Benchmark of Goal Oriented Sensitivity Analysis Methods using Ishigami Function

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Abstract: - The present paper deals with the Goal Oriented Sensitivity Analysis. The median-oriented sensitivity analysis (MSA) is presented which measures sensitivities by applying absolute distances between model outputs Y and median of Y. The median is used as the alternative central parameter to arithmetical mean applied by the established Sobol sensitivity analysis (SSA). General agreements and differences between MSA and SSA are studied by applying the Ishigami function. The paper shows that the sensitivity analysis need not necessarily be based on the analysis of variance known as ANOVA, but that there exist alternate approaches, too. CPU demanding character of MSA is approximately identical to that of SSA. The proposed MSA is efficient and practical for the problems in which it is necessary to quantify the importance of each input variable with respect to the median.

Key-Words: - Sensitivity analysis, interaction, mean value, median, model, stochastic, uncertainty, Sobol

1 Introduction

Sensitivity analysis is the study of how uncertainty in the output of a model (numerical or otherwise) can be apportioned to different sources of uncertainty in the model input factors, factors from now on [1]. The sensitivity analysis (SA) is an important part of industrial applications where engineers solve difficult computer codes, often in large input [2, 3]. The reduction of the input dimension is inevitable so that the computer applications were satisfactorily effective at observing the optimum accurateness of outputs.

One among the well introduced SA methods is the Sobol sensitivity analysis (SSA) [4, 5]. SSA is very popular and widely used approach which gives the contribution of the variability of each input w.r.t the outputs. SSA is the so-called global sensitivity analysis (GSA).

The GSA is appropriate, above all, for the computation models which are nonlinear and non-monotonic, and which are characterized by high number of dimensions and high-order interactions. GSA can help researchers find the crucial and unimportant input variables, measure relative contributions of the uncertainties of input variables to the uncertainty of model output or detect the interaction effect between different input variables [6]. Nevertheless, SSA need not always be the appropriate sensitivity indicator. Both pros and cons of the SSA are the sensitivity measurements by applying conditional mean values. It seems very intuitive that to estimate mean values could involve very different variables than to estimate e.g. medians. The importance of an input variable may vary depending on what the quantity (mean value or median) of interest is. For the evaluation of GSA, alternative solutions for the SSA require the finding of alternative approaches.

In this paper, the application of a fairly new type of global median-oriented sensitivity analysis (MSA) is presented. The MSA is a subgroup of the so-called Goal Oriented Sensitivity Analysis methods [7]. The MSA is appropriate for the applications in which it is necessary to quantify the significance of each input variable owing to the median rather than to mean value. This paper is not oriented to practical problems of engineering reliability, but to Benchmark of MSA and SSA using Ishigami Function. The application to the Ishigami test function is discussed with the purpose of illustrating the properties of the new MSA.

A numerical example of SA of Ishigami function is adopted at first. The MSA is compared to the SSA by research into sensitivity measurements of Ishigami Function [8]. Ishigami benchmark functions [9-12] and numerical aspects are investigated, and the results demonstrate that the proposed MSA is efficient and practical.

2 Brief overview of SA methods

The model is represented by a mapping f (a deterministic or stochastic function) which relates the inputs domain to the output space:

$$Y = f\left(X_1, X_2, ..., X_M\right) \tag{1}$$

where Y is a scalar. The input factors (X_1, X_2, . . ., X_M) are supposed to be random variables described by identified probability density functions which reflect the uncertain knowledge of the system under analysis [13].

Sensitivity analysis relates to the problem of investigating the contribution of the uncertainty in the input factors to the uncertainty in the model output Y [14]. Through SA, it is possible to decompose the model output uncertainty back to the input sources of uncertainty. The quantification of the signification of input uncertainties is useful for the identification of factors which must be measured with precision to reach the given accurateness of the model output.

As far as the SA methodologies are concerned, the distinction can be made between qualitative and quantitative methods [13]. Qualitative methods are oriented to the identification of significant and insignificant factors using screening, but these provide only soft observations of the relative difference of importance [15]. Quantitative technologies can be designed and adapted so to provide information on the uncertainty quantity explained by each factor [16]. Mostly, a variance is considered [17] as the degree of uncertainty; nevertheless, also other degrees of sensitivity between statistic interferences are possible [7, 18].

Another classification of available SA methodologies makes distinction between local and global methods [13, 16].

