Primary Decomposition of Ideals Arising from Hankel Matrices

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Abstract: Hankel matrices have many applications in various fields ranging from engineering to computer science. Their internal structure gives them many special properties. In this paper we focus on the structure of the set of polynomials generated by the minors of generalized Hankel matrices whose entries consist of indeterminates with coefficients from a field k. A generalized Hankel matrix M has in its j^{th} codiagonal constant multiples of a single variable X_j. Consider now the ideal $I_r(M)$ in the polynomial ring $k[X_1, \dots, X_{m+n-1}]$ generated by all $(r \times r)$-minors of M. An important structural feature of the ideal $I_r(M)$ is its *primary decomposition* into an intersection of primary ideals. This decomposition is analogous to the decomposition of a positive integer into a product of prime powers. Just like factorization of integers into primes, the primary decomposition of an ideal is very difficult to compute in general. Recent studies have described the structure of the primary decomposition of $I_2(M)$. However, the case when $r > 2$ is substantially more complicated. We will present an analysis of the primary decomposition of $I_3(M)$ for generalized Hankel matrices up to size 5×5.

Key-Words: Hankel, matrix, decomposition, ideal, Gröbner, basis, prime, primary, SINGULAR, polynomial, ring, minor, coefficient, Noetherian, field, commutative, algebra, irredundant, radical, factorization, component, algebraic, geometry, generator, normal, isolated, embedded

1 Introduction

The properties of the ideals generated by the minors of matrices whose entries are linear forms are hard to describe, unless the forms themselves satisfy some strong condition. Here we compute a primary decomposition for ideals in polynomial rings that are generated by minors of Hankel matrices. To be precise, let k be a field, and let $2 \le m \le n$ be integers. A generalized Hankel Matrix is defined as $M =$

$$\begin{bmatrix} r_{11}X_1 & r_{12}X_2 & \cdots & r_{1,n}X_n \\ r_{22}X_2 & r_{23}X_3 & \cdots & r_{2,n+1}X_{n+1} \\ & & \vdots & \\ r_{mm}X_m & r_{m,m+1}X_{m+1} & \cdots & r_{m,m+n-1}X_{m+n-1} \end{bmatrix}$$

where the X_i are indeterminates and the r_{ij} are nonzero elements of a field k. In the present work we analyze the structure of an $m \times n$ generalized Hankel matrix M, with $m \ge 3$. In particular we determine the minimal primary decomposition of ideals generated by the 3×3 minors of M. By $I_3(M)$ we denote the ideal in the polynomial ring $F[X_1, \dots, X_{m+n-1}]$ which is generated by the 3×3 minors. We denote $I_2(M)$ the ideal in the polynomial ring $F[X_1, \dots, X_{m+n-1}]$ which is generated by the 2×2 minors. Let $I_n(M)$ be the primary decomposition of ideals generated by $n \times n$ minors of a generalized Hankel matrix M. In previous research the structure of $I_2(M)$ has been described. However little is known about the cases of minors with $n \ge 3$. In our research we have analyzed $I_3(M)$ for 3×4 matrices, for 4×4 matrices, and 5×5 matrices. In Section 2 we describe the primary decomposition of ideals

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and definitions related to the understanding of $I_3(M)$. In Section 3 we give the structure of $I_2(M)$. In Section 4 we prove that $I_3(M)$ for a 3 × 4 matrix is prime. In Section 5 we give several examples and conjectures for $I_3(M)$ for a 4 × 4 matrix. In Section 6 we discuss the symmetry of $I_3(M)$ for some examples with 5 × 5 matrices. In Section 6 we have further thoughts over the project and possible future work.

2 Primary Decomposition of Ideals

The primary decomposition of an ideal in a polynomial ring over a field is an essential tool in commutative algebra and algebraic geometry. The process of computing primary decom--positions of ideals is analogous to the factorization of positive integers into powers of primes. Just like factoring an integer into powers of primes, finding the primary decom--position of an ideal is generally very difficult to compute. In this section we will provide the reader with some basic properties of ideals and their primary decompositions. We will first introduce several basic terms and concepts associated to ideals followed by the definition of a primary decomposition and examples.

Definition 1. *Let R be a commutative ring and I be an ideal.*
1. *An ideal $I \subset R$ is irreducible if it is not the intersection of strictly larger ideals.*
2. *R is Noetherian if every increasing chain of ideals $I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots$ eventually be--comes constant.*
3. *I is primary if, whenever $ab \in I$ and $a \notin I$, then $b^n \in I$ for some positive integer n.*
4. *$I \subset k[x_1, \ldots, x_n]$ is prime if whenever $f, g \in k[x_1, \ldots, x_n]$ and $fg \in I$, then either $f \in I$ or $g \in I$.*
5. Let $I \subset R$ be an ideal. The radical rad(I) is the ideal $\sqrt{I} = \{r \in R : r^n \in I \text{ for some } n \in \mathbb{Z}^+\}$.

Lemma 2. *If I is primary, then $\sqrt{I}$ is prime.*

Proof: See [4].

Example 3. 1. *$R = Z$. The only primary ideals are those of the form (p^n) for a prime number p, and the zero ideal. The radical of (p^n) is equal to (p), which is a prime ideal.*
2. *Let $R = k[x, y, z]/(xy - z^2)$, and let $P = (\bar{x}, \bar{z}) \subset R$. Then P is prime because $R/P = k[y]$ is a domain. Then $\bar{x}\bar{y} = \bar{z}^2 \in P^2$, but $\bar{x} \notin P^2$. Futhermore, $\bar{y} \notin rad(P^2) = P$. Hence, P^2 is not primary. Note, a power of a prime need not be primary, even though its radical is prime.*

Definition 4. *A primary decomposition of an ideal $I \subset R$ is a decomposition of I as an intersection $I = I_1 \cap \cdots \cap I_r$ of primary ideals with pairwise distinct radicals, which is irredundant.*

Corollary 5. *If R is a Noetherian ring, then every ideal has a primary decomposition.*

Thus, we see that the intersection of ideals is similar to the factorization of integers into their primes, since every integer has a prime factorization. However, we don't get uniqueness of the decomposition in full generality.

