A pseudo zeta function and Riemann hypothesis

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Abstract: This work is dedicated to the promotion of the results Hadamard, Landau E., Walvis A., Estarmann T and Paul R. Chernoff for pseudo zeta functions. The properties of zeta functions are studied, these properties can lead to new regularities of zeta functions.

Key–Words: Euler product, Dirichlet, Riemann, hypothesis, zeta function,Hadamard, Landau E., Walvis A., Estarmann T,Paul R. Chernoff

1 INTRODUCTION

In this work we are studying the properties of modified zeta functions. Riemann's zeta function is defined by the Dirichlet's distribution

$$\varsigma(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \;\; s = \sigma + it \tag{1}$$

absolutely and uniformly converging in any finite region of the complex s-plane, for which $\sigma \geq 1+\epsilon, \epsilon > 0$. If $\sigma > 1$ the function is represented by the following Euler product formula

$$\varsigma(s) = \prod_p \left[1 - \frac{1}{p^s}\right]^{-1} \tag{2}$$

where p is all prime numbers. $\varsigma(s)$ was firstly introduced by Euler 1737 in [1], who decomposed it to the Euler product formula (2). Dirichlet and Chebyshev, studying the law of prime numbers distribution, had considered this in [2] . However, the most profound properties of the function $\varsigma(z)$ had only been discovered later, when the function had been considered as a function of a complex variable. In 1876 Riemann was the first who in [3] that :
$\varsigma(s)$ allows analytical continuation on the whole z-plane in the following form

$$\pi^{-s/2}\Gamma(s/2)\varsigma(s) = 1/(s(s-1)) + \tag{3}$$

$$\int^{+\infty} (x^{s/2} - 1 + x^{(1-s)/2} - 1)\theta(x)dx$$

where $\Gamma(z)$- gamma function,
$\theta(x) = \sum_{n=1}^{\infty} exp(-\pi n^2 x)$.
$\varsigma(s)$ is a regular function for all values of s, except s=1, where it has a simple pole with a deduction equal to 1, and satisfies the following functional equation

$$\pi^{-s/2}\Gamma(s/2)\varsigma(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\varsigma(1-s) \tag{4}$$

This equation is called the Riemann's functional equation.
The Riemann's zeta function is the most important subject of study and has a plenty of interesting generalizations. The role of zeta functions in the Number Theory is very significant, and is connected to various fundamental functions in the Number Theory as Mobius function, Liouville function, the function of quantity of number divisors, the fucntion of quantity of prime number divisors. The detailed theory of zeta functions is showed in [4]. The zeta function spreads to various disciplines and now the function is mostly applied in quantum statistical mechanics and quantum theory of pole[5-7]. Riemann's zeta function is often introduced in the formulas of quantum statistics. A well-known example is the Stefan-Boltzman law of a black body's radiation. The given aspects of the zeta function reveal global necessity of its further investigation.
We are interested in the analytical properties of the following generalizations of zeta functions:

$$P(s) = \sum_{j\geq 1} \frac{1}{p_j^s},$$

where p_j runs through all prime numbers in ascending order. The forms of the given function P (s) allow to assume that they possess the same properties as the zeta function, but it is not quite obvious, considering

$$ln(\varsigma(s)) = \sum_{n=1}^{\infty} P(ns)/n,\ f(s) = ln(\varsigma(s)) - P(s), \tag{5}$$

Hadamard was first who began to apply P (s) for studying the zeta function in [8]. Chernoff has made significant progress in the Riemann hypothesis, studying the function P (s) in [9].We repeat the results of Shernoff with some modifications . This paper completes Chernoff 's research for the pseudo-zeta function. Chernoff gave an equivalent formulation of the Riemann hypothesis in terms of a pseudo-zeta function.
THEOREM. (Chernoff) Let $C(s) = \prod_{n>1}\left[1 - \frac{1}{(nln(n))^s}\right]^{-1}$ Then C(s) continues analytically into the critical strip and has no zeros there. Significance of the theorem: If the primes were distributed more regularly (i.e., if $p_n \equiv nlogn$), then the Riemann hypothesis would be trivially true.
Developing the works of Chernoff and Hadamard, we formulate the following question: Does the pseudo zeta function P(s) continues analytically into the critical strip?
We note that for the first time the analytic P (s) were studied by Landau E., Walvis A. in [10] and Estarmann T [11], [12] but they could not obtain effective estimates for P (s). Our work is devoted to the solution of this question.

