# A Numerical Study of Steady Water Absorption by Plant Roots in Heterogeneous Soils

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*Abstract:* - In this paper, problems involving steady infiltration in two-layered soils with water uptake by plant roots are studied. Two types of two-layered soil are considered. To solve the problems, the governing equation of the problems, which is a Richards equation, is transformed into a modified Helmholtz equation using a set of transformation. The modified Helmholtz equation is solved numerically by employing a Dual Reciprocity Method (DRM) with a predictor-corrector scheme. The solutions obtained are used to compute the amount of water absorbed by plant roots numerically.

*Key-Words:* - Infiltration, Helmholtz equation, dual reciprocity method, predictor-corrector, water absorption, heterogeneous soil.

## 1 Introduction

Studies of infiltration problems in homogeneous porous medium have been conducted by numerous researchers. Such researchers are Batu [3, 4], Waechter and Philips [16], Pullan [8], Basha [2], Clements and Lobo [5], and Solekhudin [11, 12]. In these studies the infiltration processes are in homogeneous porous mediums, which are homogeneous soils.

None of the studies mentioned above report investigations about water infiltration in heterogeneous soil. Hence, a possible direction to continue the studies is studying water infiltration in layered soil, as in real life situation, soils may contain few layers. In this paper, we consider two infiltration problems in two-layered soil with water absorption by plant roots. In the first of the two problems, the soil in the upper layer or the first layer is finer than that in the second layer. In the other problem, soil in the first layer is coarser than that in the second layer. The purpose of this research is to investigate effects of the layers to water content in the soil and amount of water absorbed by the plant roots.

## 2 Problem Formulation

We consider two-layered soil. The upper layer or the first layer has depth of 50 cm. On the surface of the first layer, a periodic trapezoidal channels are created. The surface area of the channels is 100 cm$^2$ per centimeter of channel's length. The width and the depth of the channels are $200/\pi$ cm and $150/2\pi$ cm, respectively. The distance between the centers of two consecutive channels is 200 cm. It is assumed that the channels are very long and there are big number of such channels. The channels are filled with water completely and kept filled all the time. At the middle of soil surface between any two adjacent channels, a row of crops are planted. It is assumed that the distribution of the plant roots does not alter in the direction along the channels. The depth ($Z_m$) and the width ($2X_m$) of the root zone are the same, which is 100 cm. Given this situation, we would like to determine distribution of water content and amount of water absorbed by plant roots, especially in the root zone.

For the purpose of solving the problems, a Cartesian coordinates *OXYZ*, with *O* is at the center of a channel, is considered. The *Z*-axis is positively downward. Using this Cartesian coordinate, an illustration of the problem described above is shown in Figure 1.

From the assumptions above and the symmetry of the problems, it is sufficient to consider the semi-infinite region defined by $0 \leq X \leq 100$ cm and $Z \geq 0$. This region is denoted by *R*, bounded by curve *C*. The fluxes across the surface of the channels and soil surface outside the channels are $v_0$ and 0, respectively. Since the problems are symmetrical about $X = 0$ and $X = 100$ cm, there are no fluxes across these two lines. A typical region $R$ bounded by $C$ is shown in Figure 2.

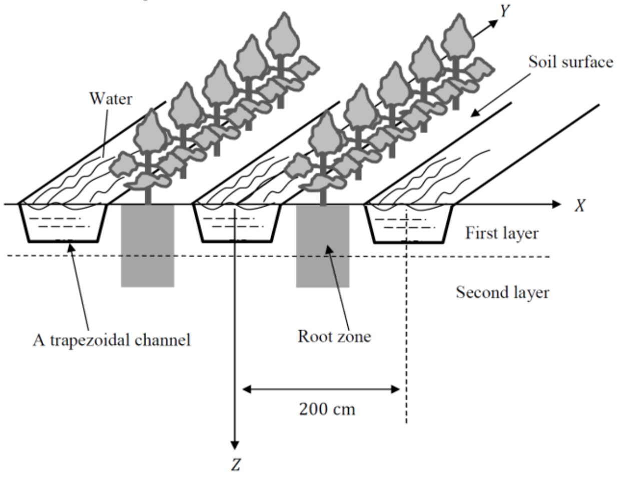


Figure 1 Geometry of periodic trapezoidal channels with rows of crops.

