Necessary existence condition for equivariant simple singularities

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Abstract: The concept of an equivariant map naturally arises in the study of manifolds with actions of a fixed group: equivariant maps are maps that commute with group actions on the source and target. An equivariant automorphism of the source of equivariant maps preserves their equivariance. Therefore, the group of equivariant automorphisms of a manifold acts on the space of equivariant maps of this manifold. The structure of orbits of this action is often complicated: it can include discrete (finite or countable) families of orbits as well as continuous ones. An orbit is called *equivariant simple* if its sufficiently small neighborhood intersects only a finite number of other orbits. In this paper we study singular multivariate holomorphic function germs that are equivariant simple with respect to a pair of actions of a finite cyclic group on the source and target. We present a necessary existence condition for such germs in terms of dimensions of certain vector spaces defined by group actions. As an application of this result, we describe scalar actions of finite cyclic groups for which there exist no equivariant simple singular function germs.

Key–Words: Equivariant topology, singularity theory, classification of singularities, simple singularities.

1 Introduction

In the study of manifolds with actions of a fixed group it is natural to consider maps that commute with group actions on the source and target.

Definition 1 *Given two actions of a group G on sets M and N, we call a map $f\colon M \to N$ **equivariant** if the condition $f(\sigma \cdot m) = \sigma \cdot f(m)$ holds for all $\sigma \in G$, $m \in M$.*

In particular, this definition can be applied to germs of holomorphic functions $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ and of biholomorphic automorphisms $\Phi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ whenever actions of G are defined on $\mathbb{C}^n$ and $\mathbb{C}$.

The group $\mathcal{D}_n^{GG}$ of equivariant biholomorphic germs $\Phi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ acts on the space $\mathcal{O}_n^{GG}$ of equivariant holomorphic function germs $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. This infinite-dimensional space is split into orbits of this action, and so are finite-dimensional spaces $j_r\mathcal{O}_n^{GG}$ of r-jets at 0 of germs from $\mathcal{O}_n^{GG}$. We introduce an equivalence relation on the space $\mathcal{O}_n^{GG}$: two germs will be called equivalent if they belong to the same orbit.

Definition 2 *Two germs $f, g \in \mathcal{O}_n^{GG}$ are called **equivariant right equivalent** if there exists a germ $\Phi \in \mathcal{D}_n^{GG}$ such that $g = f \circ \Phi$.*

It is of interest to study the orbits of the action of $\mathcal{D}_n^{GG}$ on $\mathcal{O}_n^{GG}$, or, in other terms, to classify equivariant function germs with respect to equivariant right equivalence. In the description of the structure of the orbit space, which is often complicated, the following notion is used.

Definition 3 *An orbit $\mathcal{D}_n^{GG}(j_r g) \subset j_r\mathcal{O}_n^{GG}$ is said to be **adjacent** to the orbit $\mathcal{D}_n^{GG}(j_r f)$ if any neighborhood of some point in $\mathcal{D}_n^{GG}(j_r f)$ intersects $\mathcal{D}_n^{GG}(j_r g)$.*

The orbits of equivariant function germs can include both discrete (finite or countable) and continuous families of orbits. Discrete families make up the "simplest" part of the orbit space. This motivates the following definition.

Definition 4 *A germ $f \in \mathcal{O}_n^{GG}$ is called **equivariant simple** if for all $r \in \mathbb{N}$ the orbit $\mathcal{D}_n^{GG}(j_r f) \subset j_r\mathcal{O}_n^{GG}$ has a finite number of adjacent orbits, and this number is bounded from above by a constant independent of r.*

It is worth mentioning that an equivariant nonsingular function germ is always equivariant right equivalent to its linear part, and therefore all nonsingular equivariant germs are equivariant simple. This is why we are only interested in studying equivariant simple germs with a critical point $0 \in \mathbb{C}^n$.

There exists a general problem to classify equivariant simple singular function germs up to equivariant right equivalence for a given finite abelian group G

and a pair of its actions on the source and target. This problem naturally generalizes a similar one for the non-equivariant case solved by V. I. Arnold in 1972 (cf. [1]).

