Obtaining Consensus of Singular Multi-agent Linear Dynamic Systems

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Abstract: There is much literature about the study of the consensus problem in the case where the dynamics of the agents are linear systems, but the problem is still open for the case where the dynamic of the agents are singular linear systems. In this paper the consensus problem for singular multi-agent systems is considered, in which all agents have an identical linear dynamic mode that can be of any order. A generalization to the case all agents are of the same order but do not have the same linear dynamic is also analyzed.

Key–Words: Singular multi-agent systems, consensus, control.

1 Introduction

It is well known the great interest created in many research communities about the study of control multi-agents system, as well as the increasing interest in distributed control and coordination of networks consisting of multiple autonomous (potentially mobile) agents. There are an amount of literature as for example [6, 17, 21, 23, 15, 20]. It is due to the multi-agents appear in different areas as for example in consensus problem of communication networks [17], or formation control of mobile robots [4].

Jinhuan Wang, Daizhan Cheng and Xiaoming Hu in [21], study the consensus problem in the case of multiagent systems in which all agents have an identical linear dynamics and this dynamic is a stable linear system. M.I. García-Planas in [6], generalize this result to the case where the dynamic of the agents are controllable.

Despite the overall progress some problems of the consensus theory still remain unexplored for the agents with dynamics defined as a singular linear systems. In this paper multiagent singular systems consisting of $k+1$ agents with dynamics

$$\left.\begin{array}{rl} E_1\dot{x}^1 & = A_1x^1 + B_1u^1 \\ \vdots & \\ E_k\dot{x}^k & = A_kx^k + B_ku^k \end{array}\right\}$$

where $E_i, A_i \in M_n(\mathbb{C})$, $B_i \in M_{n\times 1}(\mathbb{C})$, $C_i \in M_{1\times n}(\mathbb{C})$, for the cases

i) all agents have an identical linear dynamic mode, (i.e. E_i, $A_i = A$, $B_i = B$ for all i).

ii) all agents are of the same order but do not have the same linear dynamic.

are considered.

Wei Ni and Daizhan Cheng in [14], analyze the standard case where $E_1 = \ldots = E_k = I_n$, $A_1 = \ldots = A_k$ and $B_1 = 0$, $B_2 = \ldots = B_k$ this particular case has practical scenarios as the flight of groups of birds. It is obvious that in this case the mechanic of the first system is independent of the others, then consensus under a fixed topology can be easily obtained and it follows from the motion of the first equation. This consensus problem is known as leader-following consensus problem ([14], [10]).

2 Preliminaries

2.1 Algebraic Graph theory

We consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of order k with the set of vertices $\mathcal{V} = \{1, \ldots, k\}$ and the set of edges $\mathcal{E} = \{(i,j) \mid i,j \in \mathcal{V}\} \subset \mathcal{V} \times \mathcal{V}$.

Given an edge (i,j) i is called the parent node and j is called the child node and j is in the neighbor of i, concretely we define the neighbor of i and we denote it by $\mathcal{N}_i$ to the set $\mathcal{N}_i = \{j \in \mathcal{V} \mid (i,j) \in \mathcal{E}\}$.

The graph is called undirected if verifies that $(i,j) \in \mathcal{E}$ if and only if $(j,i) \in \mathcal{E}$. The graph is called connected if there exists a path between any two vertices, otherwise is called disconnected.

Associated to the graph we consider the matrix $G = (g_{ij})$ called (unweighted) adjacency matrix defined as follows $g_{ii} = 0$, $g_{ij} = 1$ if $(i,j) \in \mathcal{E}$, and $g_{ij} = 0$ otherwise.

In a more general case we can consider that a weighted adjacency matrix is $G = (g_{ij})$ with $g_{ii} = 0$, $g_{ij} > 0$ if $(i,j) \in \mathcal{E}$, and $g_{ij} = 0$ otherwise.

