Oscillation Conditions for a Class of Lienard Equation

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Abstract: - The aim of this paper is to investigate oscillatory properties of a class of a generalized Lienard equation .Several oscillation conditions are presented that improve the results obtained in the literature. The results obtained here are new and further improve and complement some known results in the literature. We extend and improve the oscillation criteria of several authors. Moreover, two examples are presented to demonstrate the main results.

Key-Words: - Oscillatory, Lienard equation, Second order differential equations, Non-Linear.

1 Introduction

In this paper we are concerned with the oscillation of a class of Lienard equation of the form

$$\ddot{x}(t) + f\big(x(t)\big)(\dot{x}(t))^2 + g\big(x(t)\big)\,\dot{x}(t) = 0, \quad (1.1)$$

where $f\big(x(t)\big)$ and $g\big(x(t)\big)$ are continuously differentiable functions on R .

Many criteria have been found which involves the behavior of the integral of a combination of the coefficients of second order nonlinear differential. This approach has been motivated by many authors (for example see [1],[2],[3],[4],[5],[6],[7] ,[8],[9] ,[10],[11],[13],[14]and [15] and the authors therein).Which often studied by reducing the problem to the estimate of suitable first integral Our attention is concentrated to such solutions $x(t)$ of (1.1) which exists on the interval [β,∞) for β > α. Where α is non negative real number.

Definition 1: A nontrivial solution $x(t)$ of differential equation (1.1) is said to be oscillatory if it has arbitrarily large zeroes on [β,∞), for β > α otherwise it said to be " non oscillatory.

It is well known (see [12] Reid) that either all solutions of (1.1) are nonoscillatory, or all the solutions are oscillatory . In the former case, we call the differential (1.1) nonoscillatory and in the later case is oscillatory.

2 Main Results

We prove the following theorem

Theorem 1: If

$$\lim_{x\to\infty}\left(-\int_{\beta}^{\infty}\left\{(f(y))^2 + \frac{df(y)}{dy}\right\}ds\right) = \infty, \quad (2.1)$$

and

$$\lim_{x\to\infty}\left(\int_{\beta}^{\infty}\left(\frac{(g(y))^2}{4((f(y))^2 +)} + \frac{df(y)}{dy}\right)ds\right) = \infty. \quad (2.2)$$

Then the differential equation (1.1) is oscillatory.

Proof: Let $x(t)$ be a nonoscillatory solution of (1.1) on the interval [α,∞), , without loss of generality its solution can be supposed such that $x(t) > 0$ on [α,∞).

We define

$$w(t) = \dot{x}(t)(f(x(t))^{-1}.$$

Then $w(t)$ is well defined and satisfies the equation

$$\dot{w}(t) = -\left(\Big(f\big(x(t)\big)\Big)^2 + \frac{df\big(x(t)\big)}{dx}\right)\big(w(t)\big)^2 - g\big(x(t)\big)\,w(t), \quad (2.3)$$

rewriting equation (2.3) we have

$$\dot{w}(t) = -\left\{\left(\Big(f\big(x(t)\big)\Big)^2 + \frac{df\big(x(t)\big)}{dx}\right)^{\frac{1}{2}} w(t) + \frac{g\big(x(t)\big)}{2\left(\Big(f\big(x(t)\big)\Big)^2 + \frac{df\big(x(t)\big)}{dx}\right)^{\frac{1}{2}}}\right\}^2 + \frac{\Big(g\big(x(t)\big)\Big)^2}{4\left(\Big(f\big(x(t)\big)\Big)^2 + \frac{df\big(x(t)\big)}{dx}\right)}.$$

Integrating both sides of the above equation from α to *t* we get

$$w(t) = w(\alpha) - \int_{\alpha}^{\infty}\left\{\left(\left(f(y(s))\right)^2 + \frac{df(y(s))}{dy}\right)^{\frac{1}{2}} w(s) + \frac{g(y(s))}{2\left(\left(f(y(s))\right)^2 + \frac{df(y(s))}{dy}\right)^{\frac{1}{2}}}\right\}^2 ds + \int_{\alpha}^{\infty} \frac{\left(g(y(s))\right)^2}{4\left(\left(f(y(s))\right)^2 + \frac{df(y(s))}{dy}\right)} ds. \quad (2.4)$$

Using the hypotheses (2.1) of the theorem there exist β >α we get

$$w(t) \geq -\int_{\beta}^{\infty}\left\{\left(\left(f(y(s))\right)^2 + \frac{df(y(s))}{dy}\right)^{\frac{1}{2}} w(s)\right\}^2 ds.$$

Define

$$R(t) = -\int_{\beta}^{\infty}\left\{\left(\left(f(y(s))\right)^2 + \frac{df(y(s))}{dy}\right)^{\frac{1}{2}} w(s)\right\}^2 ds. \quad (2.5)$$

Thus $w(t) \geq R(t)$.

