Direct-Approximate Methods for Solution of Singular Integral Equations with Complex Conjugation Defined on the System of Fejer Points on Contour Γ in Generalized Holder Spaces

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Abstract: The main propose of this research is the investigation and theoretical background of the direct-approximate methods for the numerical solution of singular integral equations with translation and conjugation of the unknown function defined on the smooth contours of the Lyapunov type. The equations are defined on the system of Fejér points. The numerical schemes of collocation and mechanical quadrature methods for the equations with conjugations of the unknown function and for the equations with translation are elaborated. This problem has been well studied for the case of functions defined on standard contours (a straight line segment, the unit circle, and so on). In the case of an arbitrary closed smooth contour in the complex plane, the problem is not studied enough. We suggest the numerical schemes of the Lagrange interpolation polynomials for the approximate solution of weakly SIE defined on smooth closed contours in the complex plane. Our approach is based on the Zolotarevski theory. The theorems of convergence for research methods are proved in Generalized Hölder spaces.

Key–Words: Singular Integral Equations, Fejér points, Generalized Hölder spaces, Collocation Methods, Mechanical Quadrature Methods

1 Introduction

This article is dedicated to the problem of numerical solution for Singular Integral Equations (SIE) and the theoretical background of elaborated methods in Generalized Holder spaces. The problem for finding the exact solution for SIE appears in many fields of science and engineering, for example (the SIE model the lifting force for the aircraft wing, in queuing analysis, they describe the fulfilment of priority telephone commands, in mechanics SIE model the processes from elasticity theory [1]-[3]).

In the monographs of mathematicians N. Mushelesvili [4] and I. Vekua [5] have been proved the exact solution for SIE can be found in some particular cases. Even in these particular cases, the formula for the exact solution of SIE can be very complex. It can be represented by nonlinear formulas with multiple singular integrals.

That is why it is important for both theory and practices to elaborate on the algorithms which would permit to find the approximative solutions for SIE and with the evaluation of approximative errors.

The problem for the approximative solution for SIE was studied in the scientific literature of mathematicians V. Ivanov, B. Gadfulhaev, I Gohberg, V. Zolotarevski and others [6]-[21].

At the same time, the collocation methods and mechanical quadrature methods have been applied to the approximative solving of SIE, when equations are defined on the unit circle or real axis [28], [2], [9], [8], [20]. The case when the SIE are defined on the arbitrary smooth closed contours have not been studied enough.

We should mention only the articles [22]-[24].

The case of SIE non-elliptic, such as SIE with translation and conjugation of the unknown function defined on the smooth contours has been studied in this article. The SIE is defined in Generalized Holder spaces.

The main purpose of this article consists of the approximate solution of SIE with translation and conjugation for the unknown function.

We should solve the following problems:

1. Demonstration of the compatibility of the elaborated algorithms and fixing the values for the approximation numbers. Beginning with numbers starting these algorithms are compatible.
2. Establishing the convergence of approximate solutions to the exact solutions.
3. Getting the convergence error in Generalized Holder spaces.
4. Researching methods to optimality and stability.

2 Definitions of Function Spaces and Notations

Let Γ be an arbitrary smooth closed contour bounding a simply-connected region D^+ of the complex plane, let $t = 0 \in D^+$, $D^- = C \setminus \{D^+ \cup \Gamma\}$, where C is the complex plane. We consider the point $z = \infty \in D^-$.

We denote by Λ the class of the smooth closed countours Γ. For these contours the Riemann function $z = \psi(w)$ exists. This function is satisfied the condition: for two positive constants $m_1(\Gamma)$ and $M_2(\Gamma)$ the following inequality holds:

$$0 < m_1(\Gamma) \leq |\psi^{'}(w)| \leq M_1(\Gamma) \leq \infty, |w| = 1.$$

In this paragraph, the generalized Holder spaces H_ω determined by the continuity modulus ω are described. These spaces were introduced in [25]. Also in this paragraph some results are presented concerning the properties of the singular integral equations, studied in H_ω. These results are applied for obtaining the theoretical substantiation of collocation and mechanical quadrature methods.

By d we denote the size $d = diam D^+ = \max \left| t^{'} - t^{''} \right|$, $t^{'}, t^{''} \in \Gamma$. Further more, we assume that $\omega(\delta)$ is certain continuity module, and

$$\omega(f, \delta) = \sup_{|t^{'} - t^{''}| \leq \delta} |f(t^{'}) - f(t^{''})|, \quad ,\delta \in [0; d]$$

is the continuity modulus of the function $f(t)$, $t \in \Gamma$.