The Laplacian matrix of the graph is

$$\mathcal{L} = (l_{ij}) = \begin{cases} |\mathcal{N}_i| & \text{if } i = j \\ -1 & \text{if } j \in \mathcal{N}_i \\ 0 & \text{otherwise} \end{cases}$$

Remark 1 *i) If the graph is undirected we have that the matrix $\mathcal{L}$ is symmetric, then there exist an orthogonal matrix P such that $P\mathcal{L}P^t = \mathcal{D}$.*

ii) If the graph is undirected then 0 is an eigenvalue of $\mathcal{L}$ and $\mathbf{1}_k = (1,\dots,1)^t$ is the associated eigenvector.

iii) If the graph is undirected and connected the eigenvalue 0 is simple.

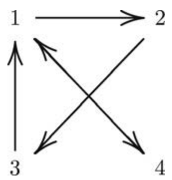

Figure 1: Undirected connected graph

For more details about graph theory see [9] and [22] for example.

2.2 Kronecker product

Remember that given two matrices $A = (a_{ij}) \in M_{n\times m}(\mathbb{C})$ and $B = (b_{ij}) \in M_{p\times q}(\mathbb{C})$ the Kronecker product $A \otimes B$ is defined as follows.

Definition 2 *Let $A = (a^i_j) \in M_{n\times m}(\mathbb{C})$ and $B \in M_{p\times q}(\mathbb{C})$ be two matrices, the Kronecker product of A and B, write $A \otimes B$, is the matrix*

$$A\otimes B = \begin{pmatrix} a^1_1 B & a^1_2 B & \dots & a^1_m B \\ a^2_1 B & a^2_2 B & \dots & a^2_m B \\ \vdots & \vdots & & \vdots \\ a^n_1 B & a^n_2 B & \dots & a^n_m B \end{pmatrix} \in M_{np\times mq}(\mathbb{C})$$

Kronecker product verifies the following properties

1) $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$
2) $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$
3) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$
4) $(A \otimes B)^t = A^t \otimes B^t$
5) If $A \in Gl(n;\mathbb{C})$ and $B \in Gl(p;\mathbb{C}))$, then $A \otimes B \in Gl(np;\mathbb{C}))$ and $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$
6) If the products AC and BD are possible, then $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

Corollary 3 *The vector $\mathbf{1}_k \otimes v$ is an eigenvector corresponding to the zero eignevalue of $\mathcal{L} \otimes I_n$.*

Proof:

$$(\mathcal{L} \otimes I_n)(\mathbf{1}_k \otimes v) = \mathcal{L}\mathbf{1}_k \otimes v = 0 \otimes v = 0$$

□

Consequently, if $\{e_1, \dots, e_n\}$ is a basis for $\mathbb{C}^n$, then $\mathbf{1}_k \otimes e_i$ is a basis for the nullspace of $\mathcal{L} \otimes I_n$.

Associated to the Kronecker product, can be defined the vectorizing operator that transforms any matrix A into a column vector, by placing the columns in the matrix one after another.

Definition 4 *Let $X = (x^i_j) \in M_{n\times m}(\mathbb{C})$ be a matrix, and we denote $x_i = (x^1_i, \dots, x^n_i)^t$ for $1 \le i \le m$ the i-th column of the matrix X. We define the vectorizing operator vec, as*

$$\begin{array}{rcl} vec : M_{n\times m}(\mathbb{C}) & \longrightarrow & M_{nm\times 1}(\mathbb{C}) \\ X & \longrightarrow & \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \end{array}$$

Obviously, vec is an isomorphism.

For more information see P. Lancaster, M. Tismenetsky in [11], or J.W. Brewer in [1] for example.

2.3 Controllability and stability

Definition 5 *We recall that a system is called controllable (see [3]) if, for any $t_1 > 0$, $x(0) \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$, there exists a control input $u(t)$ such that $x(t_1) = w$.*

This definition requires only that any initial state $x(0)$ can be steered to any final state x_1 at time t_1. However, the trajectory of the dynamical system between 0 and t_1 is not specified. Furthermore, there is no constraints posed on the control vector $u(t)$ and the state vector $x(t)$.