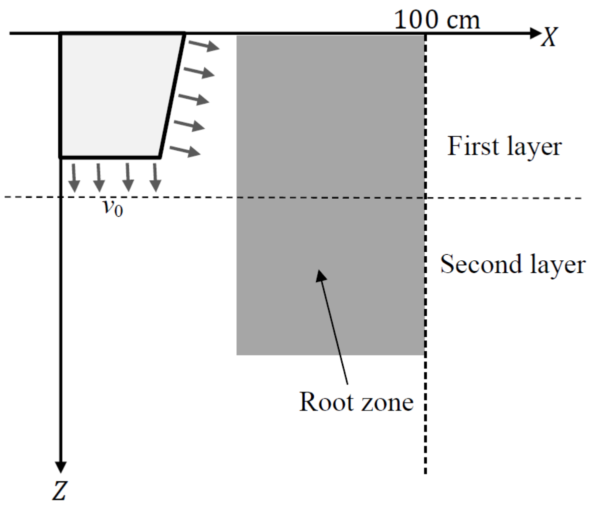


Figure 2 Geometry of semi-infinite region.

## 3 Basic Equations

In this section, basic equations related to problem involving infiltration in heterogeneous soils are presented. Before we proceed to the infiltration in heterogeneous problems, we recall the governing equation of the homogeneous problems. The infiltration problems may be modelled mathematically as

$$\frac{\partial}{\partial X}\left(K\frac{\partial\psi}{\partial X}\right)+\frac{\partial}{\partial Z}\left(K\frac{\partial\psi}{\partial Z}\right)-\frac{\partial K}{\partial X}=S(X,Z,\psi), \quad (1)$$

where $K$ is the hydraulic conductivity, $\psi$ is the suction potential, and $S$ is the root-water uptake function [10]. Function $S$ is adopted from Vrugt et. al. study [14], defined as

$$S(X,Z,\psi)=\gamma(\psi)\frac{L_t\beta(X,Z)T_{pot}}{\int_0^{Z_m}\int_{100-X_m}^{100}\beta(X,Z)dXdZ} \quad (2)$$

where $L_t$ is the width of the soil surface associated with transpiration process, $\beta$ is the spatial root-water uptake distribution, formulated as

$$\beta(X,Z)=\left(1-\frac{100-X}{X_m}\right)\left(1-\frac{Z}{Z_m}\right)e^{-T},$$
$$100-X_m\le X\le 100, \text{and } 0\le Z\le Z_m, \quad (3)$$

where

$$T=\frac{P_z}{Z_m}|Z^*-Z|+\frac{P_x}{X_m}|X^*-X|. \quad (4)$$

Here $P_Z$, $P_X$, $Z^*$, and $X^*$ are the empirical parameters. $T_{pot}$ is the transpiration potential, and $\gamma$ is the root-water stress response function as reported by Utset et. al. [13], which is illustrated in Figure 3.

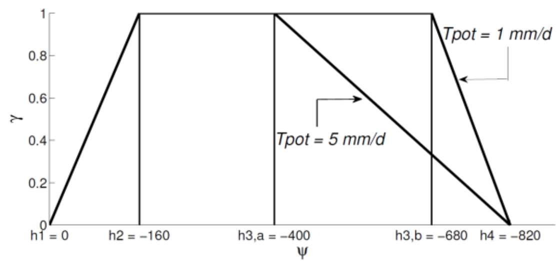


Figure 3 Root-water stress response function.

For the case of homogeneous soil, using the Kirchhoff transformation and an exponential relationship between $K$ and $\psi$ [5], Equation (1) can be written as

$$\frac{\partial^2\Theta}{\partial X^2}+\frac{\partial^2\Theta}{\partial Z^2}-\alpha\frac{\partial\Theta}{\partial Z}=\gamma(\psi)s(X,Z), \quad (5)$$

and the suction potential is formulated as

$$\psi=\frac{1}{\alpha}\ln\left(\frac{\alpha\Theta}{K_s}\right), \quad (6)$$

where $\alpha$ is a soil parameter related to the coarseness of the soil and $K_s$ is the saturated hydraulic conductivity.