Example 6. Let $R = k[x, y]$. Then
$$(x^2, xy) = (x) \cap (x, y)^2 = (x) \cap (x^2, y)$$

Fortunately, not all is lost, since the set of radical ideals associated to each primary component is unique. This motivates the following definition.

Definition 7. Let $I = Q_1 \cap \cdots \cap Q_n$ with $P_i = rad(Q_i)$ and $P_j = P_i(Q_j)$.
1. *The ideals P_i are called the primes associated to I, and the set $\{P_i\}$ is denoted by $Ass(I)$.*
2. *If a P_i does not contain any P_j, $j \neq i$, then Q_i is called an isolated component. Otherwise Q_i is called an embedded component.*

Example 8. Consider
$$I = (z^2, zx) = (z) \cap (z^2, x)$$
Here (z) is an isolated component, but (z^2, x) is embedded, since $(z, x) = \sqrt{(z^2, x)}$ contains $(z) = \sqrt{(z)}$.

Theorem 9. The isolated components of a primary decomposition are unique.

Proof: See [2].

We will close this section with an example of the computation of the primary decomposition of a monomial ideal.

Example 10. Let $I = (z^3, x^2y, yx^2z)$ be a subset of $k[x, y, z]$. Then

$$\begin{aligned} I &= (z^3, x^2, yx^2z) \cap (z^3, y, yx^2z) \\ &= (z^3, x^2, y) \cap (z^3, x^2) \cap (z^3, x^2, z) \\ &\quad \cap (z^3, y) \cap (z^3, y, x^2) \cap (z^3, y, z) \end{aligned}$$

Now observe that $(z^3, x^2) \subset (z^3, x^2, y)$ and $(z^3, y) \subset (z, y)$ and $(z^3, x^2) \subset (z, x^2)$, so we can delete (z^3, x^2, y), (z, y), (z, x^2). Thus, we get the primary decomposition

$$I = (z^3, x^2) \cap (z^3, y)$$

3 Structure of $I_2(M)$

In recent studies, Guerrieri and Swanson computed the minimal primary decomposition of ideals generated by 2 × 2 minors of generalized Hankel matrices. They showed that the primary decomposition of $I_2(M)$ is either primary itself or has exactly two minimal components and sometimes also one embedded component. They also identified two integers, *s* and *t*, intrinsic to *M*, which allow one to decide whether $I_2(M)$ is prime. To define *s* and *t* we first need to to transform *M* into a special form by scaling the variables. The scaling of the variables does not change the number of primary components, or the prime and primary properties. So, without loss of generality, *M* becomes the following generalized Hankel matrix *M* =

$$\begin{bmatrix} X_1 & X_2 & X_3 & \cdots & X_n \\ X_2 & X_3 & X_4 & \cdots & X_{n+1} \\ X_3 & r_{34}X_4 & r_{35}X_5 & \cdots & X_{n+2} \\ X_4 & r_{45}X_5 & r_{46}X_6 & \cdots & X_{n+3} \\ & & & \vdots & \\ X_m & r_{m,m+1}X_{m+1} & r_{m,m+2}X_{m+2} & \cdots & X_{m+n-1} \end{bmatrix}$$

with all r_{ij} units in *F*.

We can define *s* as:

$$s = min\{j \geq 4 : \exists i \geq 3 \text{ such that } r_{ij} \neq 1\}$$

The integer *t* is defined in a similar way to *s* for a matrix obtained from rotating *M* 180 degrees and then rescaling the variables. So without loss of generality *M* is transformed to *M*=

$$\begin{bmatrix} X_1 & r_{12}X_2 & \cdots & r_{1,n-1}X_{n-1} & X_n \\ X_2 & r_{23}X_3 & \cdots & r_{2n}X_n & X_{n+1} \\ & & \cdots & & \\ X_{m-2} & r_{m-2,\,m-1}X_{m-1} & \cdots & r_{m-2,\,m+n-4}X_{m+n-4} & X_{m+n-3} \\ X_{m-1} & X_m & \vdots & X_{m+n-3} & X_{m+n-2} \\ X_m & X_{m+1} & \cdots & X_{m+n-2} & X_{m+n-1} \end{bmatrix}$$

with all r_{ij} units in *F*.

We can define *t* as:

$$t = max\{j \leq m + n - 4 : \exists i \leq m - 2 \text{ such that } j < n + i - 1 \text{ and } r_{ij} \neq r_{i,j+1}\}$$

Now that we have *s* and *t* we can now describe the structure of $I_2(M)$ as shown in the following theorem.

Theorem 11. *Let*

$$Q_1 = I_2(M) + (X_s, \ldots, X_{m+n-1})$$
$$Q_2 = I_2(M) + (X_1, \ldots, X_t)$$
$$Q_3 = I_2(M) + (X_1^{m+n-4}, \ldots, X_{m+n-1}^{m+n-4})$$

be ideals in the ring $F[X_1, \ldots, X_{m+n-1}]$. *Then:*

I. Q_1, Q_2, Q_3 *are are primary to the prime ideals* $(X_s, \ldots, X_{m+n-1})$, $(X_1, \ldots, X_{m+n-2})$, *and* $(X_1, \ldots, X_{m+n-1})$, *respectively.*

II. If s and t do not exist, then $I_2(M)$ *is a prime ideal.*

III. If $s > t$, *then* $I_2(M) = Q_1 \cap Q_2$ *is a primary decomposition.*

IV. If $s \leq t$, *then* $I_2(M) = Q_1 \cap Q_2 \cap Q_3$ *is an irredundant primary decomposition.*

4 $I_3(M)$ for 3 × 4 Hankel Matrices

In the primary decomposition of $I_2(M)$, we saw that each primary component Q_i looks like $I_2(M) + J_i$ for some ideal J_i. In a similar way, we have the same kind of breakdown for each primary component in the primary decom- -position of $I_n(M)$.