In local approaches (also known as one-at-a-time, OAT, methods), the influence of a single factor is studied supposing that all the other factors are fixed on nominal values, see e. g. [19]. The main shortcoming of this approach is the impossibility to find the interaction between the factors, because it manifests itself, when the inputs chase simultaneously. The local sensitivity index can be obtained very intuitively by computation of derivatives, see e.g. [20]. The influence of the input factor on the model output Y is computed, for the deterministic value of input X_i^* as [13]:

$$Y'_{X_i} = \left.\frac{\partial Y}{\partial X_i}\right|_{X_i = X_i^*} \tag{2}$$

The inclusion of stochastic uncertainty of the input factor can be reached by normalizing the derivatives by the factors' standard deviations.

$$Y^{\sigma}_{X_i} = \left.\frac{\sigma X_i}{\sigma Y}\frac{\partial Y}{\partial X_i}\right|_{X_i = X_i^*} \tag{3}$$

The relations (2) and (3) provide applicable information only if the model is linear or if the range of uncertainty of the input factors is small. The interactions among factors cannot be detected.

The majority of published sensitivity analyses are either local or OAT analyses, relying on unjustified assumptions of model linearity and additivity [21]. GSA which would obviate these shortcomings, are applied by a minority of researchers [21].

Generally, GSA allow the use of model independent methods as they do not require assumptions of linearity or additivity [2, 3]. GSA can be deterministic [22], stochastic [2, 3] or based on fuzzy sets [23]. The stochastic GSA is often noticed particularly in connection with the analysis of variance (ANOVA), where the studied variance in a particular variable is partitioned into components attributable to different sources of variation [24]. More details of GSA can be found in the reviews about sensitivity analysis [21, 25-28].

In GSA, variance-based methods are commonly used [2, 4, 5] for quantifying the sensitivity of the output to the inputs in terms of a reduction in the variance of model output. Non-variance approaches to GSA are applied by a minority of researchers, and therefore it is great challenge for further research work. There exist many papers in which the objective classification of the SA method is not possible, because the term "sensitivity analysis" is generally used in the context of uncertainty analysis [21].

The paper presented deals with the study into the qualities of fairly new type of quantitative global "non-variance" GSA, called MSA in the paper, as an alternative method to be applied to classical ANOVA techniques.

3 Decomposition-based GSA

The comparison of the Sobol method (based on the measurement of distance between Y and mean of Y) with SA applying contrasting functions (distance between Y and median of Y) [7, 18] is presented in the following chapters.

3.1 Sensitivity measure - method of Sobol'

The one of traditional form of GSA is the Sobol' method [4]. The basic idea of Sobol' method [4] is to decompose the function (X_1, X_2,..., X_M) into terms of increasing dimensionality,

$$\begin{aligned} f(X_1, X_2,..., X_M) &= f_0 + \sum_{i=1}^{M} f_i(X_i) + \\ &+ \sum_{1\le i\le j\le M} f_{ij}(X_i, X_j) + ... \\ &+ f_{1,2,...,M}(X_1, X_2,..., X_M) \end{aligned} \tag{4}$$

If the input factors are statistically independent, then there exists a unique decomposition of (4) such that all the terms are mutually orthogonal. The decomposition (5) is not a series expansion, because it has 2^M (finite number) of members. The variance of the output variable Y can be decomposed into:

$$V(Y) = \sum_{i=1}^{M} V_i + \sum_{1\le i\le j\le M} V_{ij} + ... + V_{1,2,...,M} \tag{5}$$

where V_j, V_{jl},..., $V_{1,2,...M}$ denote the variance of f_i, f_{ij}, ..., $f_{1,2,...,M}$, respectively. In this approach the Sobol first-order sensitivity index for factor X_i is given by:

$$S_i = \frac{V(E(Y|X_i))}{V(Y)} \tag{6}$$

In [4], Sobol proposed an alternate definition S_i=corr^2(Y,E(Y|X_i)) based on the evaluation of the correlation between output random variable Y and the conditional random arithmetical mean $E(Y|X_i)$. The sum of all S_i is equal to 1 for additive models, and less than 1 for non-additive models. The difference $1-\sum_i S_i$ is an indicator of the presence of interactions in the model.

The second and third orders sensitivity indices can be write as

$$S_{ij} = \frac{V(E(Y|X_i, X_j))}{V(Y)} \tag{7}$$

Sensitivity index S_{ij} expresses the influence of doubles on the monitored output.

$$\begin{aligned} S_{ijk} &= \frac{V(E(Y|X_i, X_j, X_k))}{V(Y)} + \\ &- S_{ij} - S_{ik} - S_{jk} - S_k - S_j - S_i \end{aligned} \tag{8}$$

The other higher orders sensitivity indices can be written and calculated analogously.

