N-dimensional multiresolution algorithms for point values

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Abstract: Multiresolution algorithms are used in several applications in order to attain data compression, denoising or computional time reduction in algorithms dealing with large data. Our objective is to introduce nonlinear reconstructions in the N-dimensional case and compare their performances when applied with and without error control algorithms. This paper describes then the N-dimensional multiresolution algorithms with and without error control strategies in discrete point values as a generalization to N dimensions of the work done in this direction, see [13], [14], [11], [2], [16]. Some numerical experiments are included to exemplify the utility of these algorithms. In the results it can be observed that nonlinear stable methods improve their linear counterparts in presence of discontinuities in the data. Even non-stable nonlinear methods can overcome the instabilities and get better results than linear ones when used with error control.

Key–Words: Multiresolution schemes, N-dimensional, reconstructions, point values, error control, nonlinearity

1 Introduction

Multiresolution algorithms have been proved to be useful tools in several fields such as signal processing, and in particular in image compression and denoising. And why have they reached so high popularity? The answer is quite simple: they give fast and easy to program algorithms and good performances in relation to other classical approaches such as Fourier based methods.

Given the data vector $f^L = (f_1^L, \dots, f_j^L, \dots)$ where L indicates the resolution level, one can define a new representation of the original data f^L as the sequence given by $\{f^0, d^1, \dots, d^L\}$ where f^k is an approximation of f^L at resolution $k < L$ and d^{k+1} represents the details needed to recover f^{k+1} from f^k. The number of elements in the set $\{f^k, d^{k+1}\}$ is equal to that of f^{k+1} and therefore the same fact occurs between the sets $\{f^0, d^1, \dots, d^L\}$ and f^L. Besides this, it is immediate to get the decoding for a given coding algorithm.

Inside the vast variety of multiresolution algorithms, one can differentiate between linear and nonlinear ones. Wavelet decompositions are important examples of linear algorithms. They are extensively used for different purposes, for example in signal analysis ([2], [15]) and in the solution of some partial differential equations, since they lead to the solution of well-conditioned linear systems of equations and to computationally fast algorithms. The performance of wavelets in the presence of non-smooth data is limited as it is well known, and this fact comes from their intrinsic linear nature. Many nonlinear possible alternatives such as ridgelets, bandelets and curvelets have been developed, studied, and applied to different problems in the last years (see for example [22]).

A way of introducing nonlinearity without too much effort in the pyramidal multiresolution decompositions was given by Harten in [13], [14]. In Harten's framework, the different discrete resolution levels are connected by two inter-resolution operators, named decimation (from the fine scale (k) to the coarse scale $(k-1)$) and prediction (just the opposite, from coarse to fine scale). In turn, these two inter-scale operators are closely related to another group of two operators, *discretization* and *reconstruction*. These new operators act between the continuous level (where a function f, related to the discrete data, lives) and each discrete resolution level (where f^k lives). The chosen continuous level depends on the applications that one has in mind. For example to deal with data coming from an elevation map of a mountainous region the appropiate continuous level could be the continuous functions in a given two dimensional rectangle. However to work with image processing applications, one would consider the space of integrable functions instead as the continuous level. Discretization and decimation operators must be linear, but the reconstruction operator and in turn the prediction operator can be nonlinear, and this fact allows for a wide range of witty definitions in order to attain better adaption to the considered data. This adaptability of the Harten's algorithms is of tremendous importance in presence of singularities, permitting to give a data-dependent and nonlinear treatment of the data and therefore better results in applications working with such initial data. Examples of nonlinear decompositions based on Harten's multiresolution can be found for instance in [9], [6], [3], [5].

Different settings can be considered depending on the chosen continuous level and on the linear discretization operator that produces the data according to the real applications that one has in mind. The most common settings are provided by the sampling operator (point value setting) and the averaging operator (cell average setting). In this work the efforts are focused on developing the N-dimensional theory and the corresponding fast algorithms for the case of the sampling operator. The other case of cell averages was already addressed, see [16].

A similar framework for multiresolution was developed independently by Sweldens (see [19], [20], [21]).