Proposition 12. *Let M be a generalized Hankel matrix and let G be a Gröbner basis of* $I_n(M)$. *If the primary decomposition of* $I_n(M)$ *is*

$Q_1 \cap Q_2 \cap \cdots \cap Q_k$, then each Q_i is of the form $I_2(M) + J_i$ for some ideal J_i. Furthermore, if the set of generators for each Q_i is $\{h_1, h_2, \ldots, h_{m_i}\}$, then the set of generators for each J_i is precisely $\left\{\overline{h_1}^G, \overline{h_2}^G, \ldots, \overline{h_{m_i}}^G\right\}$, where each $\overline{h_j}^G$ is the normal form of h_j with respect to G.

Proof: Since $I_n(M) = Q_1 \cap Q_2 \cap \cdots \cap Q_k$, we have that $I_n(M)$ is a sub-ideal of Q_i for all i where $1 \le i \le k$. Now suppose that Q_i is the ideal generated by $\{h_1, h_2, \ldots, h_{m_i}\}$ and the Gröbner basis for $I_3(M)$ is $G = \{g_1, g_2, \ldots, g_n\}$. Then, taking the normal form of each Q_i with respect to G gives us $\left\{\overline{h_1}^G, \overline{h_2}^G, \ldots, \overline{h_{m_i}}^G\right\}$. So we have

$$\begin{aligned} Q_i &= \langle h_1, h_2, \ldots, h_{m_i}\rangle \\ &= \langle g_1, g_2, \ldots, g_n\rangle + \langle \overline{h_1}^G, \overline{h_2}^G, \ldots, \overline{h_{m_i}}^G\rangle \\ &= I_n(M) + \langle \overline{h_1}^G, \overline{h_2}^G, \ldots, \overline{h_{m_i}}^G\rangle \end{aligned}$$

Now, we have that $I_n \not\subseteq \langle \overline{h_1}^G, \overline{h_2}^G, \ldots, \overline{h_{m_i}}^G\rangle$ and $I_n \not\supseteq \langle \overline{h_1}^G, \overline{h_2}^G, \ldots, \overline{h_{m_i}}^G\rangle$. Therefore we have that each Q_i is precisely $I_n(M) + \langle \overline{h_1}^G, \overline{h_2}^G, \ldots, \overline{h_{m_i}}^G\rangle$. Thus each J_i is precisely $\langle \overline{h_1}^G, \overline{h_2}^G, \ldots, \overline{h_{m_i}}^G\rangle$. QED

The last proposition is used in our algorithms for finding the primary decomposition of $I_3(M)$. Utilizing this proposition, we now give the primary decomposition of $I_3(M)$ for any gener--alized 3 × 4 Hankel matrix *M*.

Theorem 13. If *M* is any generalized 3 × 4 Hankel matrix, then $I_3(M)$ is prime.

Proof: By Section 2 it is enough to consider the primary decomposition of a matrix of the following form:

$$M^* = \begin{bmatrix} X_1 & X_2 & X_3 & X_4 \\ X_2 & X_3 & X_4 & X_5 \\ X_3 & r_{34}X_4 & r_{35}X_5 & X_6 \end{bmatrix}$$

Now by considering r_{34} and r_{35} as variables, SINGULAR computed the primary decom--position of $I_3(M^*)$. Our output was just $I_3(M^*)$ itself – namely the ideal generated by:

$$\begin{aligned} &r_{35}x_1x_5^2 - r_{35}x_2x_4x_5 - x_1x_4x_6 \\ &\quad +x_2x_3x_6 - x_3^2x_5 + x_3x_4^2, \\ &r_{34}x_3x_4x_5 - r_{34}x_4^3 - r_{35}x_2x_5^2 \\ &\quad +r_{35}x_3x_4x_5 + x_2x_4x_6 - x_3^2x_6, \\ &r_{34}x_1x_4x_5 - r_{34}x_2x_4^2 - x_1x_3x_6 \\ &\quad +x_2^2x_6 - x_2x_3x_5 + x_3^2x_4, \\ &r_{34}x_1x_4^2 - r_{34}x_2x_3x_4 - r_{35}x_1x_3x_5 \\ &\quad +r_{35}x_2^2x_5 - x_2x_3x_4 + x_3^3 \end{aligned}$$

Hence, $I_3(M^*)$ is itself primary. Now, by Lemma 1, this implies that $\sqrt{I_3(M^*)}$ is prime. Our goal was to show that $I_3(M^*)$ is prime. However, after one more SINGULAR computation, we found that $\sqrt{I_3(M^*)} = I_3(M^*)$. Therefore $I_3(M^*)$ is prime. QED

5 $I_3(M)$ for 4 × 4 Hankel Matrix

As in the 3 × 4 Hankel matrix case we can assume for the 4 × 4 Hankel matrix that the first two rows, and the first and last columns have coefficients equal to one. The remaining four coefficients r_{ij} can assume any value. Thus we assume the 4 × 4 Hankel matrix takes on the following form:

$$M = \begin{bmatrix} X_1 & X_2 & X_3 & X_4 \\ X_2 & X_3 & X_4 & X_5 \\ X_3 & r_{34}X_4 & r_{35}X_5 & X_6 \\ X_4 & r_{45}X_5 & r_{46}X_6 & X_7 \end{bmatrix}$$

In the coefficient matrix

$$R_k(M) = \begin{bmatrix} r_{34}X_4 & r_{35}X_5 \\ r_{45}X_5 & r_{46}X_6 \end{bmatrix}$$

there are fifteen possible combinations where some $r_{ij} \neq 1$. According to many examples computed, it seems clear that the primary decomposition of $I_3(M)$ for these fifteen matrices breaks up into three cases. The primary decomposition $I_3(M)$ can be equal to one, two, or three ideal components. We conjecture that similar to previous sections, there are three possible choices for the primary decomposition of $I_3(M)$:

$$I_3(M) = Q_1$$
$$I_3(M) = Q_1 \cap Q_2$$

$$I_3(M) = Q_1 \cap Q_2 \cap Q_3$$

In the following subsections, we present each case in further detail.

