Optimal Control on Exceptional Lie Group G_2 with Kahan Integrator Analysis

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Abstract: Study of optimal control of a system on Lie groups has been a very tempting research area for mathematicians and physicist. The matrix Lie groups provide a mathematically rich definition to study variety of control problems. In this paper, controllability of the smallest exceptional Lie group G_2 is studied, along with optimal control. Finally, Kahan Integrator is applied on the dynamics of the exceptional Lie group G_2.

Key–Words: Exceptional Lie group, optimal control, numerical integration

1 Introduction

Lie groups structures a focal subject of present day mathematics and theoretical physics. They speak to the best-created hypothesis of continuous symmetry of mathematical objects and structures, and this makes them requisite tools in various parts of mathematics and physics. They give a characteristic system to analyse the continuous symmetries of structures. Particle physics, M-theory, string theory, quantum chromodynamics, quantum computation and etc. involves Lie groups. A lot of study on classical Lie group approach to different application areas such as spacecraft attitude control [1], car parking problem, underwater vehicle problem [2] and path planning of robot manipulators has been done. Control theory and Lie groups go back a long way. It was Brockett [3], who introduced the concept of Lie groups to motion control problems. His theory were on different dynamical aspects of the system, like controllability, observability and realization theory. Later Jurdjevic and Sussmann [4] studied the controllability properties for Lie groups. Control theory has its foundations in the classical calculus of variations, yet made its mark with the coming of endeavors to control and manage machinery and to create controlling systems for ships and a lot later for planes, rockets and satellites. Remsing [5] and Puta et al [6] focused on the matrix Lie group and embellished its application to various mechanical problems. Optimal control involving Lie groups was studied by Spindler [7]. Continuing on the study of optimal control on matrix Lie group, Pop [8] studied optimal control on Heisenberg Lie group, $H(3)$, and Craioveanu et al [9] focused on the special Euclidean group $SE(3,\mathbb{R})$ respectively.

The exceptional Lie groups are interesting, symmetries emerging as groups of invariants of many physical models proposed for fundamental interactions. The smallest exceptional Lie group G_2 was discovered by Friedrich Engel. In a note to the Royal Saxonian Academy of Sciences, he wrote: ”Moreover, we hereby obtain a direct definition of our 14-dimensional simple group (G_2) which is as elegant as one can wish for” [10]. The smallest exceptional group G_2, the automorphism group of octonion algebra, turned out to be the best candidate as a holonomy group of the 7-dimensional manifold for the compactification of M-theory [11]. M-theory compactified on seven-dimensional manifolds of G_2 holonomy gives rise to four-dimensional theories with $\mathcal{N} = 1$ supersymmetry. G_2 holonomy is the condition for unbroken supersymmetry in four dimensions, Michael Atiyah [12] analyzed M-theory dynamics on a seven-manifold of G_2 holonomy [12]. Recently, G_2 has got lots of attention from physicist. Since G_2 has trivial center, Yang-Mills theory with gauge group G_2 is interesting. Here in this paper we have studied the controllability of and optimal control of the smallest exceptional Lie group G_2.

2 Exceptional Lie Group G_2 [13]

The smallest of the five exceptional simple Lie groups G_2 is 14-dimensional. G_2 can be described as the automorphism group of the octonions. As a proper subgroup of the double cover $spin(7)$ of special orthogonal group $SO(7)$, G_2 is the subgroup that preserves

