$y(x) = a_0 + \alpha^x + a_1x + a_2 \sin(\beta x^2 + k)$$

$$y(x_{n-1}) = a_0 + \alpha^{x_{n-1}} + a_1x_{n-1} + a_2 \sin(\beta x_{n-1}^2 + k)$$

$$y(x_n) = a_0 + \alpha^{x_n} + a_1x_n + a_2 \sin(\beta x_n^2 + k)$$

$$y(x_{n+1}) = a_0 + \alpha^{x_{n+1}} + a_1x_{n+1} + a_2 \sin(\beta x_{n+1}^2 + k)$$

$$y(x_{n+1}) - 2y(x_n) + y(x_{n-1}) = (a_0 - 2a_0 + a_0) + (\alpha^{x_{n-1}} - 2\alpha^{x_n} + \alpha^{x_{n+1}}) + a_1(x_{n-1} + x_{n+1} - 2x_n) + a_2[(\sin(\beta x_{n-1}^2 + k) + \sin(\beta x_{n+1}^2 + k) - 2\sin(\beta x_n^2 + k))] \quad (20)$$

Let $P_n = [(\sin(\beta x_{n-1}^2 + k) + \sin(\beta x_{n+1}^2 + k) - 2\sin(\beta x_n^2 + k))]$

Substitute $x_{n-1} = a + nh - h, \quad x_{n+1} = a + nh + h,$ and $-2x_n = -2a - 2nh$ then

$$y(x_{n+1}) - 2y(x_n) + y(x_{n-1}) = \alpha^{a+nh} (\alpha^h + \alpha^{-h} - 2) + a_2[P_n]$$

At the point of coincidence with the exact solution
 $y(x_{n+1}) \equiv y_{n+1}$, $y(x_n) \equiv y_n$ and $y(x_{n-1}) \equiv y_{n-1}$

$$y_{n+1} - 2y_n + y_{n-1} \equiv \alpha^{a+nh} (\alpha^h + \alpha^{-h} - 2) + a_2[P_n]$$

$$P_n = [(\sin(\beta x_{n-1}^2 + k) + \sin(\beta x_{n+1}^2 + k) - 2\sin(\beta x_n^2 + k))]$$

$$\beta x_{n-1}^2 + k = \beta x_n^2 + \beta(h^2 - 2hx_n) + k$$

$$\beta x_{n+1}^2 + k = \beta x_n^2 + \beta(h^2 + 2hx_n) + k$$

$$\text{Let } P = \beta(x_n^2 + h^2) + k, Q = 2\beta hx_n \text{ and } R = (\beta x_n^2 + k) \tag{21}$$

Then

$$P_n = [(\sin(P + Q) + \sin(P - Q) - 2\sin(R))]$$

$$P_n = 2 \sin(P) \cos(Q) - 2\sin(R) \tag{22}$$

$$P_n = 2 \sin(\beta(x_n^2 + h^2) + k) \cos(2\beta hx_n) - 2\sin(\beta x_n^2 + k) \tag{23}$$

$$\frac{y_{n+1} - 2y_n + y_{n-1} = \alpha^{a+nh} (\alpha^h + \alpha^{-h} - 2) + \frac{\{y'' - \alpha^x(\log \alpha)^2\}[P_n]}{[-(2\beta x)^2 \sin(\beta x^2 + k) + 2\beta \cos(\beta x^2 + k)]}}{\tag{24}}$$

$$y_{n+1} - 2y_n + y_{n-1} = \frac{\alpha^{a+nh} (\alpha^h + \alpha^{-h} - 2) + \frac{\{y'' - \alpha^x(\log \alpha)^2\}[2 \sin(\beta(x_n^2 + h^2) + k) \cos(2\beta hx_n) - 2\sin(\beta x_n^2 + k)]}{[-(2\beta x)^2 \sin(\beta x^2 + k) + 2\beta \cos(\beta x^2 + k)]}}{\tag{25}}$$

Applying nonstandard dynamic step size φ instead of h

The second derivative

$$\equiv \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \equiv \frac{y_{n+1} - 2y_n + y_{n-1}}{\varphi^2}$$

The Standard scheme developed in equation (25) will be named NEW h

The hybrid scheme is obtained by substituting h for $\varphi = \sin(h)$ and $\varphi = \frac{(e^{\lambda h} - 1)}{\lambda}$ which will be named NEW SIN, NEW EXP respectively