In this study, since the soils are not homogeneous, that is $\alpha_1 \neq \alpha_2$, Equation (5) may not be applied to solve the problems described in the preceding section. Hence, to overcome this issue, we apply the method presented by Watson and Whisler [15], Malysa [7], and Purnama [9]. The value of $\alpha$ in Equation (3) is substituted by $\alpha^*$ defined by

$$\alpha^* = \frac{\alpha_1 + \alpha_2}{2}. \quad (7)$$

Now, we have the following equation.

$$\frac{\partial^2\Theta}{\partial X^2} + \frac{\partial^2\Theta}{\partial Z^2} - \alpha^* \frac{\partial\Theta}{\partial Z} = \gamma(\psi)s(X,Z). \quad (8)$$

Substituting dimensionless variables

$$x = \frac{\alpha^*}{2}X;\ z = \frac{\alpha^*}{2}Z; \Phi = \frac{\pi\Theta}{50v_0}; \quad (9)$$

into Equation (8) yield

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial z^2} - 2\frac{\partial\Phi}{\partial z} = \gamma^*(\Phi)s^*(x,z), \quad (10)$$

where

$$\gamma^*(\Phi) = \gamma\left[\frac{1}{\alpha^*}\ln\left(\frac{50\alpha^* v_0 \Phi}{\pi K_s}\right)\right], \quad (11)$$

$$s^*(x,z) = \frac{2\pi T_{pot}}{50\alpha v_0} \times \frac{l_t \beta^*(x,z)}{\int_0^{z_m}\int_{50\alpha^*-x_m}^{50\alpha^*} \beta^*(x,z)\,dx\,dz}. \quad (12)$$

Here $l_t = \frac{\alpha^* L_t}{2}$, $x_m = \frac{\alpha^*}{2}X_m$, $z_m = \frac{\alpha^*}{2}Z_m$,

$$\beta^*(x,z) = \left(1 - \frac{50\alpha^* - x}{x_m}\right)\left(1 - \frac{z}{z_m}\right)e^{-t}, \quad 50\alpha^* - x_m \le x \le 50\alpha^*, \text{and } 0 \le z \le z_m, \quad (13)$$

where

$$t = \frac{p_z}{z_m}\left|\frac{2z^*}{\alpha^*} - \frac{2z}{\alpha^*}\right| + \frac{p_x}{x_m}\left|\frac{2x^*}{\alpha^*} - \frac{2x}{\alpha^*}\right|, \quad (14)$$

$z^* = \frac{\alpha^*}{2}Z^*$, $x^* = \frac{\alpha^*}{2}X^*$, Applying transformation

$$\Phi = \phi e^z, \quad (15)$$

into Equation (10) yields

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} - \phi = \gamma^*(\phi)s^*(x,z)e^{-z}. \quad (16)$$

In this study, soils considered consist of Pima Clay Loam (PCL) and Touchet Silt Loam (TSL). The values of $\alpha$ for PCL and TSL are 0.014 cm$^{-1}$ and 0.0156 cm$^{-1}$, respectively [1]. Using these values of $\alpha$, the value of $\alpha^*$ is 0.0148 cm$^{-1}$. The values of $K_s$ for these two types of soil are 9.9 cm/day and 41.99 cm/day [1]. Boundary conditions described in Section 2 can be written in terms of $\phi$ as follows.

$$\frac{\partial\phi}{\partial n} = \frac{\pi}{0.37}e^{-0.1767} + \phi, \quad \text{for } 0 \le x \le 0.1867 \text{ and } z = 0.1767, \quad (17)$$

$$\frac{\partial\phi}{\partial n} = \frac{\pi}{0.37}e^{-0.1767} + 0.2665\phi, \quad \text{for } 0.1867 \le x \le 0.2355 \text{ and } z = -3.6209(x - 0.2355), \quad (18)$$

$$\frac{\partial\phi}{\partial n} = -\phi, \text{for } 0.2355 \le x \le 0.74 \text{ and } z = 0, \quad (19)$$

$$\frac{\partial\phi}{\partial n} = 0, \text{for } x = 0 \text{ and } z \ge 0, \quad (20)$$

$$\frac{\partial\phi}{\partial n} = 0, \text{for } x = 0.74 \text{ and } z \ge 0, \quad (21)$$

$$\frac{\partial\phi}{\partial n} = -\phi, \text{for } 0 \le x \le 0.74 \text{ and } z = \infty, \quad (22)$$

Equation (16) subject to Boundary conditions (17) – (22) is then solved numerically using a DRBEM and a predictor-corrector scheme. Detail of the method is presented in [10].