2 RESULTS

We shall use the well-known Paley-Wiener:

Theorem 1 . *Suppose that F is supported in [A, A], so that $F \in L_2(-A, A)$. Then the holomorphic Fourier transform*

$f(\zeta) = \int_{-A}^{A} F(x)e^{ix\zeta}\,dx$ is an entire function of exponential type A, meaning that there is a constant C such that $|f(\zeta)| \le Ce^{A|\zeta|}$

Theorem 2 *Let $s = i\alpha + \tau,\ \tau > 1/2 + \epsilon$ then we have estimate*

$$sup_{\tau}|f(\tau + i\alpha)| + |sup_{\tau}|\frac{d^2 f(\tau + i\alpha)}{d\tau^2}| < C$$

Proof: we have from (5) $as\ Re(s) > 1/2 + 2\epsilon$ we have

$$|f(s)| = |\sum_{m=2}^{\infty} P(ms)/m| \le \sum_{m=2}^{\infty} |P(ms)/m| \le$$

$$C_\epsilon \sum_{m=2}^{\infty} |2^{-m\epsilon}/m| < CC_\epsilon < \infty$$

$as\ Re(s) > 1/2 + 2\epsilon$ we have

$$|\frac{df(\tau + i\alpha)}{d\tau}| = |\sum_{m=2}^{\infty} \frac{dP(m(\tau + i\alpha))}{md\tau}| \le$$

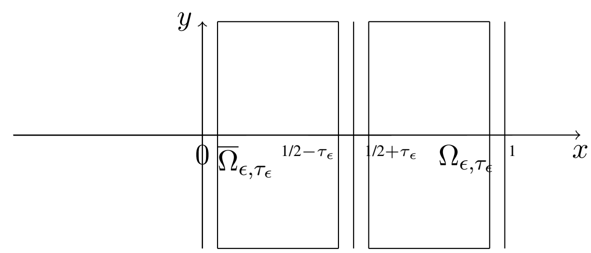
$$C_\epsilon \sum_{m=2}^{\infty} |2^{-m\epsilon}/m| < CC_\epsilon < \infty$$

$as\ Re(s) > 1/2 + 2\epsilon$ we have

$$|\frac{d^2 f(\tau + i\alpha)}{d\tau^2}| = |\sum_{m=2}^{\infty} \frac{d^2 P(m(\tau + i\alpha))}{md\tau^2}| \le$$

$$C_\epsilon \sum_{m=2}^{\infty} |2^{-m\epsilon}/m| < CC_\epsilon < \infty$$

Let introduce sets $\Omega_{\epsilon,\tau_0} = \{1/2 + \tau + i\alpha|\varsigma(1/2 + \tau_0 \pm \imath\alpha) = 0\,\&\,|\alpha| < \frac{1}{\epsilon}\&\tau > \tau_0\}$.
$\overline{\Omega}_{\epsilon,\tau_0} = \{1/2 - \tau_0 + i\alpha|\varsigma(1/2 - \tau \pm \imath\alpha) = 0\,\&\,|\alpha| < \frac{1}{\epsilon}\&\tau > \tau_0\}$.
$\Delta_{\epsilon,\tau_0} = \{1/2 - \tau_0 + i\alpha_0|\varsigma(1/2 + \tau_0 \pm \imath\alpha_0) = 0\&|\alpha| < \frac{1}{\epsilon}\}$
and $\tau_\epsilon = \sup_{1/2-\tau_0+i\alpha_0,\in\Delta_{\epsilon,\tau_0}} \tau_0$



Using the properties of the set $\Omega_{\epsilon,\tau_\epsilon}$ where the zeta function is not zeroed, we will use the logarithm of the zeta function as the principal value Log is the logarithm whose imaginary part lies in the interval$(\pi, \pi]$

Lemma 3 *For $P(s)\,, s \in \Omega_{\epsilon,\tau_\epsilon}$ we have an analytic continuation defined by the following formula*
$P(s) = ln(\varsigma(s)) - f(s)$

Proof: According to the definition of the set of $\Omega_{\epsilon,\tau_\epsilon}$, we have a single-valued analytic function for the logarithm of the zeta function on this set. The same is true for the function f

Lemma 4 *For $P(s)\,, s \in \overline{\Omega}_{\epsilon,\tau_\epsilon}$ we have an analytic continuation for defined by the following formula*
$P(s) = ln(\varsigma(s)) - f(s)$

Proof: According to the definition of the set of $\overline{\Omega}_{\epsilon,\tau_\epsilon}$, we have a single-valued analytic function for the logarithm of the zeta function on this set. The same is true for the function f

