Application of Comparison Standards for Solving of Decision-Making Problems

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Abstract: - Proposed are four decision-making technologies based on the use of comparison standards: one - for optimal strategies in cooperative n-person games determination, the second – for summarizing the results of voting, the third - for processing of experts assessments and the fourth – for the use of comparison standards to solve extreme multi-criteria problems. The first one is an alternative to the use of guaranteeing strategies whereas the second approach can be an alternative to the use of traditional technologies, selecting winner by relative or absolute majority and the third one is a further development of a method of binary comparisons. The fourth direction is the alternative to the methods of lexicographical ordering of criteria and their weighted sum. The main provisions of the proposed approach are illustrated by examples.

Keywords: - matrix of game; comparison standard; prize of game; distance in the coordinate system; guaranteeing strategies, voting processing, experts estimations, multi-criteria problem.

1. Introduction

Solution of the matrix of an antagonistic two-person game with complete information and zero-sum as a rule is based on the one hand on the application of guaranteeing strategies, and on the other – on the ban on players to enter into coalitions [1], [2]. Below is proposed: a) to abandon both above conditions [3], [4]; b) to use the standards [4], [5] when searching for optimal strategies for players.

In other words this approach assumes that the players tend to negotiate, thus providing a specific procedure of a preliminary agreement, following which the players get a price of the game, the value of which may differ from that which is determined by guaranteeing strategies. As the interests of the players are opposite, the formal statement of the problem of determining the optimal prize of such game is multiobjective. The following symbols and definitions are used.

2 Symbols and definitions

M - matrix of game, which rows correspond to strategies of maximizing player and columns – of minimizing;

y_j - j-th strategy of minimizing player;

x_i - i-th strategy of maximizing the player;

h - max win of maximizing the player;

g - the minimum loss of minimizing player;

$$h = \max_i \max_j M(i, j);$$

$$g = \min_i \min_j M(i, j).$$

The values of “h” and “g” are the comparison standards that allow each player to evaluate the deviation of the current win / loss from the corresponding best value.

Other symbols and definitions are introduced further on in the article.

3 The use of standards in cooperative games

Below are described using standards technologies searching optimal pure strategies in relation to the co-operative games of two and "n" individuals. This approach differs from that proposed in [3] in that a solution is found in pure, and not in the mixed strategies.

3.1 The formal statement of the two players problem and its solution

Formally, the problem of searching for optimal pure strategies of players can be reduced to the problem of multiobjective discrete programming with Boolean variables:

$$\begin{cases} \sum_i \sum_j M(i,j) x_i y_j \to \max; \\ \sum_i \sum_j M(i,j) x_i y_j \to \min; \\ \sum_i x_i = 1; \\ \sum_j y_j = 1; \\ \forall i : x_i = 1,0; \\ \forall j : y_j = 1,0. \end{cases} \quad (1)$$

Using comparison standards for converting system (1) into a single-criterion problem, we get:

$$\begin{cases} \left(\sum_i \sum_j M(i,j) x_i y_j - h \right)^2 + \left(\sum_i \sum_j M(i,j) x_i y_j - g \right)^2 \to \min; \\ \sum_i x_i = 1; \\ \sum_j y_j = 1; \\ \forall i : x_i = 1,0; \\ \forall j : y_j = 1,0. \end{cases} \quad (2)$$

The solution of system (2) is defined by the following theorem:
Theorem 1: The optimum value of the game price, corresponding to (2), is determined by cell M (p, q) of the game matrix M, for which holds:

$$\left[M(p,q) - \frac{h+g}{2} \right]^2 = \min_i \min_j \left[M(i,j) - \frac{h+g}{2} \right]^2. \quad (3)$$

The proof of Theorem 1 is contained in Appendix 1.

Thus, with regard to the matrix M of

M =

2	10	8
12	6	3
4	9	14

antagonistic game Γ, whose rows correspond to the strategies of maximizing player and columns – to losses of minimizing one, the use of guarantying strategies leading to a price of game to be 9, optimal is combination of the third strategy of maximizing player and the second strategy of the minimizing player. However, if matrix M corresponds to the cooperative game, the use of comparison standards and (3) leads to the different results: as h=14 and g=2, value of the game equals to 8, thus meeting optimal combination of the first strategy of the maximizing player and of the third strategy of the minimizing one.