5 Qualitative properties of the new scheme

Definition ([3])

Any algorithm for solving a differential equation in which the approximation y_{n+1} to the solution at x_{n+1} can be calculated iff x_n, y_n and h are known is called a one step method. It is a common practice to write the functional dependence y_{n+1} on the quantities x_n, y_n and h in the form $y_{n+1} = y_n + \phi(x_n, y_n, h)$

Where $\phi(x_n, y_n, h)$ is the incremental function

Theorem ([3])

Let the incremental function of the scheme defined in the one step scheme above be continuous and jointly as a function of its arguments in the region defined by $x \in [a, b]$ and $y \in (-\infty, \infty)$, $0 \leq h \leq h_0$ Where $h_0 > 0$ and let there exists a constant L such that $|\phi(x_n, y_n, h) - \phi(x_n, y_n^*, h)| \leq L|y_n - y_n^*|$ for all (x_n, y_n, h) and (x_n, y_n^*, h) in the region just defined, Then the relation $(x_n, y_n, 0) = (x_n, y_n^*)$ is a necessary condition for the convergence of the new scheme

Definition ([4])

A numerical scheme with an incremental $\phi(x_n, y_n, h)$ is said to be consistent with the initial value problem $y' = f(x, y), y(x_0) = y_0$ if the incremental function is identically zero at t_0 when $h = 0$,

Theorem ([4])

Let $y_n = y(x_n)$ and $p_n = p(x_n)$ denote two different numerical solution of the differential equation with the initial condition specified a

$$y_0 = y(x_0) = \xi \text{ and } p_0 = p(x_0) = \xi^* \text{ respectively such that } |\xi - \xi^*| < \epsilon \text{ } \epsilon > 0$$

If the two numerical estimates are generated by the integration scheme, we have

$$y_{n+1} = y_n + h\phi(x_n, y_n, h)$$

$$p_{n+1} = p_n + h\phi(x_n, p_n, h)$$

The condition that $|y_{n+1} - p_{n+1}| \leq K |\xi - \xi^*|$ is the necessary and sufficient condition for the stability and convergence of the schemes.

5.1 Proof of Convergence

Let

$$P_n = 2 \sin(\beta(x_n^2 + h^2) + k) \cos(2\beta h x_n) - 2\sin(\beta x_n^2 + k)$$

$$y' = f_n, y'' = f'_n \text{ and } y''' = f''_n$$

$$D_n = [-(2\beta x_n)^2 \sin(\beta x_n^2 + k) + 2\beta \cos(\beta x_n^2 + k)]$$

$$T_n = -\alpha^x (\log \alpha)^2$$

$$K_n = \alpha^{a+nh} (\alpha^h + \alpha^{-h} - 2)$$

$$y_{n+1} = 2y_n + y_{n-1} + K_n + \left\{ \frac{f'_n - T_n [P_n]}{D_n} \right\} \quad (26)$$

For small h the nonlocal approximation of $2y_n - y_{n-1} \cong y_n$
 Simplify to obtain

$$y_{n+1} = y_n + K_n + \left\{ \frac{f'_n - T_n [P_n]}{D_n} \right\}$$

$$y_{n+1} = y_n + \left\{ K_n - \frac{T_n [P_n]}{D_n} \right\} + \left\{ \frac{P_n}{D_n} \right\} f'_n \quad (27)$$

The incremental function can be written as

$$\phi(x_n, y_n, h) = \left\{ K_n - \frac{T_n [P_n]}{D_n} \right\} + \left\{ \frac{P_n}{D_n} \right\} f'_n \quad (28)$$

$$\phi(x_n, y_n, h) = A + B f'_n$$

The value of A is fixed for every finite ($n \ll \infty$)

$$\phi(x_n, y_n, h) - \phi(x_n, y_n^*, h) =$$

$$B[f'(x_n, y_n, h) - f'(x_n, y_n^*, h)]$$

$$= B[f'(x_n, y_n) - f'(x_n, y_n^*)]$$

$$= B \left[\frac{\partial f'(x_n, \bar{y})}{\partial y_n} (y_n - y_n^*) \right]$$

$$L = \text{SUP}_{(x_n, y_n) \in D} \frac{\partial f'(x_n, \bar{y})}{\partial y_n}$$