An equivalent definition is given by the following result

Theorem 6 ([3]) *The system $E\dot{x} = Ax + Bu$ is controllable if and only if*

$$\begin{array}{l}\operatorname{rank}\begin{pmatrix} E & B\end{pmatrix} = n,\\ \operatorname{rank}\begin{pmatrix} \lambda E - A & B\end{pmatrix} = n, \text{ for all } \lambda \in \mathbb{C}.\end{array}$$

This result is a generalization of a similar one given for linear systems, (for more details see [5]).

Proposition 7 *A necessary condition for controllability is that the system be standardizable.*

Theorem 8 ([7]) *The system $E\dot{x} = Ax + Bu$ is controllable if and only if the rank of the matrix*

$$\begin{pmatrix} E & 0 & 0 & \dots & 0 & B & 0 & 0 & \dots & 0 & 0\\ A & E & 0 & \dots & 0 & 0 & B & 0 & \dots & 0 & 0\\ 0 & A & E & \dots & 0 & 0 & 0 & B & \dots & 0 & 0\\ & & \ddots & \ddots & & & & & \ddots & & \\ 0 & 0 & 0 & \dots & E & 0 & 0 & 0 & \dots & B & 0\\ 0 & 0 & 0 & \dots & A & 0 & 0 & 0 & \dots & 0 & B\end{pmatrix}$$
$$\in M_{n^2\times((n-1)n+nm)}(\mathbb{C})$$

is n^2

Corollary 9 *Suppose that E is an invertible matrix then, the system $E\dot{x} = Ax+Bu$ is controllable if and only if, the system $\dot{x} = E^{-1}Ax + E^{-1}Bu$ is controllable.*

Proof:

$$\operatorname{rank}\begin{pmatrix} E & 0 & 0 & \dots & 0 & B & 0 & 0 & \dots & 0 & 0\\ A & E & 0 & \dots & 0 & 0 & B & 0 & \dots & 0 & 0\\ 0 & A & E & \dots & 0 & 0 & 0 & B & \dots & 0 & 0\\ & & \ddots & \ddots & & & & & \ddots & & \\ 0 & 0 & 0 & \dots & E & 0 & 0 & 0 & \dots & B & 0\\ 0 & 0 & 0 & \dots & A & 0 & 0 & 0 & \dots & 0 & B\end{pmatrix} =$$
$$\operatorname{rank}\begin{pmatrix} I_n & & & & & \\ & \ddots & & & & \\ & & I_n & & & \\ & & & (E^{-1}A)^{n-1}B & \dots & (E^{-1}A)B & B\end{pmatrix}.$$

□

The controllability indices can be computed in the following manner.

We consider the following sequences of ranks r_i of matrices

$$M_i \in M_{(i+1)n\times(in+(i+1)m)}(\mathbb{C}).$$

$$M_0 = \begin{pmatrix} B\end{pmatrix},$$

$$M_1 = \begin{pmatrix} E & B & 0\\ A & 0 & B\end{pmatrix},$$

$$M_2 = \begin{pmatrix} E & 0 & B & 0 & 0\\ A & E & 0 & B & 0\\ 0 & A & 0 & 0 & B\end{pmatrix},$$

$$\vdots$$

$$M_\ell = \begin{pmatrix} E & 0 & 0 & \dots & 0 & B & 0 & \dots & 0 & 0\\ A & E & 0 & \dots & 0 & 0 & B & \dots & 0 & 0\\ & & \ddots & \ddots & & & & \ddots & & \\ 0 & 0 & 0 & \dots & E & 0 & 0 & \dots & B & 0\\ 0 & 0 & 0 & \dots & A & 0 & 0 & \dots & 0 & B\end{pmatrix}.$$

and, we define the following collection of ρ-numbers that permit to deduce the controllability indices of a controllable triple.

Definition 10 *Let r_i be the ranks of the matrices M_i,*

$$r_i = \operatorname{rank} M_i$$

.

Then, we define the ρ_i numbers as:

$$\begin{aligned}\rho_0 &= r_0\\ \rho_1 &= r_1 - r_0 - n\\ \rho_2 &= r_2 - r_1 - n\\ &\vdots\\ \rho_s &= r_{s-1} - r_s - n.\end{aligned}$$

It is easy to prove the following proposition.