A fundamental aspect in applying nonlinear multiresolution decompositions is the matter of stability, in the sense that is explained next.

The multiresolution representation $\{f^0, d^1, \dots, d^L\}$ of the initial data is well adapted to compression procedures, since many of the details d^k are zero or close to zero. This multi-scale representation is then processed (truncation or quantization for instance) and the final result of this step is a modified multi-scale representation $\{\hat{f}^0, \hat{d}^1, \hat{d}^2, \dots, \hat{d}^L\}$ which is *close* to the original one, i.e. such that (in some norm)

$$||\hat{f}^0 - f^0|| \leq \epsilon_0 \qquad ||\hat{d}^k - d^k|| \leq \epsilon_k \quad 1 \leq k \leq L,$$

where the truncation parameters $\epsilon_0, \epsilon_1, \dots, \epsilon_L$ are chosen according to some criteria specified by the user, depending for example on the application.

After decoding the processed representation, and recovering an approximation $\hat{f}^L$ to the original data f^L, one would desire some kind of stability, i.e.

$$||\hat{f}^L - f^L|| \leq \sigma(\epsilon_0, \epsilon_1, \dots, \epsilon_L), \qquad (1)$$

where $\sigma(\cdot, \dots, \cdot)$ verifies

$$\lim_{\epsilon_l \to 0,\ 0 \leq l \leq L} \sigma(\epsilon_0, \epsilon_1, \dots, \epsilon_L) = 0.$$

It is well known that linear multiresolution schemes are stable, and that one can easily derive stability bounds in these cases (see [7] for example). However this fact is unfortunately not true in general for the nonlinear schemes. Many results in this sense have been obtained in the last two decades, see for example [12], [17], [18], [8]. In whatever case the stability for nonlinear multiresolution algorithms is normally very difficult to prove, and not always possible to obtain. In fact there are important and well known nonlinear schemes which are not stable at all ([3]). The good news are that stability can always be attained by using the error control algorithms introduced by Harten ([14], [2],[1]). A closely related concept named synchronization deserves also to be mention (see [10]). These algorithms implement a modified decoding algorithm which is able to keep track of the total cumulative error, and therefore it is possible to obtain explicit error bounds dependent only on the specific truncation or quantization parameters chosen by the user.

The algorithms presented in this paper concern the point value setting, so completing the work already done for the cell averages [16]. Both contributions could be considered as much a generalization of Harten's error control algorithms in 1D for their respective setting as a particularization to the N-dimensional case in a rectangle of the completely general case studied by Harten ([13], [14]). The relevance of dealing with the details of each dimension are let clear in the 2D, and 3D cases in [2], [4], where also some important applications are given. The principal aim of this paper is to provide a detailed analysis of the N-dimensional multiresolution algorithms in the framework of point values, with and without error control. The applications range from one devoted to compression and reconstruction of contour maps in two dimensions representing the topography of mountainous terrains, where cliffs could be found to another having to do with the treatment of a temperature field in a cylindrical pipe in a heat exchanger which occupies a lot of memory inside a finite elements calculation.

The paper is organized as follows: in Section 2 N-dimensional multiresolution algorithms in point values without error control are presented. In Section 3 multiresolution transformations using error control strategies are then introduced. In Section 4 an interesting numerical experiment is presented. Finally, Section 5 contains the conclusions.

2 N-dimensional multiresolution algorithms without error control in the point values setting

For a more detailed description on Harten's framework for multiresolution read [9]. Here only the aspects of the multiresolution framework that are relevant to the rest of the paper are described.