## 4 Results and Discussion

In this section, the method described in the preceding section is applied to problems involving steady infiltration in two layered soil with root-water uptake. Parameters of root-water uptake function appear in *S*, as reported by Solekhudin and Ang (2015), are $X_m$ = 50 cm, $Z_m$ = 100 cm, $X^*$ = 25 cm, $Z^*$ = 20 cm, $P_Z$ = 5.00, $P_X$ = 2.00, $T_{pot}$ = 0.4 cm/day and $L_t$ = 50 cm. Value of $T_{pot}$ used in this study is as that reported by Li, Jong, and Boisvert (2001). Two problems or cases involving infiltration in two-layered soils are to be solved. In the first case, the upper and second layers are PCL and TSL, respectively. The layered soil of this problem is denoted by PCL-TSL. The other problem is to solve infiltration problem in TSL-PCL.

To implement the method, curve *C* must be a close and bounded curve. Hence, an imposed boundary is needed, which is obtained after some computational experiment. The imposed boundary is $z$ = 4. The curve or boundary *C* is then discretized into a number elements, and a number of collocation points are chosen in region *R*. The boundary is descretized into 393 elements, and 400 interior collocation points are chosen. Employing the DRM with a predictor corrector, values of $\phi$ may be obtained.

In the case of homogeneous soil, from Equation (6), we may compute values of suction potential, $\psi$, in terms of $\phi$ using the formula

$$\psi = \frac{1}{\alpha}\ln\left(\frac{50v_0\alpha\phi e^z}{\pi K_s}\right). \qquad (23)$$

Flux at the surface of the channels, $v_0$, is assumed to be 0.75 times the value of saturated hydraulic conductivity [2].

In the case of two-layered soil, computing values of $\psi$ is distinguished into two states. For PCL-TSL case, the value of hydraulic conductivity in PCL is smaller than that in TSL. This imply that infiltration rate in the upper layer is smaller than that in the second layer. Hence, the value of $v_0$ of the first layer, which is PCL, may be used to compute values of in the second layer, which is TSL. For the case of TSL-PCL, the value of infiltration rate at the surface of channels, $v_0$, exceeds the maximum infiltration rate in the second layer. Hence, to compute $\psi$ in PCL, the value of $v_0$ is replaced by 0.75 times the value saturated hydraulic conductivity of PCL. Thus, $\psi$ is computed using the formula

$$\psi = \frac{1}{\alpha_i}\ln\left(\frac{50v_0\alpha_i\phi e^z}{\pi K_i}\right), \qquad (24)$$

for PCL-TSL case, and

$$\psi = \frac{1}{\alpha_i}\ln\left(\frac{50\times 0.75\alpha_i\phi e^z}{\pi}\right), \qquad (25)$$

for TSL-PCL case. Here $\alpha_i$ and $K_i$ are values of $\alpha$ and $K_s$ in Layer-$i$, $i$ = 1, 2. Employing Equations (24) and (25), numerical values of $\psi$ can be computed. Using these values of $\psi$, corresponding values of root-water uptake function, $S$, can be obtained using Equation (2). Some of the results in present study are presented in Figure 4 - Figure 6 and Table 1.

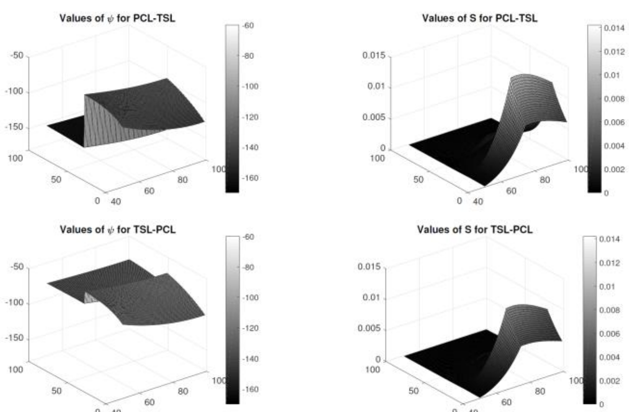


Figure 4 Values of $\psi$ and $S$ over root zone.