Several results are known in the equivariant setting for the group $G = \mathbb{Z}_2$. In [2] simple singularities of functions on manifolds with boundary are classified, and the complex analogue of this result is the classification of simple singularities that are even in the first coordinate (i.e., equivariant with respect to the action of $\mathbb{Z}_2$ on $\mathbb{C}^n$ in the first coordinate and the trivial action on $\mathbb{C}$). A similar problem arises in [3] in connection with the classification of simple functions on space curves. In [4] the classification of odd (i.e., equivariant with respect to non-trivial scalar actions of $\mathbb{Z}_2$ on $\mathbb{C}^n$ and on $\mathbb{C}$) simple germs is given (it is proved, in particular, that no such germs exist for $n \geq 3$). In [5] and [6] the problem is solved for bivariate functions that are equivariant simple with respect to certain actions of $\mathbb{Z}_3$.

Certain calculation techniques for the classification singularities with special attention to the equivariant case are presented in [7], [8], [9]. Some recent results connected with classification of equivariant maps, vector fields and differential equations can be found in [10]-[14].

In this paper we study conditions on finite cyclic group actions under which there exist no equivariant simple singularities. A necessary condition for existence of equivariant simple singularities is given. As an application we describe scalar actions of finite cyclic groups for which there exist no equivariant simple singular functions.

In Section 2 we describe equivariance conditions for holomorphic function and automorphism germs. In Section 3 a necessary condition for existence of equivariant simple singularities in terms of dimensions of certain vector spaces defined by group actions is given. In Section 4 we study equivariant simple singularities for the scalar action of $\mathbb{Z}_m$ on $\mathbb{C}^n$. In Section 5 we sum up the results of the paper and list some open questions.

2 Equivariant function and automorphism germs

It is known that an action of a finite group on a vector space can be linearized in a suitable system of coordinates due to a particular case of Bochner's linearization theorem (cf. [15]). Throughout this paper we assume that the generator $\sigma \in G = \mathbb{Z}_m$ acts on $\mathbb{C}^n$ and on $\mathbb{C}$ in the following way:

$$\begin{gathered} \sigma \cdot (z_1, \ldots, z_n) = (\tau^{p_1} z_1, \ldots, \tau^{p_n} z_n), \\ \sigma \cdot z = \tau^q z, \end{gathered} \tag{1}$$

where $\tau = \exp\left(\frac{2\pi i}{m}\right)$ and the integers $p_1, \ldots, p_n, q$ are considered modulo m. In fact we will always choose $0 < p_1, \ldots, p_n, q \leq m$.

Remark 5 *Without loss of generality we can assume that* $\gcd(p_1, \ldots, p_n, q) = 1$. *If* $\gcd(p_1, \ldots, p_n, q) = d > 1$ *and* $d \nmid m$, *then one can divide all* p_s *and* q *by* d *and obtain a pair of actions that is equivalent to the original one (these two cases coincide up to the choice of generator in* $\mathbb{Z}_m$*). If* $\gcd(p_1, \ldots, p_n, q) = d > 1$ *and* $d \mid m$, *then the given actions of the group* $\mathbb{Z}_m$ *can be considered as actions of its subgroup* $\mathbb{Z}_{m/d}$. *Moreover, we can assume that* $\gcd(p_1, \ldots, p_n) = 1$. *If* $\gcd(p_1, \ldots, p_n) = d > 1$, *but* $\gcd(p_1, \ldots, p_n, q) = 1$, *then* $d \nmid q$, *which implies that no monomials are equivariant with respect to actions* (1).