Let it be considered a mesh in $[0,1]^N$ given by:

$$X^k = \{x^k_{i_1}, x^k_{i_2}, \ldots, x^k_{i_N}\}^{J_k}_{i_1,\ldots,i_N=0}, \quad J_k = 2^k J_0,$$
$$J_0 \text{ integer}, \quad h_k = \frac{1}{J_0 2^k},$$
$$x^k_{i_s} = i_s h_k, \quad s = 1, \ldots, N,$$

and the point values discretization operator given by

$$\mathcal{D}_k : \begin{cases} C([0,1]^N) & \to V^k, \\ f & \mapsto f^k = (f^k_{i_1,\ldots,i_n})^{J^N_k}_{i_1,\ldots,i_N=0}, \end{cases} \tag{2}$$

where $f^k_{i_1,\ldots,i_N}$, $0 \le i_s \le J_k$, $s = 1,\ldots,N$, is defined by

$$f^k_{i_1,\ldots,i_N} := f(x^k_{i_1}, \ldots, x^k_{i_N}), \tag{3}$$

$C([0,1]^N)$ is the space of continuous functions in $[0,1]^N$ and V^k is the space of real sequences with dimension $(J_k+1)^N$ related to the resolution of X^k. A reconstruction operator $\mathcal{R}_k$ associated to this discretization is any right inverse of $\mathcal{D}_k$, which means that for all $f^k \in V^k$, $\mathcal{R}_k f^k \in C([0,1]^N)$ and

$$f^k_{i_1,\ldots,i_N} = (\mathcal{R}_k f^k)(x_{i_1}, \ldots, x_{i_N}). \tag{4}$$

The sequences $\{\mathcal{D}_k\}$ and $\{\mathcal{R}_k\}$ define a multiresolution transform and the prediction operator $P^k_{k-1} := \mathcal{D}_{k+1}\mathcal{R}_k : V^k \to V^{k+1}$, defines a subdivision scheme. If $\mathcal{R}_k$ is a nonlinear reconstruction operator, the corresponding subdivision scheme is also nonlinear. The decimation operator $D^{k-1}_k : V^k \to V^{k-1}$ is always linear and in our case reads

$$f^{k-1}_{i_1,\ldots,i_N} = (D^{k-1}_k f^k)_{i_1,\ldots,i_N} = f^k_{2i_1,\ldots,2i_N}. \tag{5}$$

We also need to define the error, which is given by

$$e^k_{j_1,\ldots,j_N} := f^k_{j_1,\ldots,j_N} - (P^k_{k-1} f^{k-1})_{j_1,\ldots,j_N}. \tag{6}$$

It is easy to prove that the error belongs to the null space of the decimation operator D^{k-1}_k and thus

$$e^k_{2i_1,\ldots,2i_N} = 0, \tag{7}$$

what in practice implies that the only errors necessary to recover are the impair ones.

We are now in possession of all the needed ingredients to give the expression of the encoding and decoding multiresolution algorithms. First let us denote,

$$\begin{aligned} J : &= \{(j_1,\ldots,j_N) : j_s \in \{2i_s, 2i_s+1\}, \\ & \quad s = 1,\ldots,N\}, \\ J' : &= J \backslash \{j_1 = 2i_1, \ldots, j_N = 2i_N\}. \end{aligned}$$

Then, the algorithms take the following form

Algorithm 1 $\mu(f^L) = Mf^L$ *(Encoding)*

for $k = L, \ldots, 1$
  for $i_1, \ldots, i_N = 0, \ldots, J_{k-1}$
    $f^{k-1}_{i_1,\ldots,i_N} = f^k_{2i_1,\ldots,2i_N}$
    for $(j_1,\ldots,j_N) \in J'$
      $e^k_{j_1,\ldots,j_N} = f^k_{j_1,\ldots,j_N} - (P^k_{k-1} f^{k-1})_{j_1,\ldots,j_N}$
    end
  end
end

Algorithm 2 $f^L = M^{-1}\mu(f^L)$ *(Decoding)*

for $k = 1, \ldots, L$
  for $i_1, \ldots, i_N = 0, \ldots, J_{k-1}$
    for $(j_1,\ldots,j_N) \in J'$
      $f^k_{j_1,\ldots,j_N} = (P^k_{k-1} f^{k-1})_{j_1,\ldots,j_N} + e^k_{j_1,\ldots,j_N}$
    end
    $f^k_{2i_1,\ldots,2i_N} = f^{k-1}_{i_1,\ldots,i_N}$
  end
end

In order to have control of the error after the process of compression a modified decoding algorithm is introduced in the next section.