then

$$\phi(x_n, y_n, h) - \phi(x_n, y_n^*, h) = B[L(y_n - y_n^*)]$$

$$\text{Let } M = |B.L2|$$

$$\phi(x_n, y_n, h) - \phi(x_n, y_n^*, h) \leq M|y_n - y_n^*| \quad (29)$$

which is the condition for convergence

5.2 Consistency of the Schemes

$$y_{n+1} = y_n + \left\{ K_n - \frac{T_n [P_n]}{D_n} \right\} + \left\{ \frac{P_n}{D_n} \right\} f'_n$$

$$y_{n+1} = y_n + \{A\} + \{B\}f'_n$$

When $h = 0$ $K_n = 0$, $P_n = 0$, and $A = 0$ $B = 0$

$\Rightarrow y_{n+1} = y_n$ and the incremental function is identically zero when $h = 0$ (30)

$$\Rightarrow \phi(x_n, y_n, 0) \equiv 0$$

5.3 Stability of the Schemes

Consider the equation

$$y_{n+1} = y_n + \{A\} + \{B\}f'_n(x_n, y_n)$$

$$\text{Let } S_{n+1} = S_n + \{A\} + \{B\}f'_n(x_n, S_n)$$

$$y_{n+1} - S_{n+1} = y_n - S_n + \{A - A\} + \{B\}[f'_n(x_n, y_n) - f'_n(x_n, S_n)] \quad \{A - A\} \quad (31)$$

$$= y_n - S_n + B \left[\frac{\partial f'(x_n, S_n)}{\partial p_n} (y_n - S_n) \right]$$

$$L = \text{SUP}_{(x_n, y_n) \in D} \frac{\partial f'(x_n, S_n)}{\partial p_n}$$

$$y_{n+1} - S_{n+1} = y_n - S_n + B.L(y_n - S_n)$$

$$|y_{n+1} - S_{n+1}| = |y_n - S_n| + [B.L]|(y_n - p_n)|$$

$$\text{Let } N = |1 + [B.L]|$$

$$|y_{n+1} - S_{n+1}| \leq N |y_n - S_n|$$

Let $y_0 = y(x_0) = \xi$ and $S = S(x_0) = \xi^*$ then

$$|y_{n+1} - S_{n+1}| \leq K |\xi - \xi^*| \quad (32)$$

6. Application of the Model to the Harmonic Oscillator Equation

From equation (12), we have

$$y'' = -2\varepsilon(y') - y \quad (33)$$

$$y''' = -2\varepsilon(y'') - y' \quad (34)$$

From the Nonstandard theory $y' = \frac{y_{n+1} - y_n}{\psi}$

$$f'_n = -2\varepsilon \left(\frac{y_{n+1} - y_n}{\psi} \right) - y_n$$

$$f'_n = -2\varepsilon \left(\frac{y_{n+1}}{\psi} \right) + 2\varepsilon \left(\frac{y_n}{\psi} \right) - y_n$$

The standard scheme equation (25) is

$$y_{n+1} = y_n + \{A\} + \{B\}f'_n(x_n, y_n)$$

$$y_{n+1} = y_n + \left\{ K_n - \frac{T_n P_n}{D_n} \right\} - \left\{ \frac{P_n}{D_n} \right\} \left\{ \left(\frac{2\varepsilon}{\psi} \right) \right\} y_{n+1} + \left\{ \left(\frac{2\varepsilon}{\psi} \right) - 1 \right\} \left\{ \frac{P_n}{D_n} \right\} y_n \tag{35}$$

$$\left\{ \frac{\psi D_n + 2\varepsilon P_n}{\psi D_n} \right\} y_{n+1} = \left\{ K_n - \frac{T_n P_n}{D_n} \right\} + \left\{ 1 + \frac{2\varepsilon P_n}{\psi D_n} - \frac{P_n}{D_n} \right\} y_n$$

$$y_{n+1} = \left\{ \frac{\psi D_n}{\psi D_n + 2\varepsilon P_n} \right\} \left\{ K_n - \frac{T_n P_n}{D_n} \right\} + \left\{ \frac{\psi D_n}{\psi D_n + 2\varepsilon P_n} \right\} \left\{ 1 + \frac{2\varepsilon P_n}{\psi D_n} - \frac{P_n}{D_n} \right\} y_n \tag{36}$$