Now differentiating equation (2.5) with respect to t we get

$$\dot{R}(t) \geq -\left(\left(f(y(s))\right)^2 + \frac{df(y(s))}{dy}\right)\left(R(t)\right)^2.$$

Therefor

$$-\left(\left(f(y(s))\right)^2 + \frac{df(y(s))}{dy}\right) \leq \frac{\dot{R}(t)}{\left(R(t)\right)^2}.$$

Integrating both sides of this inequality with respect to t (with t replaced by s) from β to t for t > β we get

$$-\int_{\beta}^{\infty}\left(\left(f(y(s))\right)^2 + \frac{df(y(s))}{dy}\right) ds \leq \frac{1}{R(\beta)}.$$

Which contradicts the hypothesis of the theorem. Hence the differential equation (1.1) is oscillatory. This completes the proof.

Theorem 2: If

$$\lim_{t\to\infty}\int_{\beta}^{t} \frac{(g(y(s)))^2}{\left(f(y(s))g(y(s)) + \frac{dg(y(s))}{dy}\right)} ds = \infty, \quad (2.6)$$

and

$$-\lim_{t\to\infty}\int_{\beta}^{t}\left(f(y(s))g(y(s)) + \frac{dg(y(s))}{dy}\right) = \infty. \quad (2.7)$$

Then the differential equation (1.1) is oscillatory.

Proof: Let $x(t)$ be a nonoscillatory solution of (1.1) on the interval [α,∞), , without loss of generality its solution can be supposed such that $x(t)$> 0 on [α,∞).

We define

$$w(t) = \dot{x}(t)\left(g(x(t))\right)^{-1}.$$

Then $x(t)$ is well defined and satisfies the equation

$$\dot{w}(t) = -\left(f(y(t))g(y(t)) + \frac{dg(y(t))}{dy}\right)\left(w(t)\right)^2 - g(y(t))w(t). \quad (2.8)$$

Rewriting equation (2.8) we have

$$\dot{w}(t) = -\left(\left(f(y(t))g(y(t)) + \frac{dg(y(t))}{dy}\right)^{\frac{1}{2}} w(t) + \frac{g(y(t))}{2\left(f(y(t))g(y(t)) + \frac{dg(y(t))}{dy}\right)^{\frac{1}{2}}}\right)^2 + \frac{\left(g(y(t))\right)^2}{4\left(f(y(t))g(y(t)) + \frac{dg(y(t))}{dy}\right)}.$$

Integrating both sides of this equation from α to t we get

$$w(t) = w(\alpha) - \int_{\alpha}^{t}\left(\left(f(y(s))g(y(s)) + \frac{dg(y(s))}{dy}\right)^{\frac{1}{2}} w(t) + \frac{g(y(s))}{2\left(f(y(s))g(y(s)) + \frac{dg(y(s))}{dy}\right)^{\frac{1}{2}}}\right)^2 ds + \int_{\alpha}^{t} \frac{\left(g(y(s))\right)^2}{4\left(f(y(s))g(y(s)) + \frac{dg(y(s))}{dy}\right)} ds. \quad (2.9)$$

Using the hypothesis (2.6) of the theorem 2 there exist β > α such that

$$w(t) \geq -\int_{\alpha}^{t}\left(\left(f(y(s))g(y(s)) + \frac{dg(y(s))}{dy}\right)^{\frac{1}{2}} w(t) + \frac{g(y(s))}{2\left(f(y(s))g(y(s)) + \frac{dg(y(s))}{dy}\right)^{\frac{1}{2}}}\right)^2 ds.$$

Define

$$Z(t) - \int_{\alpha}^{t}\left(\left(f(y(s))g(y(s)) + \frac{dg(y(s))}{dy}\right)^{\frac{1}{2}} w(t) + \frac{g(y(s))}{2\left(f(y(s))g(y(s)) + \frac{dg(y(s))}{dy}\right)^{\frac{1}{2}}}\right)^2 ds. \quad (2.10)$$

Thus $w(t) \geq Z(t)$.