3 Multiresolution based compression transforms with error control strategies

Multiresolution representations lead naturally to data-compression algorithms. The simplest one is obtained by setting to zero all scale coefficients which fall below a prescribed tolerance. Let us denote

$$\hat{e}^k_i = \mathbf{tr}(e^k_i; \epsilon_k) = \begin{cases} 0 & |e^k_i| \le \epsilon_k, \\ e^k_i & \text{otherwise}, \end{cases}$$

and refer to this operation as *truncation*. This type of data compression is used primarily to reduce the "dimensionality" of the data. A different strategy, which is used to reduce the digital representation of the data is *quantization*, which can be modeled by

$$\hat{e}^k_i = \mathbf{qu}(e^k_i; \epsilon_k) = 2\epsilon_k \cdot \text{ round } \left[\frac{e^k_i}{2\epsilon_k}\right],$$

where round $[\cdot]$ denotes the integer obtained by rounding. Observe that if $|e_i^k| < \epsilon_k \Rightarrow \mathbf{qu}(e_i^k; \epsilon_k) = 0$, and that in both cases

$$|e_i^k - \hat{e}_i^k| \leq \epsilon_k.$$

In what follows we present a N-dimensional extension of the one dimensional algorithms in [13],[9]. The decoding algorithm remains equal to Algorithm 2, while the encoding algorithm is modified to keep track of the total cumulative error. Thus, one ensures that the new algorithms are stable in the sense of expression (1).

Denoted by **pr** the compression process (truncation or quantization), and by $\tilde{e}^k_{j_1,\ldots,j_N} = f^k_{j_1,\ldots,j_n} - (P^k_{k-1}\hat{f}^{k-1})_{j_1,\ldots,j_N}$ an auxiliary term.

It is sufficient to define the following error control algorithm.

Algorithm 3 *(Modified encoding for point values)*

for $k = L, \ldots, 1$
  for $i_1, \ldots, i_N = 0, \ldots, J_{k-1}$
    $f^{k-1}_{i_1,\ldots,i_N} = f^k_{2i_1,\ldots,2i_N}$
  end
end
Set $\hat{f}^0 = f^0$
for $k = 1, \ldots, L$
  for $i_1, \ldots, i_N = 0, \ldots, J_{k-1}$
    for $(j_1, \ldots, j_N) \in J'$
      $\hat{e}^k_{j_1,\ldots,j_N} = \mathbf{pr}(\tilde{e}^k_{j_1,\ldots,j_N}, \epsilon_k)$
      $\hat{f}^k_{j_1,\ldots,j_N} = (P^k_{k-1}\hat{f}^{k-1})_{j_1,\ldots,j_N} + \hat{e}^k_{j_1,\ldots,j_N}$
    end
    $\hat{f}^k_{2i_1,\ldots,2i_N} = \hat{f}^{k-1}_{i_1,\ldots,i_N}$
  end
end

4 Numerical experiments

In order to apply the multiresolution algorithms in higher dimensions an experiment with the splitted paraboloid has been carried out.The function representing the cutted paraboloid is given by

$$f : [-1, 1] \times [-1, 1] \times [-1, 1] \times [-1, 1] \to R,$$

$$f(x) = \begin{cases} x_1^2 + x_2^2 + x_3^2 + x_4^2 & \text{if } x_1 \geq \frac{1}{2}, \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 + D & \text{else,} \end{cases} \tag{8}$$

where D is the size of the discontinuity. The larger D, the more abrupt the discontinuity is. The values taken for D are 10 and 100.

The present experiment is implemented in a uniform $65 \times 65 \times 65 \times 65$ grid X. The data $F := (f(x))_{x \in X}$ are compressed after descending two multiresolution levels and later reconstructed obtaining an approximation $\tilde{F}$ to the original data.The results of measuring the errors $||F - \tilde{F}||_s,\ s = 1, 2, \infty$ of this second experiment are presented in Tables 1, 2, 3, 4.