The Sobol indices have been widely used in many contexts. Application studies generally show a common drawback, Sobol indices are based on mean value as on the central parameter. The mean value, however, is, in general, not necessarily appropriate for each statistical purpose. It seems very intuitive that the sensitivity analysis based on another central parameter can show different results.

3.2 Sensitivity measure - method of contrast

A more general sensitivity measurement is based on contrasts [7]. The choice of the contrast (loss) function can determine global sensitivity indices of different types [7]. In the present paper, the SSA will be compared to the sensitivity analysis based on the measurement of distance between Y and median of Y. Median as the alternative central parameter could involve very different variables than mean value.

The contrast (1oss) function ψ associated with median can be written with parameter μ as

$$\psi(\mu) = 0.5E|Y - \mu| \tag{9}$$

and the estimator of median μ^* is given by μ^*=Argmin $\psi(\mu)$. The first order median contrast index M_i can be written as

$$M_i = 1 - \frac{E\left(\min_{\mu} E(\psi(\mu)|X_i)\right)}{\min_{\mu} \psi(\mu)} \tag{10}$$

The second order sensitivity index can be written as

$$M_{ij} = 1 - \frac{E\left(\min_{\mu} E(\psi(Y,\mu)|X_i, X_j)\right)}{\min_{\mu} \psi(\mu)} - M_i - M_j \tag{11}$$

The higher order quantile contrast indices can be written and calculated analogously. The sum of all indices is equal to one.

$$\sum_i M_i + \sum_i \sum_{j>i} M_{ij} + \sum_i \sum_{j>i} \sum_{k>j} M_{ijk} + ... + M_{123...M} = 1 \tag{12}$$

The decomposition (12) is similar to the Sobol decomposition described in detail and discussed, e.g., in [3], nevertheless, the significance of indices can be different.

4 GSA test using Ishigami function

The nonlinear and non-monotonic Ishigami function [8] is a widely used test tool in the study of sensitivity analysis techniques [9-12]. The applications SA methods described in the Chapter 2 are studied with the purpose of illustrating the properties of the new indicators M_i, M_{ij}, M_{ijk}.

4.1 Ishigami function

The mathematical expression of the Ishigami function is

$$Y = \sin x_1 + a \sin^2 x_2 + b x_3^4 \sin x_1 \tag{13}$$

where the three inputs X_i ($i = 1, 2, 3$) are independent and follow the uniform distribution $U(-\pi, \pi)$. The step by step description is used for parameters a, b. The change of parameters a and b proceeded numerically through the intervals $a \in [0, 10]$, $b \in [0, 1]$ with steps 0.2 for a, and 0.02 for b.

The Latin Hypercube Sampling (LHS) method [29, 30] was applied to generate input random variables. Twenty one thousand LHS runs were used for the evaluation of $E(Y|X_i)$ in (6), and other twenty one LHS runs were used for the evaluation of $V(E(Y|X_i))$. The variance $V(Y)$ was evaluated with forty two thousand LHS runs. It can be noticed that SSA of (13) has the analytical solution, too [9], which is in very exact agreement with numerical results presented here. Other indices described in Chapter 2 were evaluated analogously with the above described number of LHS runs. Each index defined in 2.1 and 2.2 is computed using double loop, thus the total numerically demanding character of computation of one index is 22000^2 LHS runs.

The general agreement occurs in case of sensitivity indices which are zero value; these are $M_3=S_3=0$ and $M_{23}=S_{23}=0$. The examples in which the general agreement does not occur are presented in Figs 1 to 10 where parametric analyses of dependence of sensitivity indices on parameters a, b are presented. The courses of M_1 and S_1 are similar in shapes but they are not identical, the maximum agreement (minimum difference) is abs(M_1-S_1)=0.028 in the point b=0, a=10. The courses M_2 and S_2 are similar in shapes, and there exist pairs of parameters a, b, where M_2 and S_2 are identical. The courses M_{12} and S_{12} are identical (zero) approximately for pro b>0.25, see Fig. 3 and Fig. 6. The agreement of indices M_{13} and S_{13} is illustrated in Fig. 11.

The courses of M_{123} and S_{123} are not similar in shape, but there exist parameters a, b, where $M_{123}=S_{123}=0$. In general, it holds that the higher is the order of indices, the greater differences are between MSA and SSA.