Suppose that the actions of $G = \mathbb{Z}_m$ on $\mathbb{C}^n$ and on $\mathbb{C}$ are given by formulae (1). Any holomorphic function germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ in a neighborhood of 0 can be represented by a power series

$$f(\mathbf{z}) = \sum_{J \in \mathbb{Z}^n_{\geq 0}} a_J \mathbf{z}^J. \tag{2}$$

Here $J = (j_1, \ldots, j_n), \mathbf{z} = (z_1, \ldots, z_n), a_J \in \mathbb{C}$, $\mathbf{z}^J = z_1^{j_1} \ldots z_n^{j_n}$. It is obvious that $f \in \mathcal{O}_n^{GG}$ if and only if $a_J = 0$ whenever $\sum_{s=1}^n p_s j_s \not\equiv q \pmod{m}$.

Any germ of a biholomorphic automorphism $\Phi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ in a neighborhood of 0 can be represented by n power series of the form

$$z_k = \sum_{J \in \mathbb{Z}^n_{\geq 0}} a_{k,J} \tilde{\mathbf{z}}^J, \tag{3}$$

where $a_{k,J} \in \mathbb{C}$, $\tilde{\mathbf{z}} = (\tilde{z}_1, \ldots, \tilde{z}_n)$ are new variables, $\tilde{\mathbf{z}}^J = \tilde{z}_1^{j_1} \ldots \tilde{z}_n^{j_n}$ and the matrix of the linear part of Φ is non-degenerate. It is obvious that $\Phi \in \mathcal{D}_n^{GG}$ if and only if $a_{k,J} = 0$ whenever $\sum_{s=1}^n p_s j_s \not\equiv p_k \pmod{m}$.

The equivariance conditions for function and automorphism germs given by power series admit a geometric interpretation. To each monomial $\mathbf{z}^J = z_1^{j_1} \ldots z_n^{j_n}$ we associate the point $J = (j_1, \ldots, j_n) \in \mathbb{Z}^n_{\geq 0}$. All points in $\mathbb{Z}^n$ associated to equivariant monomials lie in hyperplanes with the normal vector $(p_1, \ldots, p_n)$. For monomials of a germ f defined by (2) these hyperplanes are given by equations of the form

$$p_1 j_1 + \ldots + p_n j_n = km + q \quad (k \in \mathbb{Z}_{\geq 0}), \tag{4}$$

while for monomials of maps $z_l = z_l(\tilde{\mathbf{z}})$ defined by (3) they are given by equations of the form

$$p_1 j_1 + \ldots + p_n j_n = km + p_l \quad (k \in \mathbb{Z}_{\geq 0}). \qquad (5)$$

Note that under the choice of $p_1, \ldots, p_n, q$ made above these hyperplanes intersect all coordinate axes at points with positive (but not necesserily integral) coordinates, and thus the intersection of such a hyperplane with the positive octant $\mathbb{Z}^n_{\geq 0}$ contains only a finite number of integral points.

Definition 6 *Given an n-tuple $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ of natural numbers (**weights**), we define the **quasi-degree with weights** $\underline{\alpha}$ of a monomial $\mathbf{z}^J = z_1^{j_1} \ldots z_n^{j_n}$ to be equal to*

$$\deg_{\underline{\alpha}} \left(\mathbf{z}^J\right) = \langle \underline{\alpha}, J \rangle = \alpha_1 j_1 + \ldots + \alpha_n j_n.$$

The quasi-degree of a polynomial is defined to be the highest quasi-degree of its monomials.

Definition 7 *The r-**quasi-jet with weights** $\underline{\alpha}$ of a germ f given by power series* (2) *is the sum of all its monomials that have quasi-degrees with weights $\underline{\alpha}$ not exceeding r:*

$$j_r^{\underline{\alpha}} f = \sum_{\substack{J \in \mathbb{Z}^n_{\geq 0}: \\ \langle \underline{\alpha}, J \rangle \leq r}} a_J \mathbf{z}^J.$$

All r-quasi-jets of holomorphic function germs with given weights $\underline{\alpha}$ form a finite-dimensional vector space, which is exactly the space of polynomials of quasi-degree r with weights $\underline{\alpha}$. We denote this space by $j_r^{\underline{\alpha}} \mathcal{O}_n$.