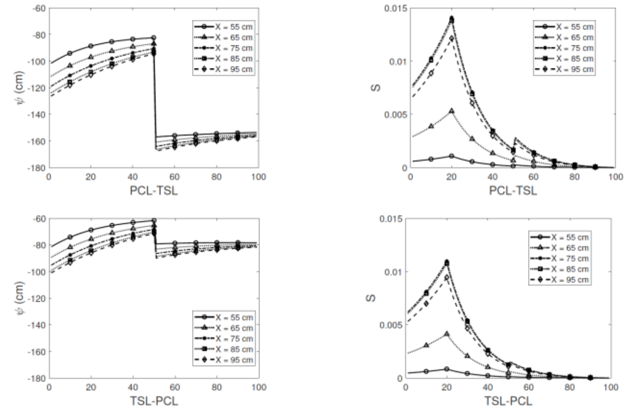


Figure 5 Graphs of $\psi$ and $S$ at selected values of $X$ along $Z$-axis.

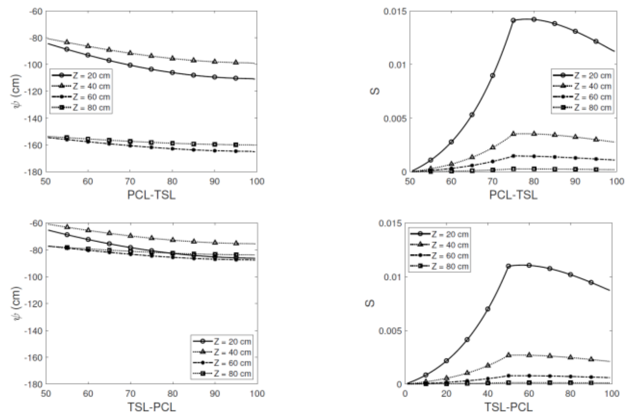


Figure 6 Graphs of $\psi$ and $S$ at selected values of $Z$ along $X$-axis.

Figure 4 shows graph of suction potential, $\psi$, and corresponding graph of root-water uptake function, $S$, over root zone, for PCL-TSL and TSL-PCL. It can be seen that suction potentials in the root zone for the case of PCL-TSL are about -100 cm for $0 \leq Z \leq 50$ cm. These values are drop significantly to about -150 cm for $Z \geq 50$ cm. In contrast, for the case of TSL-PCL, suction potentials drop slightly from about -60 cm for $0 \leq Z \leq 50$ cm to about -75 cm for $Z \geq 50$ cm. At $Z$ = 0, value of $\psi$ at $X$ = 100 cm is the lowest among other location. This result indicates that soil at $X$ = 100 cm is the driest among others, which is physically meaningful as soil at $X$ = 100 cm is the furthest from the channel and there are no fluxes across the surface of the soil outside the channels. From the graphs of values of $\psi$, since $\psi \geq$ -160 cm, from the graph in Figure 3, values of $S$ for the case of PCL-TSL are higher than those for TSL-PCL. It can also be seen that locations near (75 cm, 20 cm) have higher values of $S$ compared to other locations. These results are expected as the values of $X^* = 25$ cm and $Z^* = 20$ cm.

Figure 5 shows graphs of $\psi$ and corresponding graphs of $S$ at selected values of $X$ along $Z$ axis. The values of $X$ are specifically 55 cm, 65 cm, 75, cm, 85, cm, and 95 cm. For PCL-TSL, values of $\psi$ at Z = 50 cm jump significantly about 70 cm from between -100 cm and -85 cm to between -170 cm and -160 cm. The infiltration rate at the surface of channel for this case is 7.425 cm/day, which is much lower for TSL, since the maximum infiltration rate for TSL is 31.4925 cm/day. These results imply much low values of $\psi$ in TSL compared to those in PCL. For the case of TSL-PCL, there are jump on values of $\psi$, from between -75 cm and -60 cm to between -95 cm and -85 cm. These are due to the fact that there are decrease in the infiltration rate at $Z = 50$ cm. The upper layer soil, TSL, has much higher infiltration rate than that in the second layer, PCL.