Proposition 11 *The controllability indices $[k_1, \dots, k_p]$ of a controllable singular system, are the conjugate partition of $[\rho_0, \rho_1, \dots, \rho_s]$.*

Definition 12 *The system $E\dot{x} = Ax + Bu$ is called asymptotically stable if and only if all finite eigenvalues λ_i , $i = 1, \dots n_i$, of the matrix pencil $(\lambda E - A)$ have negative real parts.*

Definition 13 *The system $E\dot{x} = Ax + Bu$ is called asymptotically stabilizable if and only if all finite λ such that* $\operatorname{rank}\begin{pmatrix}\lambda_i E - A & B\end{pmatrix} < n$ *have negative real parts.*

Remark 14 *All controllable systems are stabilizable but the converse is false.*

It is important the following result

Theorem 15 *a) The system $E\dot{x} = Ax + Bu$ is stabilizable if and only if there exist some feedbacks F_E and F_A such that the close loop system $(E - BF_E)\dot{x} = (A - BF_A)x$ is asymptotically stable*

b) Suppose rank $\begin{pmatrix} E & B \end{pmatrix} = n$ *then the system $E\dot{x} = Ax + Bu$ is stabilizable if and only if there exist some feedbacks F_E and F_A such that $(E - BF_E)^{-1}(A - BF_A)$ is stable.*

Sensitivity and stability for singular dynamical linear systems had been studied by M.I. García-Planas in [6].

3 Consensus

Roughly speaking, we can define the consensus as a collection of processes such that each process starts with an initial value, where each one is supposed to output the same value and there is a validity condition that relates outputs to inputs. More concretely, the consensus problem is a canonical problem that appears in the coordination of multi-agent systems. The objective is that given initial values (scalar or vector) of agents, establish conditions under which through local interactions and computations, agents asymptotically agree upon a common value, that is to say: to reach a consensus.

The consensus problem appear for Example:

- when on try to Control moving a number of Aerial Vehicle's UAVs: alignment of the heading angles
- when on try to process Information in sensor networks: computing averages of initial local observations (that is to say consensus on a particular value)
- also in Design of distributed optimization algorithms: one needs a mechanism to align estimates of decision variables maintained by different agents/processors

3.1 Dynamic of singular multi-agent having identical dynamical mode

Let us consider a group of k identical agents, the dynamic of each agent is given by the following linear dynamical systems

$$\left.\begin{aligned} E\dot{x}^1 &= Ax^1 + Bu^1 \\ &\vdots \\ E\dot{x}^k &= Ax^k + Bu^k \end{aligned}\right\} \tag{1}$$

$x^i \in \mathbb{R}^n$, $u^i \in \mathbb{R}^m$, $1 \leq i \leq k$.

We consider the undirected graph $\mathcal{G}$ with

i) Vertex set: $\mathcal{V} = \{1, \ldots, k\}$

ii) Edge set: $\mathcal{E} = \{(i, j) \mid i, j \in \mathcal{V}\} \subset \mathcal{V} \times \mathcal{V}$

defining the communication topology among agents.

Definition 16 *Consider the system 1, we say that the consensus is achieved using local information if there is a state feedback*

$$u^i = K \sum_{j \in \mathcal{N}_i} (x^i - x^j),\ 1 \leq i \leq k$$

such that

$$\lim_{t \to \infty} \|x^i - x^j\| = 0,\ 1 \leq i, j \leq k.$$

The closed-loop system obtained under this feedback is as follows

$$\mathcal{E}\dot{\mathcal{X}} = \mathcal{A}\mathcal{X} + \mathcal{B}\mathcal{K}\mathcal{Z},$$

where

$$\mathcal{X} = \begin{pmatrix} x^1 \\ \vdots \\ x^k \end{pmatrix}, \quad \dot{\mathcal{X}} = \begin{pmatrix} \dot{x}^1 \\ \vdots \\ \dot{x}^k \end{pmatrix},$$

$$\begin{aligned} \mathcal{E} &= \text{diagonal}(E, \ldots, E) \\ \mathcal{A} &= \text{diagonal}(A, \ldots, A) \\ \mathcal{B} &= \text{diagonal}(B, \ldots, B) \\ \mathcal{K} &= \text{diagonal}(K, \ldots, K) \end{aligned}$$

and

$$\mathcal{Z} = \begin{pmatrix} \sum_{j \in \mathcal{N}_1} x^1 - x^j \\ \vdots \\ \sum_{j \in \mathcal{N}_k} x^k - x^j \end{pmatrix}.$$

Following this notation we can conclude the following.