For the space $\mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m}$ of function germs equivariant with respect to actions (1) there exists a natural choice of weights $\underline{\alpha}$. Namely, one can take $\alpha_s = p_s$ for $s = 1, \ldots, n$. Under this choice of weights, a monomial is equivariant if and only if its quasi-degree with weights $\underline{\alpha}$ equals $km + q$ for some $k \in \mathbb{Z}_{\geq 0}$. This implies that the corresponding r-quasi-jet spaces can only increase when r increases by m:

$$\emptyset = j_1^{\underline{\alpha}} \mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m} = \ldots = j_{q-1}^{\underline{\alpha}} \mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m} \subset j_q^{\underline{\alpha}} \mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m} = \\ = \ldots = j_{m+q-1}^{\underline{\alpha}} \mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m} \subset j_{m+q}^{\underline{\alpha}} \mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m} = \ldots = \\ = j_{2m+q-1}^{\underline{\alpha}} \mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m} \subset \ldots.$$

However, this is not the only set of weights with such a property.

Example 8 *For example, assume that the generator $\sigma \in G = \mathbb{Z}_3$ acts on $\mathbb{C}^2$ and on $\mathbb{C}$ as follows:*

$$\sigma \cdot (z_1, z_2) = (\tau z_1, \tau^2 z_2); \; \sigma \cdot z = \tau z.$$

For weights $\underline{\alpha} = (1, 2)$ suggested above, a monomial is equivariant if and only if its quasi-degree with weights $\underline{\alpha}$ equals $3k + 1$ for some $k \in \mathbb{Z}_{\geq 0}$. For weights $\underline{\beta} = (2, 1)$, a monomial is equivariant if and only if its quasi-degree with weights $\underline{\beta}$ equals $3k + 2$ for some $k \in \mathbb{Z}_{\geq 0}$). In both cases the corresponding r–quasi-jet spaces increase when r increases by 3.

This example motivates the following definition.

Definition 9 *A set of weights $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is called **admissible** with respect to actions* (1) *of $G = \mathbb{Z}_m$ on $\mathbb{C}^n$ and on $\mathbb{C}$ if it satisfies the following conditions:*
1) $\gcd(\alpha_1, \ldots, \alpha_n) = 1$;
2) for all $s \in [1, n]$ the inequalities $1 \leq \alpha_s \leq m$ hold;
3) a monomial is equivariant with respect to actions (1) *if and only if its quasi-degree (with weights $\underline{\alpha}$) has a certain excess* $\mathrm{mod}\, m$.

Remark 10 *In the case of actions* (1) *the choice of weights $\alpha_s = p_s$ $(s = 1, \ldots, n)$ suggested above provides an admissible set of weights. The fact that this set of weights satisfies conditions 1 and 2 of the definition above follows from the assumption* $\gcd(p_1, \ldots, p_n) = 1$ *explained in Remark 5 and the choice of $p_1, \ldots, p_n, q$ made prior to that remark.*

3 Necessary existence condition

Remark 11 *It is worth mentioning that equivariant simple singularities can be defined in terms of quasi-jet spaces $j_r^{\underline{\alpha}} \mathcal{O}_n^{GG}$ in a way similar to the original Definition 4. It is straightforward to check that equivariant simplicity in terms of quasi-jets (with any admissible set of weights) is equivalent to equivariant simplicity in terms of ordinary jets.*

In the following sections we will mostly check equivariant simplicity in terms of quasi-jets, which will be especially convenient for admissible sets of weights. Therefore we are interested in studying the way in which a biholomorphic equivariant automorphism germ of form (3) acts on quasi-jet spaces $j_r^{\underline{\alpha}} \mathcal{O}_n^{GG}$.

The classification of function germs is usually performed step by step starting from non-trivial jets of the lowest degree. In the equivariant case the same is done for quasi-jets. The following lemma describes the action of automorphism germs from $\mathcal{D}_n^{\mathbb{Z}_m \mathbb{Z}_m}$ on the non-trivial quasi-jet space of lowest quasi-degree.