5.1 $I_3(M) = Q_1$

Our analysis of 4 × 4 Hankel matrices shows only eight possible combinations of the coefficient matrix where there exists only one ideal component. These are the possible combinations of $R_k(M)$, $1 \le k \le 8$, where not all $r_{ij} = 1$:

$$\begin{bmatrix} 1 & r_{35} \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} r_{34} & 1 \\ r_{45} & 1 \end{bmatrix}, \begin{bmatrix} r_{34} & 1 \\ 1 & r_{46} \end{bmatrix}, \begin{bmatrix} r_{34} & r_{35} \\ r_{45} & 1 \end{bmatrix},$$

$$\begin{bmatrix} r_{34} & r_{35} \\ 1 & r_{46} \end{bmatrix}, \begin{bmatrix} r_{34} & 1 \\ r_{45} & r_{46} \end{bmatrix}, \begin{bmatrix} 1 & r_{35} \\ r_{45} & r_{46} \end{bmatrix}, \begin{bmatrix} r_{34} & r_{35} \\ r_{45} & r_{46} \end{bmatrix}$$

The examples computed ran on SINGULAR for specific values of r_{ij}.

Example 14.

$$M = \begin{bmatrix} X_1 & X_2 & X_3 & X_4 \\ X_2 & X_3 & X_4 & X_5 \\ X_3 & 11X_4 & 3X_5 & X_6 \\ X_4 & 4X_5 & 7X_6 & X_7 \end{bmatrix}$$

The output obtained by SINGULAR for $Q_1 = I_3(M) + J_1$ is:
$J_1 = 0$, and $I_3(M)$ is equal to:

[1]=x(5)^3+10663*x(4)*x(5)*x(6)+10664*x(3)*x(6)^2+10664*x(4)^2*x(7)-x(3)*x(5)*x(7)
[2]=x(4)*x(5)^2-10664*x(4)^2*x(6)-10664*x(3)*x(5)*x(6)+10664*x(2)*x(6)^2+10664*x(3)*x(4)*x(7)-x(2)*x(5)*x(7)
[3]=x(3)*x(5)^2+10664*x(3)*x(4)*x(6)-x(2)*x(5)*x(6)+10664*x(1)*x(6)^2+10663*x(3)^2*x(7)+x(2)*x(4)*x(7)-x(1)*x(5)*x(7)
[4]=x(4)^2*x(5)+10663*x(3)*x(4)*x(6)+10664*x(1)*x(6)^2+10664*x(3)^2*x(7)-x(1)*x(5)*x(7)
[5]=x(3)*x(4)*x(5)-x(2)*x(5)^2-10664*x(3)^2*x(6)+10664*x(1)*x(5)*x(6)+10664*x(2)*x(3)*x(7)-10664*x(1)*x(4)*x(7)
[6]=x(3)^2*x(5)+x(2)*x(4)*x(5)-2*x(1)*x(5)^2-x(2)*x(3)*x(6)+x(1)*x(4)*x(6)+x(2)^2*x(7)-x(1)*x(3)*x(7)
[7]=x(4)^3-x(2)*x(5)^2-10664*x(3)^2*x(6)-x(2)*x(4)*x(6)+10665*x(1)*x(5)*x(6)+10665*x(2)*x(3)*x(7)-10665*x(1)*x(4)*x(7)
[8]=x(3)*x(4)^2-2*x(2)*x(4)*x(5)+x(1)*x(5)^2+x(2)^2*x(7)-x(1)*x(3)*x(7)
[9]=x(3)^2*x(4)-x(2)*x(4)^2-x(2)*x(3)*x(5)+x(1)*x(4)*x(5)+x(2)^2*x(6)-x(1)*x(3)*x(6)
[10]=x(3)^3-2*x(2)*x(3)*x(4)+x(1)*x(4)^2+3*x(2)^2*x(5)-3*x(1)*x(3)*x(5)

Now we show that $\sqrt{I_3(M)}$, which is also $\sqrt{Q_1}$, is equal to the above, [1]-[10]. Now $J_1 = 0 \Rightarrow Q_1 = I_3(M) + 0 \Rightarrow Q_1 = I_3(M)$. Also, we see that $I_3(M) = \sqrt{I_3(M)}$, so by definition of prime we have that Q_1 is a prime ideal component, hence $I_3(M)$ is prime.