$$P_n = 2 \sin(\beta(x_n^2 + h^2) + k) \cos(2\beta h x_n) - 2 \sin(\beta x_n^2 + k)$$

$$D_n = [-(2\beta x_n)^2 \sin(\beta x_n^2 + k) + 2\beta \cos(\beta x_n^2 + k)]$$

$$T_n = \alpha^x (\log \alpha)^2$$

$$K_n = \alpha^{a+nh} (\alpha^h + \alpha^{-h} - 2)$$

The standard scheme will have $\varphi = h^2$ and the two hybrid schemes will be obtained by changing h to $\varphi = \sin^2(rh)$ and $\varphi = h \frac{(e^{\lambda h} - 1)}{\lambda}$, $\psi = \sin(rh)$ or $\psi = \frac{(e^{\lambda h} - 1)}{\lambda}$ $r, \lambda \in \mathbb{R}$

7. Result of Numerical Experiment

The algorithm of these new family of schemes have been coded into a software program. The schemes have been tested using step size $h=0.01$ and for about 100 iterations. The result of the numerical simulation is here presented in 3D graphs. NSTD EXACT is the exact solution as produced by [6] in 1994 . NSTD EXP is the solution with step-size substituted by exponential function NSTD SIN is the solution with step-size substituted by sine function. Likewise we have NEW SCH EXP which is the new scheme with exponential function as step-size and NEW SCH SIN is the new scheme with sine function as step-size

6.1 Graph of the new schemes for $h=0.01$, $\varepsilon = 0.0001$ and initial value $y(0) = 1$

simulation parameters $r = 0.005, \lambda = 0.005, \alpha = 0.5, \beta = -0.75,$

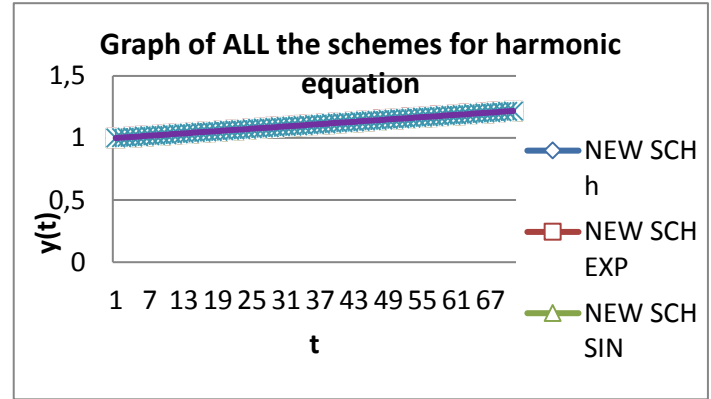


Fig 1: Graph of solution from all the new schemes and the exact solution

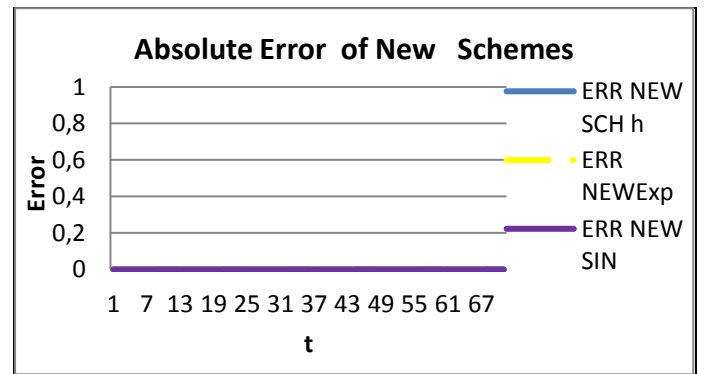


Fig 2: Graph of Error of deviation of schemes from the Mickens exact solution

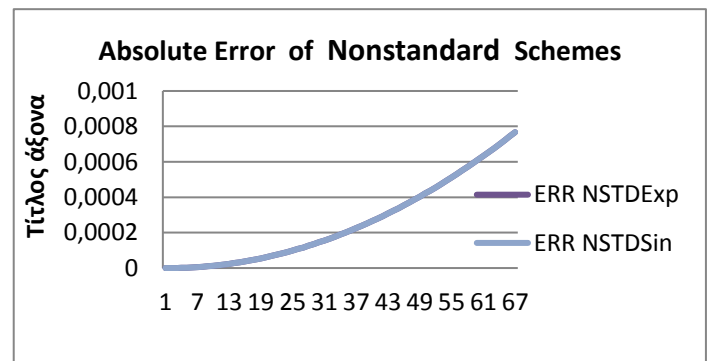


Fig 3: Graph of Error of deviation of schemes from the Mickens exact solution

**6.2 Graph of the new schemes for $h=0.001$,
 $\varepsilon = 0.001$ and initial value $y(0) = 1$
simulation parameters $r = 1, \lambda = -0.01$, $\alpha = 0.5, \beta = 0.75$,**