In general, the GSA as also the MSA, are model-free settings, because neither of them requires assumptions of additiveness or linearity [3]. Therefore, both of them can be applied with effectiveness to the reliability analysis based on many types of stochastic models [31-36]. The disadvantage of GSA is the fact that it usually leads to computationally demanding estimations, nevertheless the potential for research into the GSA method for the future is evident. It can be noted that CPU time consumption can be effectively reduced by using GSA-oriented types of meta-models [33-36].

5 Conclusion

It has been found by the Ishigami function sensitivity analyses that MSA provide relevant information which cannot be revealed by the SSA. First order sensitivity indices are similar in shape, or identical. Higher differences between results of MSA and SSA are identified for higher order sensitivity indices.

It is important to take into consideration that MSA has not any analogy with variance decomposition. The sensitivity measured by the MSA measures absolute distances from the median, whereas the SSA measures square power of the distance from mean value. It can be noticed that the absolute distance can be measured also between another point than median. MSA can be generalized for any upper or lower quantile.

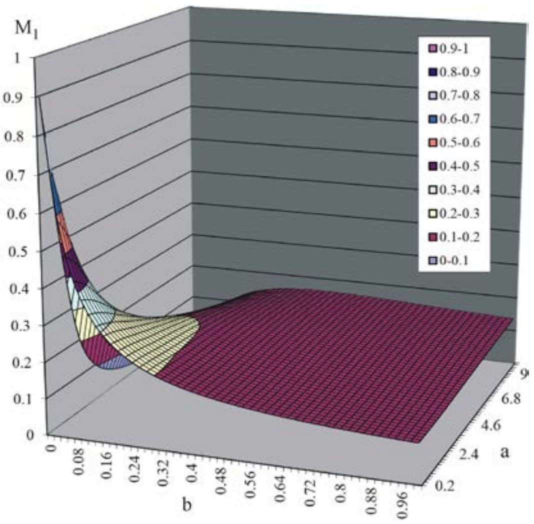

Fig.1: First order sensitivity index M_1

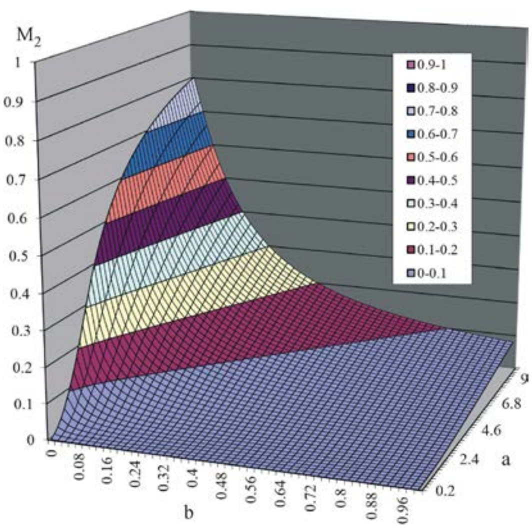

Fig.2: First order sensitivity index M_2

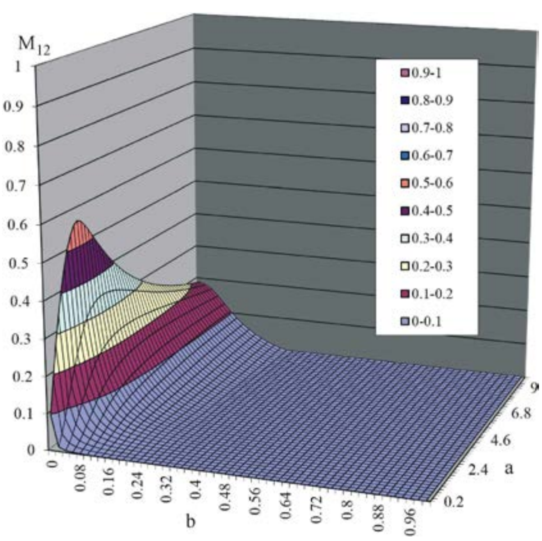

Fig.3: Second order sensitivity index M_{12}

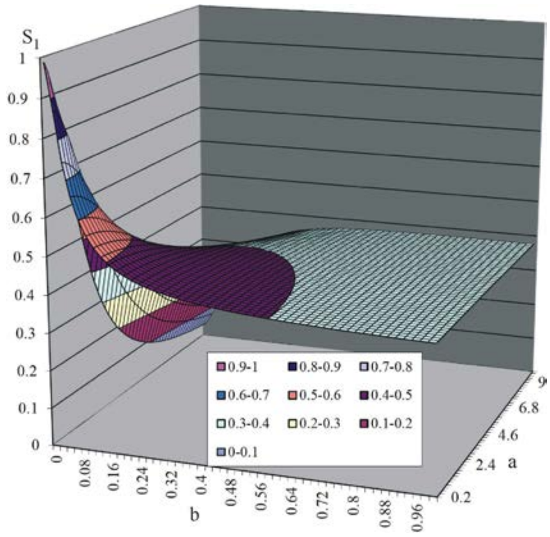

Fig.4: First order sensitivity index S_1

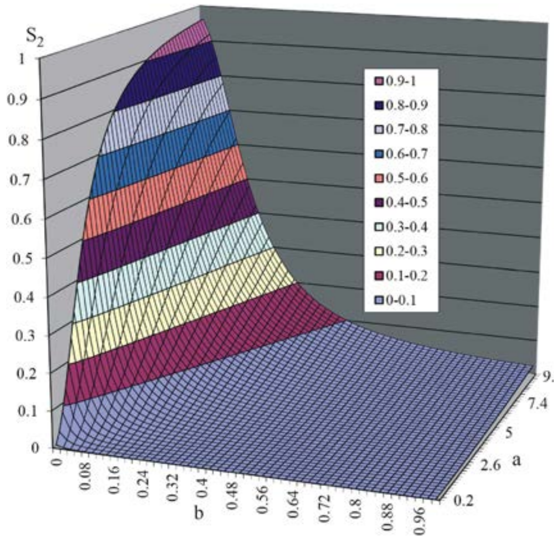

Fig.5: First order sensitivity index S_2

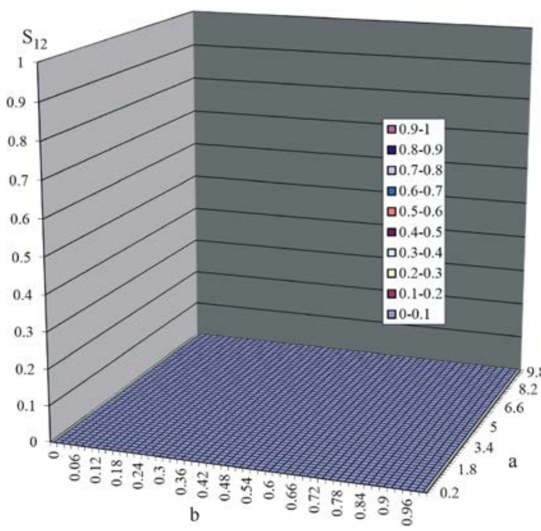

Fig.6: Second order sensitivity index S_{12}

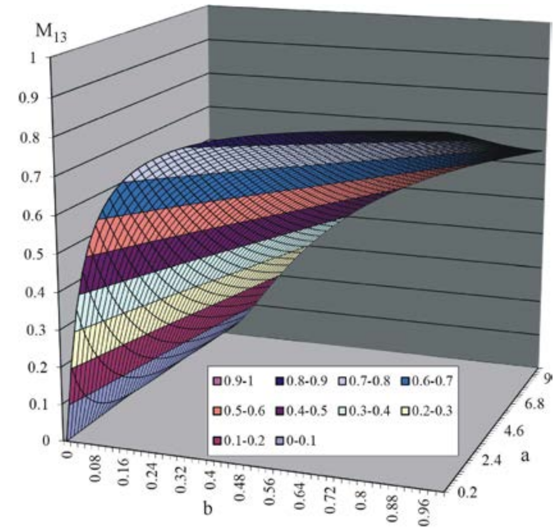

Fig.7: Second order sensitivity index M_{13}

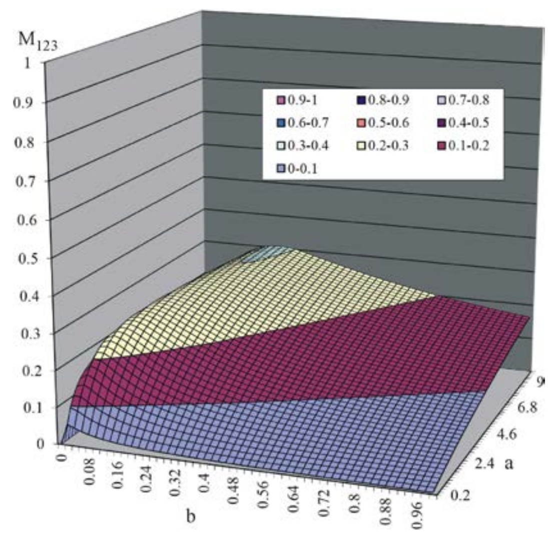

Fig.8: Third order sensitivity index M_{123}

MSA belongs into the group of global sensitivity analysis, because it is based on decomposition, and thus it is possible to compute all first and higher order indices (describe all interaction effects) the sum