5.2 $I_3(M) = Q_1 \cap Q_2$

There are five possible combinations of the coefficient matrix for there to exist two ideal components. These are the possible combin--ations of the coefficient matrix of $R_k(M)$, $1 \le k \le 5$, where not all $r_{ij} = 1$:

$$\begin{bmatrix} r_{34} & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} r_{34} & r_{35} \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ r_{45} & r_{46} \end{bmatrix},$$

$$\begin{bmatrix} 1 & r_{35} \\ r_{45} & 1 \end{bmatrix}, \begin{bmatrix} 1 & r_{35} \\ 1 & r_{46} \end{bmatrix}$$

Example 15.

$$M = \begin{bmatrix} X_1 & X_2 & X_3 & X_4 \\ X_2 & X_3 & X_4 & X_5 \\ X_3 & X_4 & 3X_5 & X_6 \\ X_4 & 4X_5 & X_6 & X_7 \end{bmatrix}$$

The output obtained by SINGULAR for $Q_1 = I_3(M) + J_1$, where J_1 is an ideal composed of the following polynomials:

[1]=x(4)*x(5)-10664*x(3)*x(6)
[2]=x(2)*x(5)-10664*x(1)*x(6)
[3]=x(4)^2-x(2)*x(6)
[4]=x(3)*x(4)-x(1)*x(6)

[5]=x(3)^2-3*x(1)*x(5)
[6]=x(2)*x(3)-x(1)*x(4)
[7]=x(1)*x(5)^2-5332*x(1)*x(4)*x(6)+7998*x(2)^2*x(7)-7998*x(1)*x(3)*x(7)
[8]=x(1)*x(3)*x(5)*x(6)+15995*x(1)*x(2)*x(6)^2-7997*x(2)^2*x(4)*x(7)+7997*x(1)^2*x(6)*x(7)
[9]=x(1)*x(2)*x(4)*x(6)-2*x(1)^2*x(5)*x(6)+15994*x(2)^3*x(7)-15994*x(1)^2*x(4)*x(7)
[10]=x(1)*x(2)^2*x(6)^2+10663*x(1)^2*x(3)*x(6)^2+15994*x(2)^3*x(4)*x(7)-15994*x(1)^2*x(2)*x(6)*x(7)

$\sqrt{Q_1}$ is equal to:

[1]=x(4)*x(5)-10664*x(3)*x(6)
[2]=x(2)*x(5)-10664*x(1)*x(6)
[3]=x(4)^2-x(2)*x(6)
[4]=x(3)*x(4)-x(1)*x(6)
[5]=x(3)^2-3*x(1)*x(5)
[6]=x(2)*x(3)-x(1)*x(4)
[7]=x(5)^3-12441*x(3)*x(6)^2-7998*x(3)*x(5)*x(7)+2666*x(2)*x(6)*x(7)
[8]=x(3)*x(5)^2-5332*x(1)*x(6)^2+7998*x(2)*x(4)*x(7)+7997*x(1)*x(5)*x(7)
[9]=x(1)*x(5)^2-5332*x(1)*x(4)*x(6)+7998*x(2)^2*x(7)-7998*x(1)*x(3)*x(7)
[10]=x(1)*x(3)*x(5)*x(6)+15995*x(1)*x(2)*x(6)^2-7997*x(2)^2*x(4)*x(7)+7997*x(1)^2*x(6)*x(7)
[11]=x(1)*x(2)*x(4)*x(6)-2*x(1)^2*x(5)*x(6)+15994*x(2)^3*x(7)-15994*x(1)^2*x(4)*x(7)
[12]=x(1)*x(2)^2*x(6)^2+10663*x(1)^2*x(3)*x(6)^2+15994*x(2)^3*x(4)*x(7)-15994*x(1)^2*x(2)*x(6)*x(7)

and the output for $Q_2 = I_3(M) + J_2$, where J_2 is an ideal composed of the following poly-nomials:

[1]=x(5)
[2]=x(3)*x(6)^2+x(4)^2*x(7)
[3]=x(1)*x(6)^2-x(3)^2*x(7)+2*x(2)*x(4)*x(7)
[4]=-3*x(2)*x(5)*x(7)
[5]=-15994*x(2)*x(5)*x(6)-15995*x(1)*x(6)^2+15995*x(3)^2*x(7)+x(2)*x(4)*x(7)+15994*x(1)*x(5)*x(7)
[6]=3*x(1)*x(5)*x(6)
[7]=x(2)*x(3)*x(6)-x(1)*x(4)*x(6)-x(2)^2*x(7)+x(1)*x(3)*x(7)
[8]=0
[9]=6*x(1)*x(5)^2+15993*x(2)*x(3)*x(6)-15993*x(1)*x(4)*x(6)-15993*x(2)^2*x(7)+15993*x(1)*x(3)*x(7)
[10]=0
[11]=-3*x(2)^2*x(5)+3*x(1)*x(3)*x(5)

$\sqrt{Q_2(M)}$ is equal to:

[1]=x(5)
[2]=x(3)*x(6)^2+x(4)^2*x(7)
[3]=x(1)*x(6)^2-x(3)^2*x(7)+2*x(2)*x(4)*x(7)
[4]=x(4)^2*x(6)-x(2)*x(6)^2-x(3)*x(4)*x(7)

[5]=x(3)*x(4)*x(6)-x(3)^2*x(7)+x(2)*x(4)*x(7)
[6]=x(3)^2*x(6)-x(2)*x(3)*x(7)+x(1)*x(4)*x(7)
[7]=x(2)*x(3)*x(6)-x(1)*x(4)*x(6)-x(2)^2*x(7)+x(1)*x(3)*x(7)
[8]=x(4)^3-x(2)*x(4)*x(6)+x(2)*x(3)*x(7)-x(1)*x(4)*x(7)
[9]=x(3)*x(4)^2+x(2)^2*x(7)-x(1)*x(3)*x(7)
[10]=x(3)^2*x(4)-x(2)*x(4)^2+x(2)^2*x(6)-x(1)*x(3)*x(6)
[11]=x(3)^3-2*x(2)*x(3)*x(4)+x(1)*x(4)^2

We conclude that $I_3(M) = (I_3(M) + J_1) \cap (I_3(M) + J_2)$, where each $J_i = \langle \overline{h_1}^G, \ldots, \overline{h_{m_i}}^G \rangle$ if G is a Gröbner basis for $I_3(M)$.