Lemma 5 *Let* $1/2+2\epsilon < x < 1-2\epsilon$ *then* $\mu_\epsilon(x) = 1$

Proof:

$$\mu_\epsilon(x) = \int \psi(s/\epsilon)\nu(x-s)ds/\epsilon =$$

$$\int_{-\epsilon}^{\epsilon} \psi(s/\epsilon)\nu(x-s)ds/\epsilon = \int_{-\epsilon}^{\epsilon} \psi(s/\epsilon)ds/\epsilon = 1$$

Theorem 6 *Let* $s = i\alpha + \tau,\ \tau > \tau_\epsilon$ *then we have equation*

$$P(\tau + i\alpha) = \overline{P(\tau - i\alpha)}$$

Proof:

$\underline{as\ Re(s)} > 1/2 + \tau_\epsilon$ we have $\underline{P(m(\tau - i\alpha))} = \overline{P(m(\tau + i\alpha)}, ln(\varsigma(\tau - i\alpha)) = \overline{\varsigma(\tau + i\alpha)}$

We can using (5) $as\ Re(s) > 1/2 + \tau_\epsilon$ and we get $P(\tau - i\alpha) = \overline{P(\tau + i\alpha)}$

Let introduce functions :

$$\begin{cases} \nu_\epsilon(s) = 0,\ \ Re(s) < \epsilon; \\ \nu_\epsilon(s) = 1,\ \ \epsilon < Re(s) < 1/2 - \epsilon; \\ \nu_\epsilon(s) = 0,\ \ 1/2 - \epsilon < Re(s) < 1/2 + \epsilon; \\ \nu_\epsilon(s) = 1,\ \ 1/2 + \epsilon < Re(s) < 1 - \epsilon, \\ \nu_\epsilon(s) = 0,\ \ Re(s) > 1 - \epsilon; \\ \psi(t) = Ce^{\frac{1}{t^2-1}},\ \ t^2 < 1; \\ \psi(t) = 0,\ t^2 \geq 1; \\ \mu_\epsilon(x) = \int \psi(s/\epsilon)\nu(x-s)ds/\epsilon; \end{cases}$$

Lemma 7 *For* μ_ϵ *we have* $\mu_\epsilon(x) = \mu_\epsilon(1-x)$

Proof:

$$\mu_\epsilon(x) = \int \psi(s/\epsilon)\nu(x-s)ds/\epsilon =$$

$$\int \psi(s/\epsilon)\nu(1-x+s)ds/\epsilon = \mu_\epsilon(1-x)$$

Lemma 8 *Let* $1/2+2\epsilon < x < 1-2\epsilon$ *then* $\mu_\epsilon(x) = 1$

Proof:

$$\mu_\epsilon(x) = \int \psi(s/\epsilon)\nu(x-s)ds/\epsilon =$$

$$\int_{-\epsilon}^{\epsilon} \psi(s/\epsilon)\nu(x-s)ds/\epsilon = \int_{-\epsilon}^{\epsilon} \psi(s/\epsilon)ds/\epsilon = 1$$

Theorem 9 *Let* $\tau_\epsilon < Re(s) < 1 - 2\epsilon$ *then* $|P(s)| < C_\epsilon$

Proof: we have from (4)

$$ln(\varsigma(s)) = ln(\varsigma(1-s)) + T(s),$$
$$T(s) = \frac{s}{2}ln(\pi) + ln(\Gamma(s)) -$$
$$\frac{1-s}{2}ln(\pi) + ln(\Gamma(1-s))$$

We can using (5) $as\ Re(s) > 1/2 + \tau_\epsilon$ therefore

$$P(s) = P(1-s) + F(s)$$
$$F(s) = T(s) - f(1-s) + f(s)$$

Using Lemma 1 we have

$$P_\epsilon(s) = P_\epsilon(1-s) + F_\epsilon(s) \quad (6)$$

$$P_\epsilon(s) = P(s)\mu_\epsilon(x),\ \ F_\epsilon(s) = F(s)\mu_\epsilon(x) \quad (7)$$

Using Furie transform we have

$$\widetilde{P_\epsilon(k)} = \frac{1}{\sqrt{2\pi}}\int_0^1 P_\epsilon(s)e^{-iks}ds = \quad (8)$$

$$\frac{e^{ik/2}}{\sqrt{2\pi}}\int_{-1/2}^{1/2} P_\epsilon(s+1/2)e^{-iks}ds = e^{ik/2}I(k) \quad (9)$$

$$\widetilde{F_\epsilon(k)} = \frac{1}{\sqrt{2\pi}}\int_0^1 F_\epsilon(s)e^{-iks}ds = \quad (10)$$

$$\frac{e^{-ik/2}}{\sqrt{2\pi}}\int_{-1/2}^{1/2} F_\epsilon(s-1/2)e^{-iks}ds \quad (11)$$