Table 1: Commutation table of exceptional Lie algebra

[.,.]	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	A_{13}	A_{14}
A_1	0	$-A_3$	0	0	A_6	0	$-2A_1$	A_7	0	A_9	0	0	$-A_{12}$	$3A_1$
A_2	A_3	0	$2A_4$	$3A_5$	0	0	A_2	0	A_{14}	$-3A_8$	$-2A_{10}$	$-A_{11}$	0	$-2A_2$
A_3	0	$-2A_4$	0	$-3A_6$	0	0	$-A_3$	A_2	$-3A_1$	$3A_7+A_{14}$	$2A_9$	0	$-A_{11}$	A_3
A_4	0	$-3A_5$	$3A_6$	0	0	0	0	0	$-2A_3$	$2A_2$	$3A_7+2A_{14}$	A_9	$-A_{10}$	$-A_4$
A_5	$-A_6$	0	0	0	0	0	A_5	0	$-A_4$	0	A_2	A_7+A_{14}	$-A_8$	$-3A_5$
A_6	0	0	0	0	0	0	$-A_6$	$-A_5$	0	A_4	$-A_3$	$-A_8$	$2A_7+A_{14}$	0
A_7	$2A_1$	$-A_2$	A_3	0	$-A_5$	A_6	0	$-2A_8$	A_9	$-A_{10}$	0	A_{12}	$-A_{13}$	0
A_8	$-A_7$	0	$-A_2$	0	0	A_5	$2A_8$	0	A_{10}	0	0	$-A_{13}$	0	$-3A_8$
A_9	0	$-A_{14}$	$3A_1$	$2A_3$	A_4	0	$-A_9$	$-A_{10}$	0	$-2A_{11}$	$3A_{12}$	0	0	$2A_9$
A_{10}	$-A_9$	$3A_8$	$-3A_7-A_{14}$	$-2A_2$	0	$-A_4$	A_{10}	0	$2A_{11}$	0	$3A_{13}$	0	0	$-A_{10}$
A_{11}	0	$2A_{10}$	$-2A_9$	$-3A_7-2A_{14}$	$-A_2$	A_3	0	0	$-3A_{12}$	$-3A_{13}$	0	0	0	A_{11}
A_{12}	0	A_{11}	0	$-A_9$	$-A_7-A_{14}$	A_8	$-A_{12}$	A_{13}	0	0	0	0	0	$3A_{12}$
A_{13}	A_{12}	0	A_{11}	A_{10}	A_8	$-2A_7-A_{14}$	A_{13}	0	0	0	0	0	0	0
A_{14}	$-3A_1$	$2A_2$	$-A_3$	A_4	$3A_5$	0	0	$3A_8$	$-2A_9$	A_{10}	$-A_{11}$	$-3A_{12}$	0	0

some non zero spinor.

The roots of G_2 are

$$\triangle = \{\alpha, \beta, \alpha+\beta, \alpha+2\beta, \alpha+3\beta, 2\alpha+3\beta, h_\alpha,\ -\alpha, -\beta, -\alpha-\beta, -\alpha-2\beta, -\alpha-3\beta, -2\alpha-3\beta \ and\ h_\beta\}$$

And hence the chevalley basis for G_2 is

$$\{e_\alpha, e_\beta, e_{\alpha+\beta}, e_{\alpha+2\beta}, e_{\alpha+3\beta}, e_{2\alpha+3\beta}, h_\alpha, e_{-\alpha}, e_{-\beta}, e_{-\alpha-\beta}, e_{-\alpha-2\beta}, e_{-\alpha-3\beta}, e_{-2\alpha-3\beta}, h_\beta\}$$

Here we take $A_1 = e_\alpha, A_2 = e_\beta, A_3 = e_{\alpha+\beta}, A_4 = e_{\alpha+2\beta}, A_5 = e_{\alpha+3\beta}, A_6 = e_{2\alpha+3\beta}, A_7 = h_\alpha, A_8 = e_{-\alpha}, A_9 = e_{-\beta}, A_{10} = e_{-\alpha-\beta}, A_{11} = e_{-\alpha-2\beta}, A_{12} = e_{-\alpha-3\beta}, A_{13} = e_{-2\alpha-3\beta}$ and $A_{14} = h_\beta$ as the generators and the commutation table is shown in Table 1.

3 Control System on Exceptional Lie group G_2

A left invariant driftless control system on exceptional Lie group (G_2) group can be defined as [14]

$$\dot{p} = Z_0(q) + \sum_{i=1}^{m} u_i'(t) Z_i(p), p(t) \in G_2, u_i'(t) \in \mathbb{R}, m \leq 14. \quad (1)$$

Here p represents the state, Z_i is any vector field on G_2 and u_i' denotes input for the control system. $Z_i(p)$ can be replaced by $T_e L_p \varrho_i$, $\forall p \in G_2$, L_p is the left translation and for some fixed $\varrho_i \in g_2$, as the vector field Z_i is left invariant. So Equation (1) can be rewritten as

$$\dot{p} = T_e L_p \left(\varrho_0 + \sum_{i=1}^{m} u_i'(t) \varrho_i \right). \quad (2)$$