5.3 $I_3(M) = Q_1 \cap Q_2 \cap Q_3$

The last two possible combinations of R_k for the 4 × 4 Hankel matrix have three ideal compon-ents for $I_3(M)$. The following are the possible $R_k(M)$, $1 \le k \le 2$, where not all $r_{ij} = 1$:

$$\begin{bmatrix} 1 & 1 \\ r_{45} & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & r_{46} \end{bmatrix}$$

Example 16.

$$M = \begin{bmatrix} X_1 & X_2 & X_3 & X_4 \\ X_2 & X_3 & X_4 & X_5 \\ X_3 & X_4 & X_5 & X_6 \\ X_4 & 4X_5 & X_6 & X_7 \end{bmatrix}$$

Each $Q_i = I_3(M) + J_i$. So, $I_3(M)$ is the same for all Q_i. Then $I_3(M)$ is:

For Q_1 we have J_1:

[1]=x(5)^2-x(4)*x(6)
[2]=x(4)*x(5)-x(3)*x(6)
[3]=x(3)*x(5)-x(2)*x(6)
[4]=x(2)*x(5)-x(1)*x(6)
[5]=x(4)^2-x(2)*x(6)
[6]=x(3)*x(4)-x(1)*x(6)
[7]=x(2)*x(4)-x(1)*x(5)
[8]=x(3)^2-x(1)*x(5)
[9]=x(2)*x(3)-x(1)*x(4)
[10]=x(2)^2-x(1)*x(3)

$\sqrt{Q_1}$ is:

[1]=x(5)^2-x(4)*x(6)
[2]=x(4)*x(5)-x(3)*x(6)
[3]=x(3)*x(5)-x(2)*x(6)
[4]=x(2)*x(5)-x(1)*x(6)
[5]=x(4)^2-x(2)*x(6)
[6]=x(3)*x(4)-x(1)*x(6)
[7]=x(2)*x(4)-x(1)*x(5)
[8]=x(3)^2-x(1)*x(5)
[9]=x(2)*x(3)-x(1)*x(4)
[10]=x(2)^2-x(1)*x(3)

For Q_2 we have J_2:

[1]=x(5)*x(6)-7998*x(4)*x(7)
[2]=x(3)*x(6)-7998*x(2)*x(7)
[3]=x(5)^2-7998*x(3)*x(7)
[4]=x(4)*x(5)-7998*x(2)*x(7)
[5]=x(3)*x(5)-7998*x(1)*x(7)
[6]=x(2)*x(5)-x(1)*x(6)
[7]=x(4)^2-x(2)*x(6)
[8]=x(3)*x(4)-x(1)*x(6)
[9]=x(3)^2-x(1)*x(5)
[10]=x(2)*x(3)-x(1)*x(4)
[11]=x(1)*x(6)^2-7998*x(2)*x(4)*x(7)
[12]=x(1)*x(4)*x(6)-7998*x(2)^2*x(7)

$\sqrt{Q_2}$ is:

[1]=x(5)*x(6)-7998*x(4)*x(7)
[2]=x(3)*x(6)-7998*x(2)*x(7)
[3]=x(5)^2-7998*x(3)*x(7)
[4]=x(4)*x(5)-7998*x(2)*x(7)
[5]=x(3)*x(5)-7998*x(1)*x(7)
[6]=x(2)*x(5)-x(1)*x(6)
[7]=x(4)^2-x(2)*x(6)
[8]=x(3)*x(4)-x(1)*x(6)
[9]=x(3)^2-x(1)*x(5)
[10]=x(2)*x(3)-x(1)*x(4)
[11]=x(1)*x(6)^2-7998*x(2)*x(4)*x(7)
[12]=x(1)*x(4)*x(6)-7998*x(2)^2*x(7)

And for Q_3 we have J_3:

[1]=x(5)
[2]=x(3)*x(6)^2+x(4)^2*x(7)
[3]=x(1)*x(6)^2-x(3)^2*x(7)+2*x(2)*x(4)*x(7)

[4]=x(2)*x(3)*x(6)-x(1)*x(4)*x(6)-x(2)^2*x(7)+x(1)*x(3)*x(7)
[5]=x(3)^2*x(4)*x(7)-x(2)*x(4)^2*x(7)+x(2)^2*x(6)*x(7)-x(1)*x(3)*x(6)*x(7)
[6]=x(3)^3*x(7)-2*x(2)*x(3)*x(4)*x(7)+x(1)*x(4)^2*x(7)
[7]=x(2)*x(3)*x(4)^2*x(7)-x(1)*x(4)^3*x(7)-x(1)*x(3)^2*x(6)*x(7)+x(1)*x(2)*x(4)*x(6)*x(7)+x(2)^3*x(7)^2-x(1)*x(2)*x(3)*x(7)^2
[8]=x(2)*x(4)^2*x(6)^2*x(7)-x(2)^2*x(6)^3*x(7)+x(3)*x(4)^3*x(7)^2-x(1)*x(4)^2*x(6)*x(7)^2
[9]=x(2)^2*x(4)^2*x(6)*x(7)-x(1)*x(3)*x(4)^2*x(6)*x(7)-x(2)^3*x(6)^2*x(7)-x(2)^2*x(3)*x(4)*x(7)^2-x(1)*x(2)^2*x(6)*x(7)^2+x(1)^2*x(3)*x(6)*x(7)^2
[10]=x(2)^2*x(4)^3*x(7)-x(1)*x(3)*x(4)^3*x(7)-x(2)^3*x(4)*x(6)*x(7)+x(1)^2*x(4)^2*x(6)*x(7)+x(2)^3*x(3)*x(7)^2-x(1)*x(2)*x(3)^2*x(7)^2

$\sqrt{Q_3}$ is:

[1]=x(5)
[2]=x(3)*x(6)^2+x(4)^2*x(7)
[3]=x(1)*x(6)^2-x(3)^2*x(7)+2*x(2)*x(4)*x(7)
[4]=x(4)^2*x(6)-x(2)*x(6)^2-x(3)*x(4)*x(7)
[5]=x(3)*x(4)*x(6)-x(3)^2*x(7)+x(2)*x(4)*x(7)
[6]=x(3)^2*x(6)-x(2)*x(3)*x(7)+x(1)*x(4)*x(7)

[7]=x(2)*x(3)*x(6)-x(1)*x(4)*x(6)-x(2)^2*x(7)+x(1)*x(3)*x(7)
[8]=x(4)^3-x(2)*x(4)*x(6)+x(2)*x(3)*x(7)-x(1)*x(4)*x(7)
[9]=x(3)*x(4)^2+x(2)^2*x(7)-x(1)*x(3)*x(7)
[10]=x(3)^2*x(4)-x(2)*x(4)^2+x(2)^2*x(6)-x(1)*x(3)*x(6)
[11]=x(3)^3-2*x(2)*x(3)*x(4)+x(1)*x(4)^2

Similar to the previous section $I_3(M) = (I_3(M)+J_1) \cap (I_3(M)+J_2) \cap (I_3(M)+J_3)$, where each $J_i = \langle \overline{h_1}^G, \ldots, \overline{h_{m_i}}^G \rangle$ if G is a Gröbner basis for $I_3(M)$.

5.3 Section Conclusions

All three cases of $I_3(M)$ for the 4×4 Hankel matrix are similar to $I_2(M)$. We see that each equality pertains to its respective subsection

$$I_3(M) = (I_3(M)+J_1)$$
$$I_3(M) = (I_3(M)+J_1) \cap (I_3(M)+J_2)$$
$$I_3(M) = (I_3(M)+J_1) \cap (I_3(M)+J_2) \cap (I_3(M)+J_3)$$

where each $J_i = \langle \overline{h_1}^G, \ldots, \overline{h_{m_i}}^G \rangle$ if G is a Gröbner basis for $I_3(M)$.

In the subsections we presented $\sqrt{Q_i}$ to show that some of the elements of a $\sqrt{Q_i}$ are contained in $\sqrt{Q_j}$ but not all, where $i \neq j$ for the same $I_3(M)$. So, from Section 2 we have that these $\sqrt{Q_i}$ are isolated ideal components.

6 5 × 5 Matrices

In this section we will analyze $I_3(M)$ for 5×5 generalized Hankel matrices. We will demonst--rate our results with an example.

Example 17. Let *A* be the following matrix

$$A = \begin{bmatrix} X_1 & X_2 & X_3 & X_4 & X_5 \\ X_2 & X_3 & X_4 & X_5 & X_6 \\ X_3 & 2X_4 & 3X_5 & 5X_6 & X_7 \\ X_4 & 7X_5 & 11X_6 & 13X_7 & X_8 \\ X_5 & 17X_6 & 19X_7 & 23X_8 & X_9 \end{bmatrix}$$

Based on a SINGULAR computation, *A* has a primary decomposition, $I = Q_1 \cap Q_2 \cap Q_3 \cap Q_4 \cap Q_5$, where $Q_1, \ldots, Q_5$ are given below.

$$Q_1 = I_3(M) + (x_1,\ x_2,\ x_3,\ x_4x_7,\ x_5x_6,\ x_4x_6,\ x_5^2,\ x_4x_5,\ x_4^2,\ x_6^2 - 4384x_5x_7 - 8206x_4x_8)$$
$$Q_2 = I_3(M) + (x_4,\ x_5,\ x_6,\ x_7^2,\ x_3^2,\ x_2x_3,\ x_7x_8,\ x_2^2 - x_1x_3,\ x_8^2 + 6954x_7x_9)$$
$$Q_3 = I_3(M) + (x_7,\ x_8,\ x_9,\ x_6^2,\ x_5x_6,\ x_3x_6,\ x_5^2,\ x_4x_5,\ x_4x_6,\ x_4^2 - 6x_3x_5 + 6x_2x_6)$$
$$Q_4 = I_3(M) + (x_1,\ x_2x_6,\ x_3x_5, x_2x_5, x_4^2,\ x_3x_4,\ x_2x_4,\ x_3^2,\ x_2x_3,\ x_2^2,\ x_8^4,\ x_4x_5 - 5466x_3x_6)$$
$$Q_5 = I_3(M) + (x_9,\ x_8^2,\ x_7x_8,\ x_6x_8, x_5x_8,\ x_4x_8,\ x_7^2,\ x_6x_7,\ x_5x_7,\ x_6^2,\ x_2^4,\ x_5x_6 + 5726x_6x_3)$$

Notice that Q_1 begins with terms of single variables (X_1, X_2, X_3), Q_2 begins with (X_4, X_5, X_6), Q_3 with (X_7, X_8, X_9), Q_4 with (X_1), and Q_5 with (X_9). If we look at the placement of these terms, also notice that those of Q_1 lie on or above the X_3 diagonal:

$$\begin{bmatrix} \boldsymbol{X_1} & \boldsymbol{X_2} & \boldsymbol{X_3} & & X_5 \\ \boldsymbol{X_2} & \boldsymbol{X_3} & & X_5 & \\ \boldsymbol{X_3} & & X_5 & & \\ & X_5 & & & \\ X_5 & & & & \end{bmatrix}$$

the terms for Q_2 lie between the X_3 and X_7 diagonals:

$$\begin{bmatrix} & & X_3 & \boldsymbol{X_4} & \boldsymbol{X_5} \\ & X_3 & \boldsymbol{X_4} & \boldsymbol{X_5} & \boldsymbol{X_6} \\ X_3 & \boldsymbol{X_4} & \boldsymbol{X_5} & \boldsymbol{X_6} & X_7 \\ \boldsymbol{X_4} & \boldsymbol{X_5} & \boldsymbol{X_6} & X_7 & \\ \boldsymbol{X_5} & \boldsymbol{X_6} & X_7 & & \end{bmatrix}$$

and the terms for Q_3 lie on or below the X_7 diagonal:

$$\begin{bmatrix} & & & & X_5 \\ & & & X_5 & \\ & & X_5 & & \boldsymbol{X_7} \\ & X_5 & & \boldsymbol{X_7} & \boldsymbol{X_8} \\ X_5 & & \boldsymbol{X_7} & \boldsymbol{X_8} & \boldsymbol{X_9} \end{bmatrix}$$