By $H_\omega = H_\omega(\Gamma)$ we denote the function space $g(t)$ continuous on Γ $(g(t) \in C(\Gamma))$ that satisfy the condition:

$$H(g; f) \equiv \sup \frac{\omega(g, \delta)}{\omega(\delta)} < \infty, \qquad ,\delta \in (0; d]$$

The norm in H_ω is defined by the equality:

$$||g||_\omega = ||g||_C + H(g; \omega), \tag{1}$$

$$||g||_C = \max_{t \in \Gamma} |g(t)|$$

Thus H_ω is nonseparable Banach space [26].

We suppose that continuity modulus ω satisfies the Bari-Stecichin condition [25]:

$$\int_0^h \frac{\omega(\xi)}{\xi} d\xi < \infty, \tag{2}$$

$$\int_0^\delta \frac{\omega(\xi)}{\xi} d\xi + \int_\delta^h \frac{\omega(\xi)}{\xi^2} d\xi = O(\omega) = \tag{3}$$

$$O(\omega(\delta)) \to 0, \delta \to +0,$$

On this case, according to [27] the singular integral operator

$$(Sg)(t) = \frac{1}{\pi i} \int_\Gamma \frac{g(\tau)}{\tau - t}, \qquad t \in \Gamma,$$

is bounded in H_ω.

Theorem 1 *Assume that the continuity modulus* $\omega_1(\delta)$ *and* $\omega_2(\delta)$ *satisfy the conditions (2)-(3), simultaneously. Then, for every function* $g(t) \in H_{\omega_2}$ *the operator* $Sg - gS$ *is bounded as an operator acting from* H_{ω_1} *to* H_{ω_2}*, and*

$$||Sg - gS||_{\omega_1 \to \omega_2} \leq constant||g||_{\omega_2}.$$

Theorem 2 *Let the continuity modulus* ω_1 *and* ω_2 *satisfy both conditions (2)-(3). If* $g(t) \in H_{\omega_2}$*, then the operator* $Sg - gS$ *mapping the* H_{ω_2} *in* H_{ω_1} *is complitely continious.*

The demonstration of these theorems is realized analogously to the similar outcome get in [25] about boudness of the operator $T = Sg - gS$ in classicla Hölder spaces $H_\beta(\Gamma)$ with conditions $g(t) \in H_\alpha(\Gamma)$.

We shoud mention that, in case, when $\omega_1(\delta) = (\delta^\beta, 0 < \beta \leq \alpha \leq 1$ the spaces H_{ω_2} and H_{ω_1} coincide with classical Hödler spaces H_β, and respectively H_α, but theorems 1 and 2 for this case obtained in [25].

Let U_n be the Lagrange interpolating polynomial

$$(U_n g)(t) = \sum_{s=0}^{2n} g(t_s) \cdot l_s(t), \tag{4}$$

$$l_j(t) = \prod_{k=0, k\neq j}^{2n} \frac{t - t_k}{t_j - t_k} \left(\frac{t_j}{t}\right)^n \equiv$$

$$\sum_{k=-n}^{n} \Lambda_k^{(j)} t^k, \; t \in \Gamma, \quad j = 0, \ldots, 2n.$$

The constant

$$\lambda_n(\Gamma) = \lambda_n = \max_{t\in\Gamma} \sum_{j=0}^{2n} |l_j(t)|$$

is the Lebesgue interpolation constant for the contour Γ for the points t_j, $j = \overline{0, 2n}$.

Theorem 3 *Let* $\Gamma \in \Lambda$ *and the points* $t_j = t_j^{(\theta)}$ $(j = \overline{0, 2n})$ *forms the system of the Fejér points on* Γ :

$$t_j^{(\theta)} = \psi(\omega_j^{(\theta)},$$

$$\omega_j^{(\theta)} = exp\left\{\left[\frac{2\pi}{2n+1}(j-n) + \theta\right] i\right\},$$

$$0 \leq \theta \leq 2\pi, j = \overline{0, 2n}, i^2 = -1. \tag{5}$$

Then, the positive constants exist $m_2(\Gamma)$, $M_2(\Gamma)$, *and* $M_2'(\Gamma)$, *so that, for* $\forall$ *natural* n, *the following relation holds for Lebegue interpolation contants* λ_n*, defined in (5).*