of which must be equal to one. The next research can concern the output sensitivity with respect to each factor individually and the total factor sensitivity inclusive of interactions. The research into the total effect can link up with extensive knowledge on total sensitivity indices realized within the framework of SSA [3]. The failure probability is the most important contribution to the reliability analysis. The above presented MSA can be modified for identification of crucial input quantities, which influence the failure probability to maximum, or, on the contrary, they are peripheral.

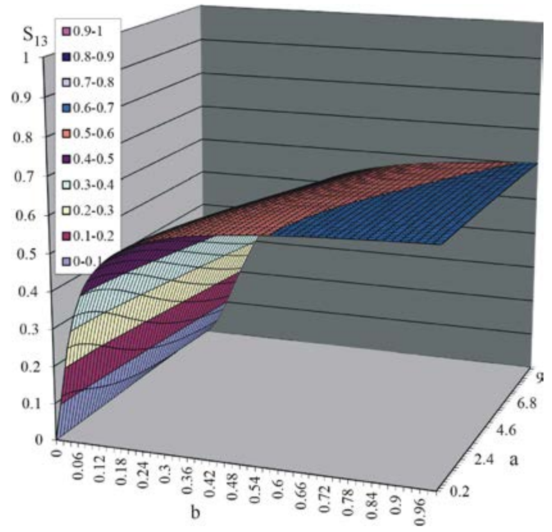

Fig.9: Second order sensitivity index S_{13}

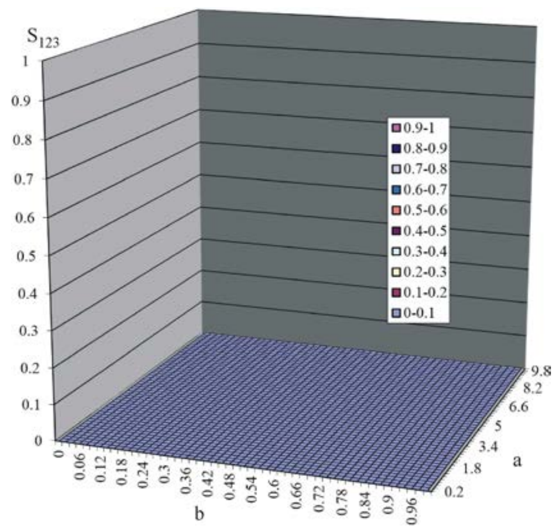

Fig.10: Third order sensitivity index S_{123}

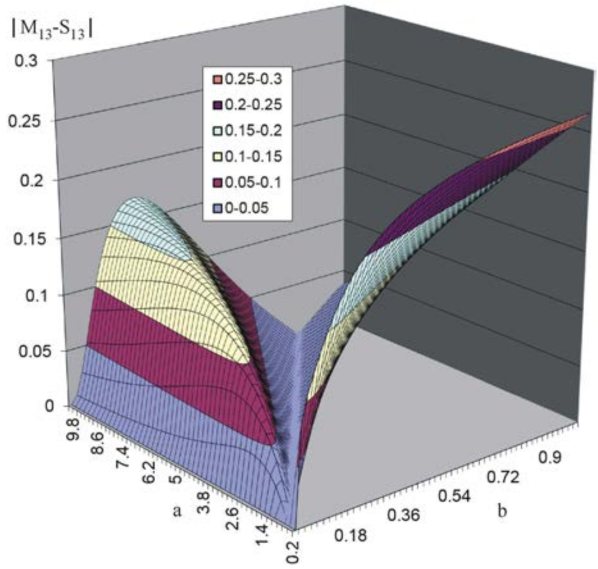

Fig.11: Difference between indexes M_{13} and S_{13}

It shows that MSA and SSA are only part of a higher whole of the Goal Oriented Sensitivity Analysis, which merits much further work to become a practical and useful tool for the future.

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