Note: The above approach does not allow one to choose strategies of gamers, if matrix M includes several cells, satisfying (3). Formally this case for two such cells M (p, q) and M (r, f) is described by the system:

$$\begin{cases} \exists p \neq r, \exists q \neq f : [M(p,q) - G]^2 = [M(r,f) - G]^2; \\ G = 0.5(h+g). \end{cases} \quad (4)$$

It is easy to see that such a case can be presented by matrix M1 below:

M1 =

1	10	8
12	6	3
4	9	13

As in this case value h=13, value g=1, optimal G value equals to 7, two sells M1(1, 3) and M1(2, 2) satisfy (3) and (4). In this case, there is an obvious need for additional conditions that will lead to a unique choice of gamers' strategies. An example of such conditions may be the selection of a pair of strategies, satisfying (3) and (4) and demonstrating minimum (or maximum) difference from the price of the game received with the guaranteeing strategies. In this case the latter are also used as comparison standards. Returning to matrix M1 above it is easy to show that the nearest optimal price of cooperative game to that of this game, determined by the guaranteeing strategies matches the contents of M1 (1, 3), and the most distant from it - the contents of M1 (2, 2).

The approach described in the previous section is summarized below to the case of n> 2 players. The ideology of this approach is close to the proposed by Shapley in [7]: the role of "expected wins" play the best values of wins for each player – comparison standards, but the difference is reflected by the following points:

- each player has the opportunity to make only a single move;
- the price of the game below is defined in a different way.

3.2 The formal statement of the problem of "n" players (n>2) and its solution

Below we study the effectiveness of the proposed above technique to find the optimum pure strategies in the zero-sum cooperative game of "n" players under the following conditions:
a) each i-th player has the opportunity to use only one of his m_i strategies (i = 1, 2, ..., n);
b) at each j-combination of strategies $(1 \le j \le m)$ win of a i-th player (i = 1, 2, ... n), is equal to $b_{i,j}$;

c) $$m = \prod_{i=1}^{n} m_i; \quad (5)$$

d) $$\forall j : \sum_i b_{i,j} = b. \quad (6)$$

Feature "d" means that the price of the game is fixed and its value is equal to "*b*", different combinations of players strategies correspond to different distributions of this value between them. This is not a strong restriction since it is easy to show that the game of this kind can be reduced to a search of pure strategies in any zero-sum game by adding dummy (n + 1)-th player for whom the following system is valid:

$$\begin{cases} b = \sum_{i \le n} \max_j b_{i,j}; \\ \forall j : b_{n+1,j} = b - \sum_{i \le n} b_{i,j} \end{cases} \quad (7)$$

Let a_i be the best winning value for each i-th player. Further, we assume that, in spite of the antagonistic nature of the game, it has the main feature of cooperative games: the players may agree on the distribution of "b" within their strategies. The aim is to find for each player such x_i wins values, for which is true:

- the difference between the values of a_i and x_i is minimal;
- $\sum_i x_i = b.$

In general, the formal statement of the problem is multiobjective:

$$\begin{cases} \forall i \le n : F_i = (a_i - x_i)^2 \to \min; \\ \sum_{i=1}^{n+1} x_i = b. \end{cases} \quad (8)$$

Summing in (8) goal functions, we obtain the single-criterion problem, the solution of which coincides with one of the Pareto - optimal system (8) solutions [3]. In this case, the system (8) is converted to:

$$\begin{cases} F = \sum_{i=1}^{n} (a_i - x_i)^2 \to \min; \\ \sum_{i=1}^{n+1} x_i = b. \end{cases} \quad (9)$$

The solution of system (9) is determined by the following theorem:
Theorem 2. System (9) optimal vector of variables is determined as follows:

$$\forall i \le n : x_i = a_i - \frac{4\left(\sum_{i \le n} a_i - b\right)}{n}. \quad (10)$$

The proof of Theorem 2 is contained in Annex 2.

3 Summarizing the results of voting by means of standards

Techniques in use today for voting results processing, such as the methods of relative and absolute majority [3], the Borda Count Method [8] and a number of others, do not guarantee an unambiguous result. For example, using the method of relative majority when counting the voting results shown in Table 1 below, we determine "a" as winner, whereas the use of the method of the absolute majority and the organization of elections in two rounds lead to the victory of "b". In the modified Borda Count Method each voter gives to each candidate (or alternative) set at the j-th place, j points. The winner is "c" – i.e. the one who got the minimum sum (see Table 1):

n_a = $1·6+3·7=27$;
n_b $=1·4+2·5+3·4=26$;
$n_c=1·3+2·8+3·2=25$.

Table 1.