$$0 < m_2(\Gamma)\ln(2n+1) \leq \lambda_n \leq$$

$$M_2(\Gamma) + M_2'(\Gamma)\ln(2n+1).$$

Theorem 4 *Let* $\omega_1(\delta)$, $\omega_2(\delta)$, $\delta \in (0; l]$, *satisfy the conditions (2)-(3), so that the function* $\Phi(\sigma) = \omega_1(\delta)/\omega_2(\delta)$ *nondecreasing on* $(0; l]$. *Then, for every function* $g(t)$ *from* $H(\omega_1$ *the following inequality takes place:*

$$||g - U_n g||_{\omega_2} \leq (d_1 + d_2|U_n||_C)\Phi(1/n)H(g; \omega_2)$$

Remark 5 *If the points* $\{t_j\}_{j=0}^{2n}$ *are form the system of Fejér points on* Γ *(5) and* $\lim_{n\to\infty}\Phi(1/n)\ln n = 0$, *then*

$$\lim_{n\to\infty} ||g - U_n g||_{\omega_2} = 0, \forall g(t) \in H(\omega_1).$$

Remark 6 *Let* $\omega(\delta)$ *is an arbitrary module of continuity, which satisfies the conditions (2)-(3), then* $||U_n|| \leq (d_{11}\ln(2n+1))$, $U_n : H(\omega) \to H(\omega)$

3 Problem Formulation. Singular Integral Equations with Complex Conjugation

In $H(\omega)$ we study the SIE which contains the uknown complex conjugation function:

$$(R\phi \equiv)c_1(t)\phi(t) + \frac{d_1(t)}{\pi i}\int_\Gamma \frac{\phi(\tau)}{\tau - t}d\tau + c_2(t)\overline{\phi(t)}+$$

$$\frac{d_2(t)}{\pi i}\int_\Gamma \frac{\overline{\phi(\tau)}}{\tau - t}d\tau + \frac{1}{2\pi i}\int\limits_\Gamma h_1(t,\tau)\phi(\tau)d\tau+$$

$$\frac{1}{2\pi i}\int\limits_\Gamma h_2(t,\tau)\overline{\phi(\tau)}d\tau = f(t), t \in \Gamma, \qquad (6)$$

where c_k, d_k and $h_k(t,\tau)$ by both variables), $k = 1, 2\ldots$, are known functions in $H(\omega)$and $\phi(t)$ is unknown function.

The general theory for (6) was studied very detailedly in the scientific literature. But, the problem for the approximate solution for this equation is not studied enough. We will elaborate on the numerical schemes: collocation methods and mechanical quadrature methods for the approximate solution of (6). The SIE is defined on an arbitrary smooth closed contour of the complex plane. The theoretical background was obtained in Generalized Holder spaces.

We introduce the new unknown functions, $\phi_1(t) = \phi(t)$ and $\phi_2(t) = \overline{\phi(t)}$, we will introduce to the equation (6), which was obtained from (6) using the complex variable. Taking into account the relations, we obtain:

$$\overline{\frac{1}{\pi i}\int_\Gamma \frac{\phi(\tau)}{\tau - t}d\tau} = \frac{1}{\pi i}\int_\Gamma \frac{\overline{\phi(\tau)}}{\tau - t}d\tau +$$

$$\frac{2}{\pi}\int_\Gamma \frac{\partial\theta}{\partial\sigma}\overline{\phi(\tau)}$$

$\theta = arg(\tau - t), \tau = \tau(\sigma)$, ($\sigma \in (0, l]; l$ is the the length for contour Γ), is the equation for contour Γ, σ is the abscisse for arcs and,

$$\overline{\frac{1}{\pi i}\int_\Gamma h(t,\tau)\phi(\tau)d\tau} = -\frac{1}{\pi i}\int_\Gamma \overline{h(t,\tau)\phi(\tau)}\overline{(\tau')^2(\sigma)}d\tau,$$

We obtain the following system of SIE which not contain conjugates:

$$c_1(t)\phi_1(t) + \frac{d_1(t)}{\pi i}\int_\Gamma \frac{\phi_1(\tau)}{\tau - t}d\tau + c_2(t)\phi_2(t)+$$

$$\frac{d_2(t)}{\pi i}\int_\Gamma \frac{\phi_2(\tau)}{\tau - t}d\tau + \frac{1}{2\pi i}\int\limits_\Gamma h_1(t,\tau)\phi_1(\tau)d\tau+$$

$$\frac{1}{2\pi i}\int\limits_\Gamma h_2(t,\tau)\phi_2(\tau)d\tau = f(t), t \in \Gamma, \qquad (7)$$