Lemma 12 *Let $\Phi \in \mathcal{D}_n^{\mathbb{Z}_m \mathbb{Z}_m}$ be a biholomorphic automorphism germ equivariant with respect to action* (1) *of the group $G = \mathbb{Z}_m$ on $\mathbb{C}^n$ defined by n power* *series of form* (3). *Let* $f \in \mathcal{O}_n^{\mathbb{Z}_m\mathbb{Z}_m}$ *be a holomorphic function germ equivariant with respect to actions* (1) *of* G *on* $\mathbb{C}^n$ *and on* $\mathbb{C}$. *Suppose that* $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$ *is a set of weights admissible with respect to actions* (1), *and* $j_{r-1}^{\underline{\alpha}} f = 0 \neq j_r^{\underline{\alpha}} f$. *Then the quasi-jet* $j_r^{\underline{\alpha}}(f \circ \Phi)$ *depends only on those terms of series* (3) *whose exponents satisfy one of the following conditions:*

$$\sum\nolimits_{s=1}^{n} \alpha_s j_s = \alpha_k. \tag{6}$$

Lemma 12 follows directly from the multiplication rule for power series.

Remark 13 *Geometrically, equation* (6) *for each* $k = 1, \dots, n$ *defines a hyperplane in* $\mathbb{Z}^n$ *with normal vector* $(\alpha_1, \dots, \alpha_n)$ *passing through the point* $(0, \dots, 1_k, \dots, 0)$.

Therefore, under the conditions of Lemma 12 a group of transformations depending on parameters acts on the quasi-jet space $j_r^{\underline{\alpha}} \mathcal{O}_n^{\mathbb{Z}_m\mathbb{Z}_m}$. The number of these parameters (i.e., the dimension of the group) equals the number of solutions in $\mathbb{Z}_{\geq 0}^n$ to system of equations (6) in the variables $\left\{ j_s^{(k)} \right\}$ $(k = 1, \dots, n)$, which is also equal to the number of integer points with non-negative coordinates in hyperplanes (6). We denote this number by $D_r^{\underline{\alpha}}$. We also put $d_r^{\underline{\alpha}} = \dim \left(j_r^{\underline{\alpha}} \mathcal{O}_n^{\mathbb{Z}_m\mathbb{Z}_m} / j_{r-1}^{\underline{\alpha}} \mathcal{O}_n^{\mathbb{Z}_m\mathbb{Z}_m} \right)$, which is the dimension of the space of quasi-degree r (with weights $\underline{\alpha}$) equivariant polynomials. It follows from Lemma 12 that if $0 = d_0^{\underline{\alpha}} = d_1^{\underline{\alpha}} = \dots = d_{r-1}^{\underline{\alpha}} \neq d_r^{\underline{\alpha}} > D_r^{\underline{\alpha}}$, then the orbits of the action of $\mathcal{D}_n^{\mathbb{Z}_m\mathbb{Z}_m}$ on $j_r^{\underline{\alpha}} \mathcal{O}_n^{\mathbb{Z}_m\mathbb{Z}_m}$ form at least $(d_r^{\underline{\alpha}} - D_r^{\underline{\alpha}})$-parameter families. This implies the following statement.

Theorem 14 *Let* $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$ *be an admissible set of weights with respect to actions* (1) *of the group* $G = \mathbb{Z}_m$ *on* $\mathbb{C}^n$ *and on* $\mathbb{C}$. *If* $0 = d_0^{\underline{\alpha}} = d_1^{\underline{\alpha}} = \dots = d_{r-1}^{\underline{\alpha}} \neq d_r^{\underline{\alpha}} > D_r^{\underline{\alpha}}$ *(in the notation chosen above), then there exist no holomorphic function germs in* $\mathcal{O}_n^{\mathbb{Z}_m\mathbb{Z}_m}$ *that are equivariant simple with respect to actions* (1).

Theorem 14 gives a necessary condition for existence of equivariant simple singular germs (or, equivalently, a sufficient condition for nonexistence of equivariant simple singular germs).