Now differentiating equation (2.10) with respect to *t* we get

$$\dot{Z}(t) \geq -\left(f(y(t))g(y(t)) + \frac{dg(y(t))}{dy}\right)\left(Z(t)\right)^2.$$

Therefor

$$-\left(f(y(t))g(y(t)) + \frac{dg(y(st))}{dy}\right) \leq \frac{\dot{Z}(t)}{\left(Z(t)\right)^2}$$

Integrating both sides of this inequality with respect to *t* (with *t* replaced by *s*) from β to t for $t > \beta$ we get

$$-\int_{\beta}^{t}\left(f(y(s))g(y(s)) + \frac{dg(y(s))}{dy}\right) ds \leq \frac{1}{Z(\beta)} - \frac{1}{Z(t)},$$

we conclude that

$$\lim_{t\to\infty}\left(-\int_{\beta}^{t}\left(f(y(s))g(y(s)) + \frac{dg(y(s))}{dy}\right) ds\right) \leq \frac{1}{Z(\beta)}.$$

Which contradicts the hypothesis of the theorem. Hence the differential equation (1.1) is oscillatory.

This completes the proof.

3 Examples

The following examples illustrate the applicability of the theorems.

Example 1: The applicability of theorem 1.

Consider the second nonlinear order differential equation

$$\ddot{x}(t) + cothx(t)\,(\dot{x}(t))^2 + e^{x(t)}\dot{x}(t) = 0, \quad (3.1)$$

for this differential equation we have $f(x(t)) = cothx(t)$ and $g(x(t)) = e^{x(t)}$.

To show the applicability the hypothesis (2.1) of theorem 1

$$\lim_{x\to\infty}\left(-\int_{\beta}^{\infty}\left\{(f(y))^2+\frac{df(y)}{dy}\right\}ds\right)$$
$$=-\lim_{t\to\infty}\int_{\beta}^{t}(coth^2y-csch^2y)ds$$
$$=\lim_{t\to\infty}\int_{\beta}^{t}-ds=\lim_{t\to\infty}s]_{\beta}^{t}=\infty\,.$$

To show the applicability of the hypothesis (2.2) of theorem 1

$$\lim_{x\to\infty}\left(\int_{\beta}^{\infty}(\frac{(g(y))^2}{4((f(y))^2+)}+\frac{df(y)}{dy})ds\right)$$
$$=\lim_{t\to\infty}\int_{\beta}^{t}\frac{e^{2y}}{4(coth^2y-csch^2y)}\;ds$$
$$=-\lim_{t\to\infty}\frac{e^{y(s)}}{8}\Big]_{\beta}^{t}=-\infty\,.$$

Therefore the theorem implies that the differential equation is oscillatory.

Example 2:

The applicability of theorem 2.

Consider the second nonlinear order differential equation

$$\ddot{x}(t)+\frac{2x(t)-1}{1+(x(t))^2}\;(\dot{x}(t))^2-(1+(x(t))^2)\dot{x}(t)=0,\qquad(3.2)$$

for this differential equation we have $f\big(x(t)\big)=\frac{2x(t)-1}{1+(x(t))^2}$ and $g\big(x(t)\big)=-(1+(x(t))^2)$.

To show the applicability the hypothesis (2.6) of theorem 2

$$\lim_{t\to\infty}\int_{\beta}^{t}\frac{(g(y(s)))^2}{\left(f\big(y(s)\big)g\big(y(s)\big)+\frac{dg(y(s))}{dy}\right)}ds$$
$$=\lim_{t\to\infty}\int_{\beta}^{t}\frac{(1+(y(s))^2)^2}{\left(\frac{2y(s)-1}{1+(y(s))^2}(-1-(y(s))^2)+2y(s)\right)}\;ds$$
$$=\lim_{t\to\infty}\int_{\beta}^{t}(1+(y(s))^2)^2ds=\infty.$$

And the applicability of the hypothesis (2.7) of theorem 2

$$-\lim_{t\to\infty}\int_{\beta}^{t}\left(f\big(y(s)\big)g\big(y(s)\big)+\frac{dg(y(s))}{dy}\right)$$
$$=\lim_{t\to\infty}\int_{\beta}^{t}\left(\frac{2y(s)-1}{(1+(y(s))^2)}(-1-(y(s))^2)+2y(s)\right)ds$$
$$=-\lim_{t\to\infty}\int_{\beta}^{t}(-1)\;ds=\infty\,.$$

Hence the theorem is applicable.

4 Conclusion

In this paper, we have discussed some conditions for oscillation of class of lienard equation (1,1), where $f\big(x(t)\big)$ and $g\big(x(t)\big)$ are continuously differentiable functions on R . Under certain assumptions, we have derived a complete characterization of an eventually positive solution $x(t)$ of (1.1). By using generalized Riccati techniques, we have proved that under a number of conditions that every solution $x(t)$ of (1.1) is oscillatory. Also, we have given two examples to illustrate the obtained results.

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