Also notice that the term of Q_4 and Q_5 are placed at opposite ends of the X_5 diagonal:

$$\begin{bmatrix} \boldsymbol{X_1} & & & & X_5 \\ & & & X_5 & \\ & & X_5 & & \\ & X_5 & & & \\ X_5 & & & & \boldsymbol{X_9} \end{bmatrix}$$

With these facts in mind, suppose that possibly some symmetry exists. Let the X_5 diagonal be the line of symmetry. If we reflect or map terms to each other along this diagonal we have the following mapping ϕ:

$$x_1 \leftrightarrow x_9$$
$$x_2 \leftrightarrow x_8$$
$$x_3 \leftrightarrow x_7$$
$$x_4 \leftrightarrow x_6$$
$$x_5 \leftrightarrow x_5$$

Using this mapping, for the terms Q_1 and Q_3 we get the following:

$Q_1 = I_3(M) +$			$Q_3 = I_3(M) +$		
$(x_1,$	$\longrightarrow$	x_9	$(x_7,$	$\longrightarrow$	x_3
$x_2,$	$\longrightarrow$	$x_8,$	$x_8,$	$\longrightarrow$	$x_2,$
$x_3,$	$\longrightarrow$	$x_7,$	$x_9,$	$\longrightarrow$	$x_1,$
$x_4x_7,$	$\longrightarrow$	$x_6x_3,$	$x_6^2,$	$\longrightarrow$	$x_4^2,$
$x_5x_6,$	$\longrightarrow$	$x_5x_4,$	$x_5x_6,$	$\longrightarrow$	$x_5x_4,$
$x_4x_6,$	$\longrightarrow$	$x_6x_4,$	$x_3x_6,$	$\longrightarrow$	$x_7x_4,$
$x_5^2,$	$\longrightarrow$	$x_5^2,$	$x_5^2,$	$\longrightarrow$	$x_5^2,$
$x_4x_5,$	$\longrightarrow$	$x_5x_6,$	$x_4x_5,$	$\longrightarrow$	$x_6x_5,$
$x_4^2,$	$\longrightarrow$	$x_6^2,$	$x_4x_6,$	$\longrightarrow$	$x_6x_4,$
x_6^2	$\longrightarrow$	x_4^2	x_4^2	$\longrightarrow$	x_6^2
-4384		-4384	-6		-6
$*x_5x_7$		$*x_5x_3$	$*x_3x_5$		$*x_7x_5$
-8206		-8206	$+6$		$+6$
$*x_4x_8)$		$*x_6x_2$	$*x_2x_6)$		$*x_8x_4$

After carefully analyzing the above table, it is clear that $\phi(Q_1) \in Q_3$ for every term in Q_1. Similarly, $\phi(Q_3) \in Q_1$ for every term in Q_3. Note, however, that there are some variations of coefficients. After performing the same proced- -ure for all Q_i we have

$$Q_1 \leftrightarrow Q_3$$
$$Q_2 \leftrightarrow Q_2$$
$$Q_4 \leftrightarrow Q_5$$

We believe that this same type of symmetry exists for all the different 5 × 5 matrices. The amount of symmetry may depend on the values of *s* and *t* which in turns depends on the amount and placement of coefficients. We hope to further investigate this in future research.

7 Conclusion and Future Work

We analyzed the primary decomposition of $I_3(M)$ for M as a 3 × 4, 4 × 4, or 5 × 5 Hankel matrix. One important result that we proved is the primary decomposition of $I_3(M_{3\times4})$. However, more work still needs to be done.

It is possible that we may be close to finality on the primary decomposition of $I_3(M_{4\times4})$. Since there are only fifteen possible primary decompositions for $I_3(M_{4\times4})$, depend- -ing on the placement of four coefficients, we hypothesize that there are eight decompositions that are prime, five that are the intersections of two ideals, and two that are the intersections of three ideals.

It is plausible that this can be proven using SINGULAR, much in the same way as was done for $I_3(M_{3\times4})$. However, at the time of this writing, SINGULAR was already computing for days on end. So it is unclear whether our conjecture is true.

Other possibilities for future work consist of analyzing the patterns inherent in the primary decompositions of $I_3(M_{n\times m})$ for $n, m \geq 5$. Specifically, for $I_3(M_{5\times5})$, are the symmetries we discussed inherent in all the primary decompositions of $I_3(M_{5\times5})$? If so, are these symmetries based on *s*'s and *t*'s? More generally, assuming that these symmetries exist, can they also be found in the primary decompositions of $I_3(M_{n\times m})$ for any $n \times m$ matrix? A progressive result would be a theorem describing the primary decomposition of $I_3(M)$ for any Hankel matrix M.

Now, supposing we find the primary decomposition of $I_3(M_{n\times m})$ for all n × m Hankel matrices, the best result possible would be a theorem describing the primary decomposition of $I_n(M)$ for any Hankel matrix M. This is our ultimate goal. However, the present work shows how complicated this is. To work on or expand on any of our questions would make for promising future research.

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