4 Scalar actions of $G = \mathbb{Z}_m$, $m \geq 3$

In this section we study equivariant simple singularities in $\mathcal{O}_n^{\mathbb{Z}_m\mathbb{Z}_m}$ in the case when the action of $G = \mathbb{Z}_m$, $m \geq 3$ on $\mathbb{C}^n$ is scalar. We will only consider the case $n \geq 2$ (the case $n = 1$ is trivial). Without loss of generality we can assume that the actions of the group on the source and target are given by the formulae

$$\begin{gathered} \sigma \cdot (z_1, \dots, z_n) = (\tau z_1, \dots, \tau z_n)\,; \\ \sigma \cdot z = \tau^q z, \end{gathered} \tag{7}$$

where $\sigma \in \mathbb{Z}_m$ is a generator, $\tau = \left(\frac{2\pi i}{m} \right)$. The result essentially depends on the excess $q \bmod m$. For the rest of this section we choose $q \in [1, m]$.

Theorem 15 *(cf. [5, Theorem 1]) Suppose the action of the group* $\mathbb{Z}_m$ *on* $\mathbb{C}^n$ *and* $\mathbb{C}$ *is given by formulae* (7) *with* $q = 1$. *For* $m \geq 3$, $n \geq 2$ *there exist no equivariant simple singular function germs in* $\mathcal{O}_n^{\mathbb{Z}_m\mathbb{Z}_m}$.

Proof: Take $\underline{\alpha} = (1, \dots, 1)$. Then $0 = d_0^{\underline{\alpha}} = d_1^{\underline{\alpha}} = \dots = d_m^{\underline{\alpha}}$, while $d_{m+1}^{\underline{\alpha}} = \binom{n+m}{n-1}$. At the same time, $D_{m+1}^{\underline{\alpha}} = n^2$, because due to Lemma 12, the $(m+1)$-jets of singular equivariant germs depend only on the linear parts of automorphism germs from $\mathcal{D}_n^{\mathbb{Z}_m\mathbb{Z}_m}$. Finally, it is straightforward to check (e.g. by induction on m) that whenever $m \geq 3$, $n \geq 2$, the inequality $\binom{n+m}{n-1} > n^2$ holds. Therefore, the statement of the theorem follows from Theorem 14. □

Remark 16 *For* $m = 2$ *equivariant simple singular germs with respect to actions* (7) *with* $q = 1$ *are classified in [4].*

Theorem 17 *Suppose the action of* $G = \mathbb{Z}_m$ *on* $\mathbb{C}^n$ *and* $\mathbb{C}$ *is given by formulae* (7) *with* $q = 2$. *A singular equivariant germ* $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *is equivariant simple with respect to the given actions if and only if it is equivalent to one of the following germs:*

$$(z_1, \dots, z_n) \mapsto z_1^{mk+2} + z_2^2 + \ldots + z_n^2 \;(k \in \mathbb{Z}_{\geq 0}). \tag{8}$$

Proof: The proof is based on the following two lemmas.

Lemma 18 *In a neighborhood of the origin there exists an equivariant change of coordinates* $x = x(\tilde{x}, \tilde{y})$, $y = y(\tilde{x}, \tilde{y})$ *that gives the germ* f *the form* $f(\tilde{x}, \tilde{y}) = \varphi(\tilde{x}) + Q(\tilde{y})$, *where* Q *is a nondegenerate quadratic form,* $\dim\{\tilde{y}\} = rk(d^2 f|_0) = \rho$, $\dim\{\tilde{x}\} = n - \rho$.