Using (6)-(9)we have

$$\widetilde{P_\epsilon(k)} = \widetilde{P_\epsilon(-k)} + \widetilde{F_\epsilon(k)} \quad (12)$$

We see from (6) and Paley - Wiener theorem $I(k)$ is an entire analytic function of exponential type 1/2, and $\widetilde{P_\epsilon(k)}$ bounded function on upper half-plane. Using the well-known arguments about the symmetry of $P_\epsilon(k),\ \overline{P_\epsilon(k)}$ relative to the real axis, we get Riemann problem-finding the analytic function that satisfies a jump condition on the real axis

$$\Psi_+(k) = P_\epsilon(k),\ \Psi_-(k) = \overline{P_\epsilon(k)} \quad (13)$$

$$\Psi_+(k) = \Psi_-(k) + \widetilde{F_\epsilon(k)} \quad (14)$$

Using the well-known formulas for the Riemann problem we get

$$P_\epsilon(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\widetilde{F_\epsilon(t)}}{t-z}dt$$

Using Theorem 1 we obtain

$$P_\epsilon(s) = \frac{1}{\sqrt{2\pi}}\lim_{\tau\to 0}\int_{-\infty}^{\infty}\frac{e^{izs}}{2\pi i}\int_{-\infty}^{\infty}\frac{\widetilde{F_\epsilon(t)}}{t-z-i\tau}dtdz = \int_{-\infty}^{\infty}\frac{e^{izs}}{2\pi i}V.p\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\frac{\widetilde{F_\epsilon(t)}}{t-z}dtdz + \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{its}\widetilde{F_\epsilon(t)}dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{its}\widetilde{F_\epsilon(t)}dt$$

Using Theorem 1 we obtain

$$|P(s)| < 2||\widetilde{F_\epsilon}||_{L^1}/|\mu_\epsilon(s)| < C_\epsilon/|\mu_\epsilon(s)|$$

From last estimate and Lemma 2 we have as $1/2 + 2\tau_\epsilon < Re(s) < 1-2\epsilon$

$$|P(s)| < C_\epsilon$$

From last estimate and conditons theorem we have proof.

Theorem 10 *The Riemann's function has nontrivial zeros only on the line Re(s)=1/2;*

Proof: For $f(s) = \sum_{m=2}^{\infty} P(ms)/m$, as $Re(s) > 1/2 + \tau_\epsilon$ we have

$$|f(s)| = |\sum_{m=2}^{\infty} P(ms)/m| \le \sum_{m=2}^{\infty}|P(ms)/m| \le$$

$$C_\epsilon \sum_{m=2}^{\infty}|2^{-m\epsilon}/m| < CC_\epsilon < \infty$$

Applying the formula from the theorem 2

$$ln(\varsigma(s)) = P(s) + \sum_{m=2}^{\infty} P(ms)/m = P(s) + f(s)$$

estimating by the module

Estimating the zeta function, potentiating, we obtain

$$|(\varsigma(s)| > \exp\left[-|P(s)| - |f(s)|\right]$$

According to Theorem 2, $|P(s)|$ limited for s from the following set

$$(s, 1/2 + 2\tau_\epsilon < Re(s) < 1-2\epsilon)$$

finally we obtain:

$$|(\varsigma(s)| > \exp\left[-C_\epsilon\right],$$

$$as\ s \in (s, 1/2 + 2\tau_\epsilon < Re(s) < 1-2\epsilon)$$

These estimations for $|P(s)|, |f(s)|$ prove that function does not have zeros on the half-plane $Re(s) > 1/2 + 2\tau_\epsilon$ due to the integral representation (3) these results are projected on the half-plane $Re(s) < 1/2$. Due to the arbitrariness of the ϵ, we obtain a proof of the Riemann hypothesis. Riemann's hypothesis is proved.

3 CONCLUSION

In this work we obtained the estimation of the Riemann's zeta function logarithm outside of the line Re(s)=1/2 and outside of the pole s=1. This work accomplishes all the works of the greatest mathematicians, applying their immense achievements in this field. Without their effort we could not even attempt to solve the problem. My research on the Riemann hypothesis was completed by reducing the Riemann hypothesis to the Riemann boundary value problem for analytic functions. The most difficult part in this matter was overcome by Wiener and Paley. It was started by Riemann himself, continued by Hadamard and many others. The closest to the idea in the work was Landau E., Walvis A, and Estarmann T, Chernoff and it was possible to complete the solution of the Riemann hypothesis by the successes of Riemann and Hilbert for the Riemann boundary value problem.

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