Place	The number of voters			
	2	3	4	4
1	a	c	a	b
2	b	b	c	c
3	c	a	b	a

Because of these differences below we consider the use of processing of voting results technology, based on the comparison standards. The simplicity of this technology is determined by a simple choice of a comparison standard: it corresponds to the case where all electors will vote for one nominee. Suppose that there are "n" locations claimed by "m" candidates, each k-th candidate is associated with vector: $v_k=\{i_1^k, i_2^k, i_3^k, ..., i_n^k\},\ k=1,2,\dots,m,$ where i_j^k – the number of votes cast for that k-th candidate takes j-th place. The following equality is then apparent:

$$\forall j \neq q: \sum_k i_j^k = \sum_k i_q^k = w, \qquad (11)$$

where "w" - the total number of electoral votes. Obviously, the standard vector is equal to: v_s = {w, 0, 0,…., 0}. With reference to Table 1 vector v_s = {13,0,0}.

The winner is the q-th pretender, satisfying the system:

$$\begin{cases} L_k=\sqrt{(w-i_1^k)^2+\sum_{j=2}^{n}(i_j^k)^2}; \\ L_q=\min_k L_k. \end{cases} \qquad (12)$$

Search of the system (12) solution can be interpreted as the choice of a point "q" located closest to the "standard" in the n-dimensional Euclidean space. It is easy to see that in accordance with (12) "a" is the winner all in accordance with Table 1.

4 Processing of expert assessments using comparison standards

In cases where the numerical characteristics of objects are missing, their ranking is done on the basis of expert estimations. Let every expert or group of experts estimate a couple of objects "c" and "d" using only binary relations: object "c" featured object "d", object "c" is equivalent to the object "d" and object "c" is worse than "d" object. The aim is to rank the objects, i.e. the definition of such a permutation $\pi=\{i_1,i_2, \dots i_n\}$ of n objects, for which holds: if $k<j$, then i_k is better than i_j. Interpretation of such ranking by binary relations with the help of graph G(X, U), where objects correspond to vertices and relations - to arcs, is illustrated below in Fig. 1.

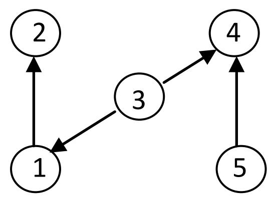

Fig. 1. Graph G(X,U)

In this case ranking corresponds to the distribution of vertices into layers (Fig. 2).

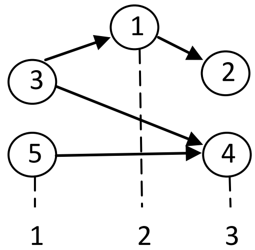

Fig. 2. Distribution of the graph G(X,U) vertices into layers.

It is easy to see that this ranking is not unique because it corresponds to the number of equivalent permutations of vertices (Table 2):

Table 2

i	i-th permutation
1	3,5,1,2,4
2	5,3,1,2,4
3	3,5,1,4,2
4	5,3,1,4,2

The situation can be improved by adding comparison standards - fictitious vertices "a" and "b": the first corresponds to the best object, the second - the worst (Fig. 3).

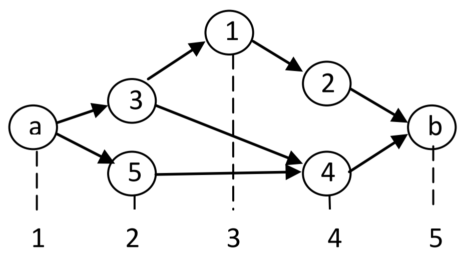

Fig. 3. Modified graph G'(X',U') and distribution of its' vertices into layers.

Let us denote the minimal number of arcs, leading from vertex "a" to the i-th vertex as r(a,i) and the maximal number of arcs, leading from vertex "i" to the vertex "b" - as r(i,b) (Table 3).

Table 3

Vertex	r(a,i)	r(i,b)
a	0	4
1	2	2
2	3	1
3	1	3
4	2	1
5	1	2
b	3	0

Define for each i-th vertex ratio $\eta(i) = r(a,i)/r(i,b)$ and create a permutation π of them in ascending order of this magnitude [4]: $\pi = \{3, 5, 1, 4, 2\}$. It is obvious that the use of comparison standards in solving this problem can significantly reduce the ambiguity of ranking.