Proof of Lemma 18: The lemma is proved similarly to [1, Lemma 4.1]. The only required modification for the equivariant case is the Morse lemma with parameter: we need to prove that a family of equivariant functions that depends analytically on the parameter and has a critical point analytically depending on the parameter with critical value 0 is equivariant right equivalent to a sum of squares. The corresponding coordinate change can be obtained in the same way as in the proof of the ordinary Morse lemma (cf. [16, Lemma 2.2]); the equivariance of this coordinate change follows from its explicit form. □

Lemma 19 *In the notation of Lemma 18 the inequality $\rho \geq n - 1$ holds.*

Proof of Lemma 19: If $\rho < n-1$, then φ is an equivariant function germ in two or more variables with a trivial $(m+1)$-jet. Therefore its lowest degree nontrivial jet is of degree greater than or equal to $m + 2 \geq 5$. But the classification of forms of degree 5 and higher in two or more variables contains moduli (i.e., continuous parameters), and therefore, in this case the germ f will not be equivariant simple. □

Now we can finish the proof of Theorem 17. If $\rho = n$, then f is a non-degenerate quadratic form in n variables that is linearly equivalent to the sum of squares, i.e., has the form (8) with $k = 0$. If $\rho = n-1$, consider the equivariant function φ in one variable. If all of its derivatives vanish at the origin, then f is not equivariant simple (all orbits $\tilde{x}^{3k+2} + Q(\tilde{y})$ are adjacent to the orbit of f.) If $\varphi^{(i)}(0) = 0$ for $0 \leq i \leq 3k + 1$, but $\varphi^{(3k+2)}(0) \neq 0\,(k \in \mathbb{N})$, then the germ f is equivalent to germ (8) with the same k. Each of germs (8) is equivariant simple: adjacent orbits in $j_r\mathcal{O}_n^{\mathbb{Z}_m\mathbb{Z}_m}$ for $r \geq 3k + 2$ are $z_1^{3l+2} + z_2^2 + \ldots + z_n^2$ with $0 \leq l \leq k$. Any two germs of form (8) with different values of k are not equivalent because they have different multiplicities of zero at the origin. □

Theorem 20 *Suppose the action of $G = \mathbb{Z}_m$ on $\mathbb{C}^n$ and $\mathbb{C}$ is given by formulae (7) with $q \geq 3$. If $q = 3$, $n = 2, 3$ or $q \geq 4$, $n \geq 2$, then there exist no equivariant simple function germs in $\mathcal{O}_n^{\mathbb{Z}_m\mathbb{Z}_m}$.*

Proof: The proof is similar to the proof of Theorem 15. Take $\underline{\alpha} = (1, \ldots, 1)$. Then (in the notation of Theorem 14) $0 = d_0^{\underline{\alpha}} = d_1^{\underline{\alpha}} = \ldots = d_{q-1}^{\underline{\alpha}}$, while $d_q^{\underline{\alpha}} = \binom{n+q-1}{q-1}$, and $D_q^{\underline{\alpha}} = n^2$. From Theorem 14 it follows that if the inequality $\binom{n+q-1}{q-1} > n^2$ is satisfied, then there exist no equivariant simple function germs in $\mathcal{O}_n^{\mathbb{Z}_m\mathbb{Z}_m}$. For $q = 3$ this inequality holds only for $n = 2$ and $n = 3$. For $q \geq 4$ and $n \geq 2$ this inequality is always true, which can be proved by induction on q. □

Remark 21 *The assumptions of Theorem 20 depend on the excess $q \bmod m$ and not on m itself. In particular, the statement of the theorem holds for $q = m$.*

5 Conclusion

We obtained a necessary condition for existence of equivariant simple singular holomorphic function germs for finite cyclic group actions and used it for the case of scalar actions on the source. This condition can be used as the first step in classifying equivariant simple singularities for all possible actions of a given finite cyclic group on the source and target. However, the sufficiency of this condition is still an open question. Moreover, the application of this result can meet some technical difficulties, because for a non-scalar action of the group on the source the calculation of dimensions that are used in the conditions amounts to find the number of non-negative integer solutions to certain systems of diophantine equations. In particular, the result of Theorem 20 is incomplete: the necessary existence condition does not allow to study the case $q = 3$, $n \geq 4$ straightforwardly. Further development of calculation techniques and possibly computer algorithms for such calculations that might help to solve this problem is the aim of our future research.

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