Values of $\psi$ in TSL-PCL are more than -160 cm. Since values of $\psi$ in PCL-TSL are smaller than those in TSL-PCL, values of root-water uptake function for the case of PCL-TSL are higher than those for TSL-PCL case. This means that the amount of water absorbed from PCL-TSL is more than those absorbed from TSL-PCL. At $Z = 50$ cm, it can be observed that there are steep increases in $S$ at some values of $X$, especially at $X = 75$ cm and $X = 85$ cm, for PCL-TSL. This is due to the jumps on the values of at $Z = 50$ cm. The jumps result in the increase in values of $S$. The values of $S$ increase almost twice, at $Z = 50$ cm.

The highest value of $S$ occur at $X = 75$ cm, especially at point (75 cm, 20 cm), as $X^* = 25$ cm and $Z^* = 20$ cm. Values of $S$ at $X = 85$ cm and $X = 95$ cm are slightly smaller than those at $X = 75$ cm. In contrast, with the same distance from $X = 75$ cm, values of $S$ at $X = 55$ cm and $X = 65$ cm are relatively small compared to those at $X = 75$ cm. These indicates that roots at positions near the plants absorb more water than those at further locations.

Figure 6 shows graphs of $\psi$ and corresponding graphs of $S$ at selected values of $Z$ along $X$ axis, specifically at $Z = 20$ cm, $Z = 40$ cm, $Z = 60$ cm and $Z = 80$ cm. For PCL-TSL, it can be seen that at $Z = 20$ cm and $Z = 40$ cm, values of $\psi$ are between -100 cm and -80 cm. For $Z = 60$ cm and $Z = 80$ cm, values of are between -160 cm and -155 cm. These imply that values of $\psi$ in soil in the upper layer are much higher than those in the second layer. For the case of TSL-PCL, it can be observed that values of $\psi$ are between -80 cm and -60 cm. These imply that there are no significant changes in $\psi$ for this case.

The corresponding values of $S$ for PCL-TSL case are greater than those for TSL-PCL. As discussed before, these are the implications of corresponding values of $\psi$. In TSL-PCL the values of are more than -160 cm, and values of in PCL-TSL case are less than those in TSL-PCL. As been discussed before, these imply that at any value of $Z$, graph of S for PCL-TSL case is above that for TSL-PCL case. It can also be observed that values of $S$ increase significantly from $X = 50$ cm to $X = 75$ cm and then decrease slowly from $X = 75$ cm to $X = 100$ cm. This imply that the highest absorption occurs at $X = 75$ cm. The amount of water absorbed at points further from the plants are smaller than that from points near the plants. Amounts of water absorbed from the root zone with volume of 1 $m^3$ are shown in Table 1.

Table 1 Volume of water absorbed by plant roots of 1 $m^3$.

| **Soil type** | **Total water absorbed ($cm^3$/day)** |
|---|---|
| Touchet Silt Loam - Pima Clay Loam | 2063.7959 |
| Pima Clay Loam - Touchet Silt Loam | 2721.1789 |

The values in Table 1 are computed using formula

$$2 \times 100 \int_{50}^{100} \int_{0}^{100} S(X, Z, \psi) dX dZ, \qquad (30)$$

numerically. The results show that crops planted on the surface of PCL-TSL soil type absorb more water than those planted on the surface of TSL-PCL soil type. The roots of plants in PCL-TSL absorb about 700 $cm^3$ more water per day.

## 5 Concluding Remarks

Two problems involving steady infiltration in two-layered soil with water absorption by plant roots are studied and solved numerically. The governing equation of the problems is transformed into a modified Helmholtz equation using a set of transformation. The modified Helmholtz equation is then solved numerically employing a DRM with a predictor-corrector scheme simultaneously. The numerical solutions obtained are then used to compute values of suction potential and root-water uptake function.

Results indicate that there are leaps in values of suction potential and root-water uptake function at $Z$ = 50 cm. The leaps in two-layered soil with finer soil in the upper layer are more significant than that with coarser soil in the upper layer.

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