Here, $T_e L_p$ is the linearization of L_p at identity element denoted as e, which is invariant to left translation. In order to improve the applicability of Equation (2) the 'drift term' $T_e L_p \varrho_0$ is omitted, and hence the control system obtained is known as 'drift-free control system'. Equation (2) reduces to

$$\dot{p} = T_e L_p \left(\sum_{i=1}^{m} u_i'(t) \varrho_i \right). \quad (3)$$

A small damping term ϵ is multiplied to each of the input term for further improvization of performance. As a result, $\epsilon u_i'$ denotes a periodic control input with a very small amplitude. Hence, Equation (3) can be stated as

$$\dot{p} = \delta T_e L_p \left(\sum_{i=1}^{m} u_i'(t) \varrho_i \right). \quad (4)$$

The left invariant driftless control system described by Equation (1) on the Lie group G_2 with chevalley basis becomes

$$\dot{X} = XU(t),\ \ U(t) = \left(\sum_{i=1}^{14} u_i(t)A_i\right). \quad (5)$$

where $X(t)$ is a curve in the Lie group G_2 and $U(t)$ is a curve in the Lie algebra g_2. $A_1, A_2, \ldots, A_m$ are the chevalley basis of g_2. So a generalized drift-free, left invariant control system on G_2 can be written as

$$\dot{X} = X\left(\sum_{i=1}^{14} A_i u_i\right). \quad (6)$$

The control system in (6) can be modelled in terms of fewer independent controls, one of such system under consideration is

$$\dot{X} = X\left(A_1u_1 + A_2u_2 + A_8u_8 + A_9u_9\right). \quad (7)$$

4 Controllability

Controllability of a system marks the existence of a steering control input $u(t)$ which steers the system from an initial state $X(t_i) = x_i$ to a desired final state $X(t_f) = x_f$. In order to analyze the system derived from a Lie group we use the result of a known standard theorem.

Rashevesky-Chow Theorem [15]

"If M is a connected manifold and the control distribution $\Delta = \text{span}\{f_1, f_2, ..., f_n\}$ is bracket generating, then the drift-free system

$$\dot{X} = \sum_{i=1}^{n} x_i f_i(x), x \in M \quad (8)$$

is controllable."

Proposition 1. *The system described by equation (7) is controllable.*

Proof. Since the span of the set of Lie brackets generated by $\{A_1, A_2, A_8, A_9\}$ coincides with the exceptional Lie algebra g_2, therefore by Rashevesky-Chow Theorem the system in (7) is controllable. □

5 Optimal Control

Controllability analysis guarantees that a control input exist which steers the system from an initial state to a desired state, but it fails to discuss anything about the uniqueness of the available input choices. There can be several steering inputs driving the control system from initial to final state. For improvised performance the input choice should be made in an optimized manner. There are several ways to optimize input choices. In this section, we take minimum effort problem and design control inputs in such a way that the input cost function is minimized. Here the input cost function is defined by

$$K(u_1, u_2, u_8, u_9) = \frac{1}{2}\int_0^{t_f} \left(c_1{u_1}^2 + c_2{u_2}^2 + c_8{u_8}^2 + c_9{u_9}^2\right) dt,$$
$$c_1, c_2, c_3, c_4 > 0.$$

In order to minimize K, we have constructed a controlled Hamiltonian which is defined by

$$\overline{H_c} = x_1u_1 + x_2u_2 + x_8u_8 + x_9u_9 - \frac{1}{2}(c_1u_1^2 + c_2u_2^2 + c_8u_8^2 + c_9u_9^2).$$

According to Krishnaprasad's theorem [16], the Hamiltonian $\overline{H_c}$ of our system, has to be partially differentiated with respect to each control input and equated to zero in order to obtain optimal inputs which will minimize the total input cost. Hence

$$\frac{\partial \overline{H_c}}{\partial u_1} = \frac{\partial \overline{H_c}}{\partial u_2} = \frac{\partial \overline{H_c}}{\partial u_8} = \frac{\partial \overline{H_c}}{\partial u_9} = 0. \quad (9)$$

Thus the optimized control inputs are

$$u_1 = \frac{x_1}{c_1}, u_2 = \frac{x_2}{c_2}, u_8 = \frac{x_8}{c_8}, u_9 = \frac{x_9}{c_9}.$$