$$\overline{c_2(t)}\phi_1(t) + \frac{\overline{d_2(t)}}{\pi i}\int_\Gamma \frac{\phi_1(\tau)}{\tau - t}d\tau + \overline{c_1(t)}\phi_2(t)-$$

$$\frac{\overline{d_1(t)}}{\pi i}\int_\Gamma \frac{\phi_2(\tau)}{\tau - t}d\tau - \frac{1}{2\pi i}\int\limits_\Gamma \overline{h_2(t,\tau)}\overline{\tau'^2(\sigma)}\phi_1(\tau)d\tau+$$

$$\frac{2d_2(t)}{\pi}\int_\Gamma \frac{\partial\theta}{\partial\sigma}\phi_1(\tau)d\tau-$$

$$\frac{1}{2\pi i}\int\limits_\Gamma \overline{h_2(t,\tau)}\overline{\tau'^2(\sigma)}\phi_2(\tau)d\tau+ \qquad (8)$$

$$\frac{2\overline{d_1(t)}}{\pi}\int_\Gamma \frac{\partial\theta}{\partial\sigma}\phi_2(\tau)d\tau = \overline{f(t)}, t \in \Gamma,$$

for the vector $\{\varphi_1(t), \varphi_2(t)\}$.

The function $\eta(\sigma, s) = \frac{\partial\theta(\sigma,s)}{\partial\sigma}$ can be represented in the form

$$\eta(\sigma, s) = \frac{k(\sigma, s)}{|\sigma - s|^\lambda}; \qquad (9)$$

$$0 \le \sigma, s \le l, \lambda \in (1 - \mu, 1),$$

and the function satisfies the Holder conditions by both variables.

We suppose that the kernels $h_1(t,\tau)$ and $h_2(t,\tau)$ satisfy on the $\Gamma \times \Gamma$ the Holder conditions. Then, the integral operator with kernels $h_k(t,\tau)$ and $\overline{h_k(t,\tau)}\overline{\tau'^2(\sigma)}$ $(k = 1, 2)$, transform from $H_{\omega_1} \to H_{\omega_2}$.

On this case the right part belongs to the Holder spaces. The SIE (6) and (7) are equilivalent in sense, if the equation (6) has an unique solution $\varphi(t)$, the system (7) has an unique solution $\left\{\varphi(t), \overline{\varphi(t)}\right\}$. Inverse, if the system (7) has an unique solution $\varphi_1(t), \varphi_2(t)$, then the SIE (6) has an unique solution $\varphi(t)$. This solution has the form:

$$\varphi(t) = \frac{1}{2}\varphi_1(t) + \frac{1}{2}\overline{\varphi_2(t)} \tag{10}$$

Taking into account the results obtained above we will construct the numerical schemes for collocation and mechanical quadrature methods for SIE (7). After that applying the relation (6) and the system (7), we will obtain the fomulae for obtaining the approximative solutions for SIE (6). We will transform the system of S.I.E. (7) in the forms of vectorial equation with matrix coefficients.

$$(Mx)(t) \equiv C(t)x(t) + \frac{D(t)}{\pi i}\int_\Gamma \frac{x(\tau)}{\tau - t}d\tau + \tag{11}$$

$$\frac{1}{2\pi i}\int_\Gamma H(t,\tau)x(\tau)d\tau = F(t), t \in \Gamma,$$

where $C(t), D(t)$ and $H(t,\tau)$ are matrix functions of second order, and $F(t)$ and $x(t)$ are vector functions of second order.

$$C(t) = \begin{bmatrix} \underline{c_1(t)} & \underline{c_2(t)} \\ c_1(t) & c_2(t) \end{bmatrix} \tag{12}$$

$$D(t) = \begin{bmatrix} \underline{d_1(t)} & \underline{d_2(t)} \\ -d_2(t) & d_1(t) \end{bmatrix}$$

$$H(t,\tau) = \left[\frac{h_1(t,\tau)}{-\overline{h_2(t,\tau)\tau^{2\prime}} + 4\pi i \overline{d_2(t)}\frac{\partial\theta}{\partial\sigma}} \right]$$

$$\left[\frac{h_2(t,\tau)}{-\overline{h_1(t,\tau)\tau^{2\prime}} + 4\pi i \overline{d_2(t)}\frac{\partial\theta}{\partial\sigma}} \right]$$

$F(t) = [f(t); \overline{f(t)}], x(t) = [x_1(t); x_2(t)]$

The system of SIE (11) is the equation of Noether type, if

$$det[C(t) \pm D(t(] \neq 0, \qquad t \in \Gamma.$$

From the condition above we can verify that

$$\Delta(t) = [c_1(t) + d_1(t)][\overline{c_1(t)} + \overline{d_1(t)}] -$$

$$[c_2(t) + d_2(t)][\overline{c_2(t)} + \overline{d_2(t)}] \neq 0. \tag{13}$$

We suppose that the condition (13) takes place. We will study the elliptic case which contains the conjugation unknown function.