5 Comparison standards in solving of extreme multi-criteria problems

Below, we consider the application of comparison standards to solve extreme multi-criteria problems in the form of:

$$\begin{cases} \forall i : F_i(\vec{X}) \to \text{extreme value}; \\ \forall j : \varphi_j(\vec{X}) \le b_j; \\ \forall \mathrm{k} : \mathrm{x_k} \in X_k; \quad \vec{X} = \{x_1,\ x_2,\ \ldots..,\ x_m.\}, \end{cases} \qquad (13)$$

where: $\vec{X}$ - vector of variables; X_k - the set of values that can accept *k*-th variable; $F_i(\vec{X})$ - *i*-th criterion; $\varphi_j(\vec{X})$ - *j*- th constraint; b_j – *j*-th constant.

Due to the sequence of proved in [6] theorems, the system (13) can be converted to the single-criterion form:

$$\begin{cases} \sum_i \left[F_i^{'}(\vec{X}) - A_i \right]^2 \to \min; \\ \forall \mathrm{i} : F_i^{'}(\vec{X}) = \dfrac{F_i(\vec{X}) - \min_{\vec{X}} F_i(\vec{X})}{\max_{\vec{X}} F_i(\vec{X}) - \min_{\vec{X}} F_i(\vec{X})}; \\ \forall j : \varphi_j(\vec{X}) \le b_j; \\ \forall \mathrm{k} : \mathrm{x_k} \in X_k; \quad \vec{X} = \{x_1,\ x_2,\ \ldots..,\ x_m.\}, \end{cases} \qquad (14)$$

where A_i is the best value (comparison standard) of the normalized i-th criterion in system (13).

The thing is that system (14) optimal vector of variables coincides with one of the Pareto-optimal vectors of variables of system (13).

As an example, below is considered 3x3 assignment problem in which each combination of job / worker is associated with two numbers: the first - the cost of this job performing by the corresponding worker, the second - the appropriate time. In the following matrices M_1 and M_2 lines correspond to the workers, and the columns – to the jobs, cell M_1 (i, j) contains the cost of j-th job performing by i-th worker, and matrix M_2 eponymous cell - the relevant runtime.

M_1=

5	7	3
2	4	6
8	1	9

M_2=

50	30	90
80	60	40
20	70	10

Corresponding to (13) and using M_1 and M_2 formal statement of this assignment problem looks as follows:

$$\begin{cases} F_1 = \sum_i \sum_j M_1(i,j) x(i,j) \to \min; \\ F_2 = \max_i \max_j M_2(i,j) x(i,j) \to \min; \\ \forall i : \sum_j x(i,j) = 1; \\ \forall j : \sum_i x(i,j) = 1; \\ \forall i, \forall j : x(i,j) = 1,0. \end{cases} \quad (15)$$

Since the lower boundaries of F_1 and F_2 are zero, whereas the upper boundary of F_1 is equal to 21 and the upper boundary of F_2 is equal to 90, going from (15) to a corresponding to (14) single-criterion formulation, we obtain:

$$\begin{cases} F = \left[\frac{1}{21} \sum_i \sum_j M_1(i,j) x(i,j) \right]^2 + \\ + \left[\frac{1}{90} \max_i \max_j M_2(i,j) x(i,j) \right]^2 \to \min; \\ \forall i : \sum_j x(i,j) = 1; \\ \forall j : \sum_i x(i,j) = 1; \\ \forall i, \forall j : x(i,j) = 1,0, \end{cases} \quad (16)$$

It is easy to see that optimal for systems (15) and (16) is the vector of variables, containing: $x_{1,1}=x_{2,3} = x_{3,2} =1$; $x_{1,2} = x_{1,3} = x_{2,1} = x_{2,2} = x_{3,1}= x_{3,3}=0$. Corresponding values of goal functions in (15) and (16) are: $F_1 = 12$; $F_2 = 70$; $F = 0.9314$.

6 Conclusions

Comparison standards are not only a powerful tool for solving problems of operations research, they also allow us to formulate new, not previously considered problems, reduce the ambiguity of the solutions obtained, and eliminate the need for additional data to solve multicriteria problems.

A few words about the applications of the proposed approaches. Antagonistic matrix two-person game with perfect information and zero-sum may be a classic example of pricing in relation to a product that can be made in several departments of the same company (maximizing player) and consumed by various departments of another company (minimizing player). If this is enough to organize the delivery of this product by only one division of the manufacturer, for a single division of consumer (in the terminology of [9] - the market of two persons), it is obvious that the decision may be obtained in pure strategies using guaranteeing strategies [2]. It is understood that the players are not inclined to cooperate, which is not always the case, so above is proposed a cooperative approach to finding prices for antagonistic games of this type, being alternative to guaranteeing strategies using. Thus, the proposed approach allows us on the one hand to expand the pricing tools, and on the other - to increase the number of persons involved in the pricing process.