The optimal Hamiltonian has been derived to be

$$H_c(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}) = \frac{1}{2}\left(\frac{x_1^2}{c_1} + \frac{x_2^2}{c_2} + \frac{x_8^2}{c_8} + \frac{x_9^2}{c_9}\right).$$

The system is forced to follow a certain dynamics while optimizing the input choices. This dynamics which can be obtained by using the results of Krishnaprasad's theorem, which states that the resulting restricted dynamics is

$$[\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4, \dot{x}_5, \dot{x}_6, \dot{x}_7, \dot{x}_8, \dot{x}_9, \dot{x}_{10}, \dot{x}_{11}, \dot{x}_{12}, \dot{x}_{13}, \dot{x}_{14}]^t = \Omega_{-} \cdot \nabla H_c$$

Here, Ω_{-} is the minus Lie Poisson matrix and is defined by

$$\Omega_{-}=\begin{bmatrix}
0 & -x_3 & 0 & 0 & x_6 & 0 & -2x_1 & x_7 & 0 & x_9 & 0 & 0 & -x_{12} & 3x_1\\
x_3 & 0 & 2x_4 & 3x_5 & 0 & 0 & x_2 & 0 & x_{14} & -3x_8 & -2x_{10} & -x_{11} & 0 & -2x_2\\
0 & -2x_4 & 0 & -3x_6 & 0 & 0 & -x_3 & x_2 & -3x_1 & 3x_7+x_{14} & 2x_9 & 0 & -x_{11} & x_3\\
0 & -3x_5 & 3x_6 & 0 & 0 & 0 & 0 & 0 & -2x_3 & 2x_2 & 3x_7+2x_{14} & x_9 & -x_{10} & -x_4\\
-x_6 & 0 & 0 & 0 & 0 & 0 & x_5 & 0 & -x_4 & 0 & x_2 & x_7+x_{14} & -x_8 & -3x_5\\
0 & 0 & 0 & 0 & 0 & 0 & -x_6 & -x_5 & 0 & x_4 & -x_3 & -x_8 & 2x_7+x_{14} & 0\\
2x_1 & -x_2 & x_3 & 0 & -x_5 & x_6 & 0 & -2x_8 & x_9 & -x_{10} & 0 & x_{12} & -x_{13} & 0\\
-x_7 & 0 & -x_2 & 0 & 0 & x_5 & 2x_8 & 0 & x_{10} & 0 & 0 & -x_{13} & 0 & -3x_8\\
0 & -x_{14} & 3x_1 & 2x_3 & x_4 & 0 & -x_9 & -x_{10} & 0 & -2x_{11} & 3x_{12} & 0 & 0 & 2x_9\\
-x_9 & 3x_8 & -3x_7-x_{14} & -2x_2 & 0 & -x_4 & x_{10} & 0 & 2x_{11} & 0 & 3x_{13} & 0 & 0 & -x_{10}\\
0 & 2x_{10} & -2x_9 & -3x_7-2x_{14} & -x_2 & x_3 & 0 & 0 & -3x_{12} & -3x_{13} & 0 & 0 & 0 & x_{11}\\
0 & x_{11} & 0 & -x_9 & -2x_7-x_{14} & x_8 & -x_{12} & x_{13} & 0 & 0 & 0 & 0 & 0 & 3x_{12}\\
x_{12} & 0 & x_{11} & x_{10} & x_8 & -2x_7-x_{14} & x_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-3x_1 & 2x_2 & -x_3 & x_4 & 3x_5 & 0 & 0 & 3x_8 & -2x_9 & x_{10} & -x_{11} & -3x_{12} & 0 & 0
\end{bmatrix}.$$