As, the kernel $H(t,\tau)$ for the system (11) contains the weak singularity of order λ $\lambda \in (1-\mu, 1)$, that is why the application of collocation methods is difficult and quadrature methods cannot be applied. We will introduce the new equation, the kernel will be in Hoder spaces.

We will study the new system of SIE. This equation is very similar to the equation (11), the kernel will not contain the singularities. We can apply the collocation and mechanical quadrature methods for this equation for this new equation.

Let ρ is an arbitrary positive number. We denote by $\eta_\rho(\sigma, s)$

$$\eta_\rho(\sigma,s) = \begin{cases} \eta(\sigma,s), \text{for} |\sigma - s| \geq \rho, \\ \rho^{-\lambda}k(\sigma,s), \text{for} |\sigma - s| < \rho \end{cases}$$

The functions $\eta(\sigma,s)$ and $k(\sigma,s)$ are defined in (9).

We denote by $H_\rho(t,\tau)$ the matrix function (m.f.) from $H(t,\tau)$ changing the function $\frac{\partial\theta}{\partial\sigma}$ by function $\eta_\rho(\sigma,s)$. We should mention that the function $H_\rho(t,\tau)$ does not contain the singularaties. It is Holder function by both variables.

We will study the system of SIE

$$(M_\rho x)(t) \equiv C(t)x(t) + \frac{D(t)}{\pi i}\int_\Gamma \frac{x(\tau)}{\tau - t}d\tau +$$

$$+\frac{1}{2\pi i}H_\rho(t,\tau)x(\tau)d\tau = F(t), \qquad t \in \Gamma, \tag{14}$$

The approximate solution for collocation methods we looking for in the form of polynomial bidimenional vector- function (v.f.)

$$x_n(t) = \sum_{k=-n}^{n} \alpha_k^{(n)} t^k = \tag{15}$$

$$\left\{ \sum_{k=-n}^{n} \alpha_{k,1}^{(n)} t^k ; \alpha_{k,2}^{(n)} t^k \right\}, t \in \Gamma,$$

with unknown coefficients $\alpha_k^{(n)} = \alpha_k = \{\alpha_{k,1}; \alpha_{k,2}\}$, $k = \overline{-n, n}$.

4 Numerical Schemes for Collocation Methods.

Let $\Theta_n(t) = M_\rho x_n(t) - F(t)$ be the residual of SIDE. The collocation method consists in setting it equal to zero at some distinct points t_j, $j = 0, \ldots, 2n$ on Γ. Thus we obtain a system of linear algebraic equations (SLAE) for the unknown complex numbers ξ_k, $(k = -n, \ldots, n)$ which can be determined by solving

$$\Theta_n(t_j) = (M_\rho x_n)(t_j) - F(t_j) = 0, \tag{16}$$

$$j = 0, \ldots, 2n.$$

From the conditions (16) we obtain the following SLAE for unkowns $\alpha_k = \{\alpha_{k,1}; \alpha_{k,2} \ k = \overline{-n, n}$:

$$\sum_{k=0}^{n} [C(t_j) + D(t_j)] t_j^k \alpha_k$$

$$\sum_{k=-n}^{-1} [C(t_j) - D(t_j)] t_j^k \alpha_k +$$

$$\sum_{k=-n}^{-1} \frac{1}{2\pi i} \int_\Gamma H_\rho(t;\tau) \tau^k d\tau \alpha_k \tag{17}$$

$$= F(t_j), j = 0, \ldots, 2n.$$

5 Numerical Schemes for Mechanical Quadrature Methods

We approximate the integrals in SLAE (17) by quadrature formula:

$$\frac{1}{2\pi i} \int_\Gamma \tau^k g(\tau) d\tau \equiv \frac{1}{2\pi i} \int_\Gamma U_n[g(\tau)] \tau^{k-1} =$$

$$\sum_{r=0}^{2n} g(t_r) t_r \Lambda_{-k}^{(r)}, k = \overline{-n, n},$$

where U_n is Lagrage interpolation polynomial (4). We obtaint the SLAE for quadrature methods:

$$\sum_{k=0}^{n} [C(t_j) + D(t_j)] t_j^k \alpha_k$$

$$\sum_{k=-n}^{-1} [C(t_j) - D(t_j)] t_j^k \alpha_k +$$

$$\sum_{k=-n}^{-1} \sum_{r=0}^{2n} H_\rho(t_j;\tau) \tau_r \Lambda_{-k}^{(r)} \alpha_k \tag{18}$$

$$= F(t_j), j = \overline{0, 2nl}$$

The numbers $\Lambda_{-k}^{(r)}$ $r = \overline{0, 2n}$ $k = \overline{-n, n}$ are defined in (4).

6 Convergence Theorems

We present the Theorems (7) and (8) which give the theoretical background for collocation and mechanical quadrature methods in Generalized Holder spaces. We suppose that the contour Γ is Lyapunov contour with smooth index $\mu(0; 1)$. It is very important for the smoothness of function $\eta(t, s)$.

Theorem 7 *Let the following conditions be satisfied:*

1) *the functions* $c_k(t)$, $d_k(t)$ *and* $h_k(t,\tau)$, $k=1,2$ *belongs to* H_{ω_1};

2) $\Delta(t)$ *satisfies the condition (13);*

3) *the left partial indices of M.F.* $[C(t)-D(t)]^{-1}[C(t)+D(t)]$ *are equal zero;*

4) $dimKerM=0$;

5) β *satisifies the condition* $\beta\in(0;\delta)$, $\delta=\min(\mu;alpha)$ *If the points* t_j *form the system of Fejer points (5) on* Γ. *Then, for the enough large values* $n\geq n_1$, *and values* ρ *small enough the SLAE (17) has an unique solution* $\alpha_k=\{\alpha_{k1},\alpha_{k2}\}$, $k=\overline{-n,n}$. *The approximate solutions* $\varphi_{n,\rho}(t)$ *calculated by formula*

$$\varphi_{n,\rho}=\frac{1}{2}\sum_{k=-n}^{n}(\alpha_{k1}t^k+\bar{\alpha}_{k2}\bar{t}^k)$$

converge when $n\to\infty$ *and* $\rho\to 0$ *in the norm of the space* $H(\omega_2)$ *for* $\forall f(t)\in H(\omega_1)$ *to the exact solution* $\varphi(t)$ *of SIE in sens that*

$$\lim_{n\to\infty}\lim_{\rho\to 0}||\varphi-\varphi_{n,\rho}||_{\omega_2}, \tag{19}$$

and the following estimation for the convergence is true:

$$||\varphi-\varphi_{n,\rho}||_{\omega_2}=(1/n)O(\Phi(1/n\ln n)+O(\rho^{\mu}).$$

The proof of this theorem is very similar to the proof from [6] for collocation methods.

Theorem 8 *Let the conditions (1-4) from the theorem 7 be satisfied. The condition 5) is changed by* $\beta\in(0;\nu)$, $\nu=\min(\gamma;\alpha),\gamma=\min(\mu;1-\mu)$. *If the points* t_j *form the system of Fejer points (5) on* Γ. *Then, for the enough large values* $n\geq n_2$, *and values* ρ *small enough the SLAE (18) has an unique solution* $\alpha_k^{\rho}=\{\alpha_{k1}^{\rho},\alpha_{k2}^{\rho}\}$, $k=\overline{-n,n}$. *The approximate solutions* $\varphi_{n,\rho}(t)$ *calculated by formula*

$$\varphi_{n,\rho}=\frac{1}{2}\sum_{k=-n}^{n}(\alpha_{k1}^{\rho}t^k+\bar{\alpha}_{k2}^{\rho}\bar{t}^k)$$

converge when $n\to\infty$ *and* $\rho\to 0$ *in the norm of the space* $H(\omega_2)$ *for* $\forall f(t)\in H(\omega_1)$ *to the exact solution* $\varphi(t)$ *of SIE in sens that*

$$\lim_{n\to\infty}\lim_{\rho\to 0}||\varphi-\varphi_{n,\rho}||_{\omega_2}, \tag{20}$$

and the following estimation for the convergence is true:

$$||\varphi-\varphi_{n,\rho}||_{\omega_2}=(1/n)O(\Phi(1/n\ln^2 n)+O(\rho^{\gamma}).$$

The proof of this theorem is very similar to the proof from [6] for mechanical quadrature methods.

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