Proposition 17 *The closed-loop system can be described as*

$$\mathcal{E}\dot{\mathcal{X}} = ((I_k \otimes A) + (I_k \otimes BK)(\mathcal{L} \otimes I_n))\mathcal{X}.$$

Taking into account that the graph is undirected, following remark 1, we have that there exists an orthogonal matrix $P \in Gl(k; \mathbb{R})$ such that $P\mathcal{L}P^t = D = \text{diag}\,(\lambda_1, \ldots, \lambda_k)$, $(\lambda_1 \geq \ldots \geq \lambda_k)$.

Corollary 18 *The closed-loop system can be described in terms of the matrices E, A, B, the feedback K and the eigenvalues of $\mathcal{L}$ in the following manner*

$$\mathcal{E}\dot{\widehat{\mathcal{X}}} = \textit{diagonal}\,(A+\lambda_1 BK,\dots,A+\lambda_k BK)\widehat{\mathcal{X}}. \quad (2)$$

Proof:

$$\begin{array}{l}(I_k\otimes BK)(\mathcal{L}\otimes I_n) = (I_k\otimes BK)(P^tDP\otimes I_n) = \\ (I_k\otimes BK)(P^t\otimes I_n)(D\otimes I_n)(P\otimes I_n) = \\ \quad (P^t\otimes BK)(D\otimes I_n)(P\otimes I_n) = \\ (P^t\otimes I_n)(I_k\otimes BK)(D\otimes I_n)(P\otimes I_n) = \\ \quad (P^t\otimes I_n)(D\otimes BK)(P\otimes I_n)\end{array}$$

$$\begin{array}{l}(I_k\otimes E) = (P^t\otimes I_n)(I_k\otimes E)(P\otimes I_n)\\ (I_k\otimes A) = (P^t\otimes I_n)(I_k\otimes A)(P\otimes I_n)\end{array}$$

Then,

$$\begin{array}{l}(P^t\otimes I_n)(I_k\otimes E)(P\otimes I_n)\dot{\mathcal{X}} = \\ (P^t\otimes I_n)(I_k\otimes A)(P\otimes I_n)\mathcal{X} + \\ \quad (P^t\otimes I_n)(D\otimes BK)(P\otimes I_n)\mathcal{X}\end{array}$$

so,

$$\begin{array}{l}(I_k\otimes E)(P\otimes I_n)\dot{\mathcal{X}} = \\ (I_k\otimes A)(P\otimes I_n)\mathcal{X} + (D\otimes BK)(P\otimes I_n)\mathcal{X}\end{array}$$

and calling $(P\otimes I_n)\mathcal{X} = \widehat{\mathcal{X}}$ we have the result. □

The system 2 can be understood as the close loop system corresponding to the system

$$\begin{pmatrix}E & & \\ & \ddots & \\ & & E\end{pmatrix}\dot{\widehat{\mathcal{X}}} = \begin{pmatrix}A & & \\ & \ddots & \\ & & A\end{pmatrix}\widehat{\mathcal{X}} + \begin{pmatrix}\lambda_1 B\\ \vdots \\ \lambda_k B\end{pmatrix}\widehat{\mathcal{U}} \quad (3)$$

after to apply the feedback $u = Kx$.

3.1.1 Consensus problem

It would seem that if the graph is connected the consensus problem would be solvable if there is a K such that the system 2 is stabilized. But taking into account that $\lambda_1 = 0$ is necessary that $E\dot{x}^1 = Ax^1$ be asymptotically stable.