With regard to the results of voting processing technology described above, we can say, that using it person can be perceived as an optimist: pessimist would rather choose a strategy corresponding to a point at maximum distance from the origin.

Finally, if the goal functions of the system (13) use identical measurement units, normalization in (14) is not necessary.

References

[1] J. Von Neumann, O. Morgenstern. Theory of Games and Economic Behavior. M .: *"Science"*, 1970. (russ).

[2] Davidov E.G. *Methods and Models of the Theory of Antagonistic Games*. Moscow

University publishing house, 1978, 208 p. (russ).

[3] Joseph J. Moder, Salah E. Elmaghraby. *Handbook of Operations Research. Vol. 1: Foundations and Fundamentals*. Van Nostrand Reinhold Company, 1978, p. 513 – 548.

[4] Groppen V.O. Principles of decision-making with the help of standards // *Automatika i telemechanika*, # 4, 2006, p. 167 – 184 (russ).

[5] Groppen V.O. New Solution Principle for Multi-criteria Problems Based on Comparison Standards: Models, Algorithms, Applications // *Applications to Industrial and Societal Problems. CIMNE*, Barcelona, Spain, 2008, p. 201 - 209.

[6] Groppen V.O. New Solution Principles of Multi-Criteria Problems Based on Comparison Standards, www.arxiv.org/ftp/math/papers/0501/0501357.pdf , 2004.

[7] Dresher M., Shapley L. S., Tucker A.W., Eds., Advances in Game Theory, *Annals of Mathematics Study* No 52, Princeton University Press, Princeton, New Jersey, 1964.

[8] Emerson, Peter. *Designing an All-Inclusive Democracy, Part 1,* Springer Verlag, 2007, p. 15-38.

[9] John L. Daly. *Pricing for Profitability: Activity-Based Pricing for Competitive Advantage*. Wiley, 2001.

Appendix 1

Proof of the Theorem 1.

The proof is based on the graphic interpretation of the choice of the optimum price of the game. Thus, matrix M of two-persons cooperative game Γ corresponds to Fig. 4, on the horizontal axis of which are projected winnings of maximizing player, and on the ordinate axis – minimizing player losses.

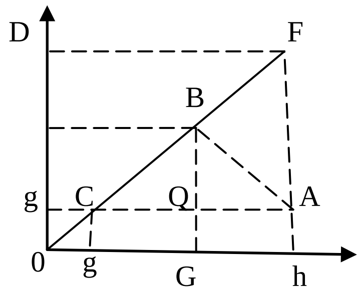

Fig. 4. Graphic interpretation of the game matrix

The comparison standard satisfying both players corresponds to point "A", whereas the points showing the actual win / loss of players belong to the segment CF of bisector of the right angle D0h. It is obvious that the shortest distance between point "A" and the bisectrix 0F is determined by the length of the segment AB perpendicular to it, which implies optimal price of game G. The latter can differ from the price of the same game, obtained using guaranteeing strategies. Since CBA triangle is a isosceles and rectangular, since the length of the hypotenuse CA is known and equal to (h - g), cathetus length is equal to BA. But at the same time line segment BA is the hypotenuse of an isosceles right triangle QBA, allowing us to determine the length of his cathetus z = 0.5 (h - g). Since the lengths of CQ, QB and QA coincide, the length of the segment 0G, determines the optimum price of the game which is equal to z + g:

$$z + g = 0.5\,(h + g). \qquad (17)$$

Needless to say that both players should follow the strategies defined by the cell M (p,q), the content of which is closest to the price of game, corresponding to the right-hand side of (17). The latter is determined by the expression (3). The theorem 1 is proved.

Appendix 2

The proof of Theorem 2. Using the method of Lagrange multipliers, on the basis of (9) we can obtain the following system:

$$\begin{cases} \forall i \le n : x_i - a_i = -\lambda / 2; \\ \sum_{i \le n+1} x_i = b. \end{cases} \qquad (18)$$

Considering that $a_{n+1} = 0$, solution of system (18) is:

$$\lambda = 2\left(\sum_{i \le n} a_i - b\right) / n;\ \forall i \le n : x_i = a_i - 2\lambda. \qquad (19)$$

It is obvious that systems (10) and (19) coincide. So, theorem 2 is proved.