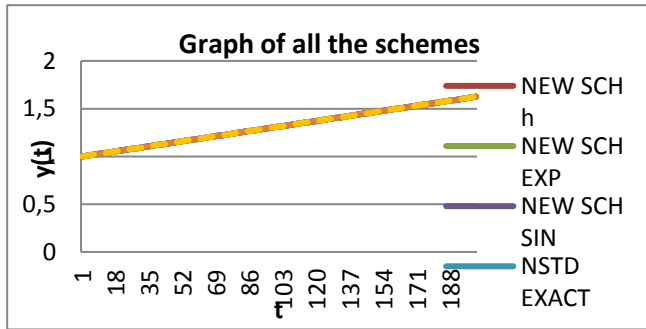


Fig 4: 3D Graph of solution from all the new schemes and the Exact solution

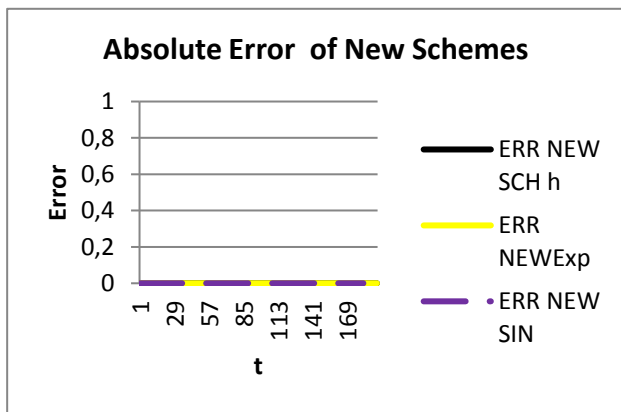


Fig 5: Graph of Error of deviation of New schemes from the Mickens exact solution

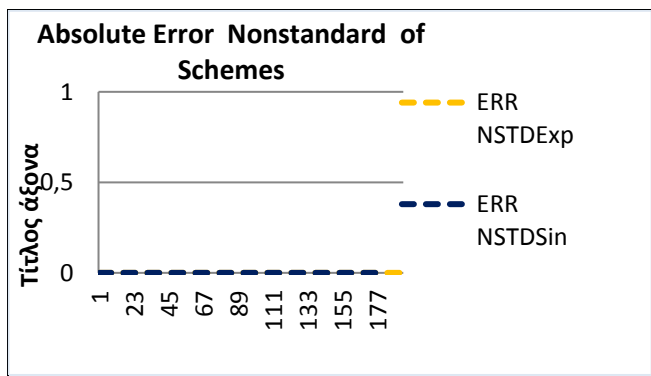


Fig 6: Graph of Error of deviation of schemes from the Mickens exact solution

8. Discussion and conclusion

We have experimented with the exact scheme obtain by Mickens in 1994 in comparison with the new schemes under the same parameter. We did for various step sizes . we observed monotonicity of solutions and total monotonicity of all solutions, monotone dependence on initial values. We observe total congruency as h becomes smaller for any finite number of iterations. The schemes converge faster depending on how small the value of α , large values of β makes the schemes to behave like very small value of β

The new Discrete Hybrid Nonstandard models NEWSH EXP and NEWSCH SIN produced the exact solution for the Harmonic Oscillator Equation as presented. The schemes have been proved to be stable, consistent and convergent analytically. The numerical experiment has shown that the absolute error of deviation for the Hybrid schemes are all zeros as h approaches zero (see Fig. 3 and 6) and this is consistent for all $h < 1$

These results shows that even though Mickens non-standard modelling rules remain a very powerful tool for discrete modelling of the true behaviour of dynamical systems, a carefully chosen interpolation function can produced an exact scheme with same property and also that interpolation schemes can be better if combined with some Nonstandard modelling techniques.

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