Suppose now, that the system (E, A, B) is controllable, so there exist $\overline{K}_E$ and $\overline{K}_A$ such that the close loop system $\overline{E}\dot{x} = (E + B\overline{K}_E)\dot{x} = (A + B\overline{K}_A)x = \overline{A}x$ is asymptotically stable and we apply all results presented in §3.1 over the group of k identical agents, where the dynamic of each agent is given by the following linear dynamical systems

$$\left.\begin{array}{ll}\overline{E}\dot{x}^1 & = \overline{A}x^1 + Bu^1\\ \vdots & \\ \overline{E}\dot{x}^k & = \overline{A}x^k + Bu^k,\end{array}\right\} \quad (4)$$

$x^i\in\mathbb{R}^n$, $u^i\in\mathbb{R}^m$, $1\le i\le k$.

Lemma 19 *Let $E\dot{x} = Ax + Bu$ be a controllable singular system and we consider the set of k-linear systems*

$$E\dot{x}^i = Ax^i + \lambda_i Bu^i,\ 1\le i\le k$$

with $\lambda_i > 0$. Then, there exist feedbacks K_E and K_A which simultaneously assign the eigenvalues of the systems as negative as possible.

More concretely, for any $M > 0$, there exist $u^i = K_Ax^i - K_E\dot{x}^i$ for $1\le i\le k$ such that

$$\textit{Re}\,\sigma(E + BK_E, A + \lambda_i BK_A) < -M,\ 1\le i\le k.$$

($\sigma(E+BK_E, A+\lambda_i BKA)$ denotes de spectrum of $(E + BK_E, A + \lambda_i BK_A)$ for each $1\le i\le k$).

Remark 20 *We observe that if $E\dot{x} = Ax + Bu$ is controllable then, $E\dot{x} = Ax + \lambda_i Bu$ is controllable being $\lambda_i\ne 0$.*

Proof:

Reducing the system to the canonical reduced form

$E = PE_cQ + PB_cF_E$, $A = PA_cQ + PB_cF_A$ and $B = PB_cR$ with $E_c = I_n$, and (A_c, B_c) is a pair in its Brunovsky canonical form.

$$\begin{array}{l}\det(s(E + BK_E) - (A + \lambda_i BK_A) = \\ \det(s(PE_cQ + PB_cF_E + PB_cRK_E) - \\ \quad (PA_cQ + PB_cF_A + \lambda_i PB_cRK_A)) = \\ \det P\det(s(E_c + B_cF_EQ^{-1} + B_cRK_EQ^{-1}) - \\ \quad (A_c + B_cF_AQ^{-1} + \lambda_i B_cRK_AQ^{-1}))\det Q = \\ \det P\det Q\det(s(E_c + B_c\widetilde{K}_E) - (A_c + B_c\widetilde{K}_A)),\end{array}$$

where $\widetilde{K}_E = F_EQ^{-1} + RK_EQ^{-1} = 0$ and $\widetilde{K}_A = F_AQ^{-1} + \lambda_i RK_AQ^{-1}$.

So, the eigenvalues of $\det(s(E + BK_E) - (A + \lambda_i BK_A)$ are the same than $\det(sI_n - (A_c + B_c\widetilde{K}_A))$.

Now, it suffices to apply the result for standard systems.

□

Remark 21 *The Kronecker reduced form of a singular controllable system, can be directly obtained from controllability indices defined in proposition 11.*

As a corollary, we can consider the consensus problem.

Corollary 22 *We consider the system 1 with a connect adjacent topology. If $E\dot{x} = Ax+Bu$ is a controllable singular system then, the consensus is achieved by means the feedback of lemma 19 and a feedback $\overline{K}$ stabilizing $E\dot{x} = Ax + Bu$.*

Proof: Taking into account that the adjacent topology is connected we can apply corollary 3: $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k$ and $(1,\ldots,1)^t = \mathbf{1}_k$ is the eigenvector corresponding to the simple eigenvalue $\lambda_1 = 0$.

On the other hand we can find $\overline{K}$ stabilizing $E\dot{x} = Ax + Bu$ and then we can find K_E and K_A stabilizing the associate system 4, and we find $\widehat{\mathcal{X}}$ such that $\lim_{t\to\infty} \widehat{\mathcal{X}} = 0$. Consequently, we can find $\mathcal{Z}$ such that $\lim_{t\to\infty} \mathcal{Z} = 0$.