In an explicit form the dynamics can be written as

$$\begin{cases}
\dot{x}_1 &= -\frac{1}{c_2}x_2x_3+\frac{1}{c_8}x_7x_8\\
\dot{x}_2 &= \frac{1}{c_1}x_1x_3+\frac{1}{c_9}x_9x_{14}\\
\dot{x}_3 &= -\frac{2}{c_2}x_2x_4+\frac{1}{c_8}x_2x_8-\frac{3}{c_9}x_1x_9\\
\dot{x}_4 &= -\frac{3}{c_2}x_2x_5-\frac{2}{c_9}x_3x_9\\
\dot{x}_5 &= -\frac{1}{c_1}x_1x_6-\frac{1}{c_9}x_4x_9\\
\dot{x}_6 &= -\frac{1}{c_8}x_5x_8\\
\dot{x}_7 &= \frac{2}{c_1}x_1^2-\frac{1}{c_2}x_2^2-\frac{2}{c_8}x_8^2+\frac{1}{c_9}x_9^2\\
\dot{x}_8 &= -\frac{1}{c_1}x_1x_7+\frac{1}{c_9}x_9x_{10}\\
\dot{x}_9 &= -\frac{1}{c_2}x_2x_{14}-\frac{1}{c_8}x_8x_{10}\\
\dot{x}_{10} &= -\frac{1}{c_1}x_1x_9+\frac{3}{c_2}x_2x_8+\frac{2}{c_9}x_9x_{11}\\
\dot{x}_{11} &= \frac{2}{c_2}x_2x_{10}-\frac{3}{c_9}x_9x_{12}\\
\dot{x}_{12} &= \frac{1}{c_2}x_2x_{11}+\frac{1}{c_8}x_8x_{13}\\
\dot{x}_{13} &= \frac{1}{c_1}x_1x_{12}\\
\dot{x}_{14} &= -\frac{3}{c_1}x_1^2+\frac{2}{c_2}x_2^2+\frac{3}{c_8}x_8^2-\frac{2}{c_9}x_9^2.
\end{cases} \tag{10}$$

6 Numerical Integration of Dynamics

It is very difficult to find the analytical solution of the dynamical system described by equation (10), as it involves simultaneous nonlinear ordinary differential equations. So, numerical technique have been used to solve the system of nonlinear ordinary differential equations. An unconventional integrator have been implemented and the subsequent results have been analyzed.

Poisson preservation

An integrator $\psi : \mathbb{R}^m \to \mathbb{R}^m$ is said to be Poisson preserving if it satisfies the following condition[17]:

$$\psi_z(z)P(z)\psi_z^t(z) = P(\psi(y)) \tag{11}$$

Let $z(t) = X^n = [x_1^n \ x_2^n \ \cdots\cdots \ x_m^n]$.

For one step, with h as the step length it can be stated that

$$z(t+h) = X^{n+1}$$

hence,

$$P' = \psi_z(z) = \frac{\partial(z(t+h))}{\partial(z(t))} = \frac{\partial X^{n+1}}{\partial X^n} \tag{12}$$

Thus Equation (11) reduces to

$$P' \cdot P(X^n) \cdot (P')^T = P(X^{n+1}). \tag{13}$$

where P is the Poisson tensor (matrix) and P' is the Fréchet derivative.

Hamiltonian or Energy preservation

Hamiltonian H is said to be preserved if H is constant along the solution of dynamics, $z(t)$, or

$$\frac{d}{dt}H(z(t)) = 0. \tag{14}$$

In the discrete form it can be written as

$$\begin{aligned} &\frac{H(z(t+h)) - H(z(t))}{h} = 0 \\ \Rightarrow \quad & H(z(t+h)) - H(z(t)) = 0 \\ \Rightarrow \quad & H(X^{n+1}) = H(X^n). \end{aligned} \tag{15}$$

6.1 Kahan's Integrator

An unconventional discretization was proposed by Kahan [18]. It inherits various integrability properties from Runge-Kutta methods, i.e., it preserves all affine symmetric integrals, foliations and affine reserving symmetries. In this section Kahan's integrator has been applied to the symplectic Poisson system and properties related to integrability as mentioned above has been studied.

Kahan's integrator can be written in the following form:

$$
\begin{cases}
x_1^{n+1}-x_1^n & = -\frac{h}{2c_2}(x_2^{n+1}x_3^n+x_3^{n+1}x_2^n)+ \\
& \frac{h}{2c_8}(x_7^{n+1}x_8^n+x_8^{n+1}x_7^n) \\
x_2^{n+1}-x_2^n & = \frac{h}{2c_1}(x_1^{n+1}x_3^n+x_3^{n+1}x_1^n)+ \\
& \frac{h}{2c_9}(x_9^{n+1}x_{14}^n+x_{14}^{n+1}x_9^n) \\
x_3^{n+1}-x_3^n & = -\frac{h}{2c_2}(x_2^{n+1}x_4^n+x_4^{n+1}x_2^n)+ \\
& \frac{h}{2c_9}(x_9^{n+1}x_{14}^n+x_{14}^{n+1}x_9^n) \\
x_4^{n+1}-x_4^n & = -\frac{3h}{2c_2}(x_2^{n+1}x_5^n+x_5^{n+1}x_2^n)- \\
& \frac{h}{c_9}(x_3^{n+1}x_9^n+x_9^{n+1}x_3^n) \\
x_5^{n+1}-x_5^n & = -\frac{h}{2c_1}(x_1^{n+1}x_6^n+x_6^{n+1}x_1^n)- \\
& \frac{h}{2c_9}(x_4^{n+1}x_9^n+x_9^{n+1}x_4^n) \\
x_6^{n+1}-x_6^n & = -\frac{h}{2c_8}(x_5^{n+1}x_8^n+x_8^{n+1}x_5^n) \\
x_7^{n+1}-x_7^n & = \frac{h}{c_1}(x_1^{n+1}x_1^n)-\frac{h}{2c_2}(x_2^{n+1}x_2^n)- \\
& \frac{h}{c_8}(x_8^{n+1}x_8^n)+\frac{h}{2c_9}(x_9^{n+1}x_9^n) \\
x_8^{n+1}-x_8^n & = -\frac{h}{2c_1}(x_1^{n+1}x_7^n+x_7^{n+1}x_1^n)+ \\
& \frac{h}{2c_9}(x_9^{n+1}x_{10}^n+x_{10}^{n+1}x_9^n) \\
x_9^{n+1}-x_9^n & = -\frac{h}{2c_2}(x_2^{n+1}x_{14}^n+x_{14}^{n+1}x_2^n)- \\
& \frac{h}{2c_8}(x_8^{n+1}x_{10}^n+x_{10}^{n+1}x_8^n) \\
x_{10}^{n+1}-x_{10}^n & = -\frac{h}{2c_1}(x_1^{n+1}x_9^n+x_9^{n+1}x_1^n)+ \\
& \frac{3h}{2c_2}(x_2^{n+1}x_8^n+x_8^{n+1}x_2^n)+ \\
& \frac{h}{c_9}(x_9^{n+1}x_{11}^n+x_{11}^{n+1}x_9^n) \\
x_{11}^{n+1}-x_{11}^n & = \frac{h}{c_2}(x_2^{n+1}x_{10}^n+x_{10}^{n+1}x_2^n)- \\
& \frac{3h}{2c_9}(x_9^{n+1}x_{12}^n+x_{12}^{n+1}x_9^n) \\
x_{12}^{n+1}-x_{12}^n & = \frac{h}{2c_2}(x_2^{n+1}x_{11}^n+x_{11}^{n+1}x_2^n)+ \\
& \frac{h}{2c_8}(x_8^{n+1}x_{13}^n+x_{13}^{n+1}x_8^n) \\
x_{13}^{n+1}-x_{13}^n & = \frac{h}{2c_1}(x_1^{n+1}x_{12}^n+x_{12}^{n+1}x_1^n) \\
x_{14}^{n+1}-x_{14}^n & = -\frac{3h}{2c_1}(x_1^{n+1}x_1^n)+\frac{h}{c_2}(x_2^{n+1}x_2^n)+ \\
& \frac{3h}{2c_8}(x_8^{n+1}x_8^n)-\frac{h}{c_9}(x_9^{n+1}x_9^n)
\end{cases}
\tag{16}
$$

Proposition 2. *Kahan's integrator has the following properties:*

1. *It does not preserve the Poisson structure.*
2. *It does not preserve the Hamiltonian H_c of the system.*

Proof. In Equation (16), simultaneous equations are solved and elements of X^{n+1} are written in terms of elements of X^n, then it has been explicitly computed and found that

$$P' \cdot P(X^n) \cdot (P')^T \neq P(X^{n+1}).$$

Hence it is not Poisson preserving. Also,

$$H_c(X^{n+1}) \neq H_c(X^n).$$

Hence it doesn't preserve the Hamiltonian. □

7 Conclusion

In this paper a generalized left-invariant drift-free control system has been established on the exceptional Lie group G_2. Controllability and optimal control has been studied for this system by minimizing the Langrangian. Finally numerical integration has been analyzed via unconventional Kahan integrator. This work emphasizes on optimal control of the left invariant control system on G_2, which may give a superior understanding to mathematicians and physicist to actualize our proposed plan in specific optimal control problems.

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