Algorithm	**EC**		
Method	Lagrange	ENO	PPH
Com. (%)	94.31	94.31	97.39
$\|\|F - \tilde{F}\|\|_1$	0	0	$3.00 \cdot 10^{-5}$
$\|\|F - \tilde{F}\|\|_2$	0	0	$2.92 \cdot 10^{-8}$
$\|\|F - \tilde{F}\|\|_\infty$	0	0	$9.77 \cdot 10^{-4}$
CPU time	80.59	181.56	94.31

Table 1: Norms of errors in compression and reconstruction processes with error control strategies for the splitted paraboloid in equation (8) with size of the discontinuity $D = 10$ for Lagrange, ENO and PPH reconstructions. Multiresolution levels $L = 2$.

Algorithm	**Without EC**		
Method	Lagrange	ENO	PPH
Com. (%)	94.31	94.31	97.39
$\|\|F - \tilde{F}\|\|_1$	$2.45 \cdot 10^{-7}$	$2.45 \cdot 10^{-7}$	$3.00 \cdot 10^{-5}$
$\|\|F - \tilde{F}\|\|_2$	$1.53 \cdot 10^{-7}$	$1.53 \cdot 10^{-7}$	$2.93 \cdot 10^{-8}$
$\|\|F - \tilde{F}\|\|_\infty$	0.63	0.63	$4.85 \cdot 10^{-3}$
CPU time	75.27	176.17	89.92

Table 2: Norms of errors in compression and reconstruction processes without error control strategies for the splitted paraboloid in equation (8) with size of the discontinuity $D = 10$ for Lagrange, ENO and PPH reconstructions. Multiresolution levels $L = 2$.

In Tables 1 and 2 it can be immediately appreciated the better behavior of the nonlinear methods with respect to the linear ones in presence of discontinuities, i.e, nonlinear stable reconstructions improve the performance of their linear counterparts in these cases. It can be also observed that error control algorithms apart of allowing to keep track of the cumulative error, they get better compression rates and do not increase the time cost too much, under 5% approximately in these experiments.

Algorithm	EC		
Method	Lagrange	ENO	PPH
Com.(%)	94.31	94.31	97.39
$\|\|F-\tilde{F}\|\|_1$	0	0	$3.00 \cdot 10^{-5}$
$\|\|F-\tilde{F}\|\|_2$	0	0	$2.93 \cdot 10^{-8}$
$\|\|F-\tilde{F}\|\|_\infty$	0	0	$9.77 \cdot 10^{-4}$
CPU time	74.36	167.69	84.30

Table 3: Norms of errors in compression and reconstruction processes with error control strategies for the splitted paraboloid in equation (8) with size of the discontinuity $D = 100$ for Lagrange, ENO and PPH reconstructions. Multiresolution levels $L = 2$.

Algorithm	Without EC		
Method	Lagrange	ENO	PPH
Com. (%)	94.31	94.31	97.39
$\|\|F-\tilde{F}\|\|_1$	$2.45 \cdot 10^{-6}$	$2.45 \cdot 10^{-6}$	$3.01 \cdot 10^{-5}$
$\|\|F-\tilde{F}\|\|_2$	$1.53 \cdot 10^{-5}$	$1.53 \cdot 10^{-5}$	$2.94 \cdot 10^{-8}$
$\|\|F-\tilde{F}\|\|_\infty$	6.25	6.25	$4.88 \cdot 10^{-3}$
CPU time	70.81	163.00	83.14

Table 4: Norms of errors in compression and reconstruction processes without error control strategies for the splitted paraboloid in equation (8) with size of the discontinuity $D = 100$ for Lagrange, ENO and PPH reconstructions. Multiresolution levels $L = 2$.

In Table 3 and 4 it can be seen that increasing the size of the discontinuity affects much more the linear Lagrange method and the nonstable nonlinear ENO method than the stable nonlinear method PPH. All the other previous comments are consistent also with this new experiment.

5 Conclusion

N-dimensional multiresolution algorithms for the point values discretization with and without error control strategies have been presented. While classical algorithms fixed the number of coefficients to be stored, error control algorithms allow to specify the quality in the reconstructed data. Apart of keeping track of the reconstruction error, these error control algorithms allow the application of nonlinear schemes that are not stable when applied directly without these strategies.

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