Using $\mathcal{Z} = (\mathcal{L}\otimes I_n)\mathcal{X} = (\mathcal{L}\otimes I_n)(P^t\otimes I_n)\widehat{\mathcal{X}}$ we have that $\lim_{t\to\infty} \mathcal{X}$ is an eigenvector of $\mathcal{L}\otimes I_n$, that is to say $\lim_{t\to\infty} \mathcal{X} = \mathbf{1}_k \otimes v$ for some vector $v \in \mathbb{R}^n$ and the consensus is obtained. □

Example 1.

We consider three singular identical agents with the following dynamics of each agent

$$\begin{aligned} E\dot{x}^1 &= Ax^1 + Bu^1 \\ E\dot{x}^2 &= Ax^2 + Bu^2 \\ E\dot{x}^3 &= Ax^3 + Bu^3 \end{aligned} \tag{5}$$

with $E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

It is easy to generate using the Matlab tool all possible graphs for $k = 3$, then select those that are indirect and connected, among of them, the communication topology that we chose in this example is defined by the graph $(\mathcal{V},\mathcal{E})$:

$\mathcal{V} = \{1,2,3\}$

$\mathcal{E} = \{(i,j) \mid i,j \in \mathcal{V}\} = \{(1,2),(1,3)\} \subset \mathcal{V}\times\mathcal{V}$

and the adjacency matrix:

$$G = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The neighbors of the parent nodes are $\mathcal{N}_1 = \{2,3\}, \mathcal{N}_2 = \{1\}, \mathcal{N}_3 = \{1\}$.

The Laplacian matrix of the graph is

$$\mathcal{L} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

with eigenvalues $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 3$.

$$u^i = K(\sum_{j\in\mathcal{N}_i} (x^i - x^j)) = Kz^i \tag{6}$$

$$\begin{aligned} u^1 &= K((x^1-x^2)+(x^1-x^3)) = \\ &= K(2x^1 - x^2 - x^3), \\ u^2 &= K(x^2 - x^1), \\ u^3 &= K(x^3 - x^1). \end{aligned}$$

First of all we observe that with the derivative feedback $\overline{K}_E = \begin{pmatrix} 0 & 1 \end{pmatrix}$ we obtain $\overline{E} = I$ and the new multiagent system is $\dot{x}^i = Ax^i + Bu^i$.

Taking into account that the system $\dot{x}^1 = Ax^1$ is not stable but (A,B) is a controllable system, we consider $\overline{A} = A + B\overline{K} = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}$ with appropriate values for a and b.

Then, the close loop system of 1 with control 6 is

$$\begin{aligned} \dot{x}^1 &= \overline{A}x^1 + BK(2x^1 - x^2 - x^3) = \\ &= (\overline{A} + 2BK)x^1 - BKx^2 - BKx^3 \\ \dot{x}^2 &= \overline{A}x^2 + BKx^2 - x^1) = (\overline{A} + BK)x^2 - BKx^1 \\ \dot{x}^3 &= \overline{A}x^3 + BKx^3 - x^1) = (\overline{A} + BK)x^3 - BKx^1 \end{aligned} \tag{7}$$

Or in a (formal)-matrix form:

$$\dot{\mathcal{X}} = \begin{pmatrix} \overline{A} + 2BK & -BK & -BK \\ -BK & \overline{A} + BK & 0 \\ -BK & 0 & \overline{A} + BK \end{pmatrix} \mathcal{X}.$$

The basis change matrix diagonalizing the matrix $\mathcal{L}$ is

$$P = \begin{pmatrix} 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix},$$

and we obtain the following equivalent system

$$\dot{\widehat{\mathcal{X}}} = \begin{pmatrix} \overline{A} & & \\ & \overline{A} + BK & \\ & & \overline{A} + 3BK \end{pmatrix} \widehat{\mathcal{X}}.$$

The eigenvalues are in fonction of a,b,c,d, concretely:

$\lambda_1, \lambda_2 = \frac{b\pm\sqrt{b^2+4a}}{2}$,

$\lambda_3, \lambda_4 = \frac{b+d\pm\sqrt{b^2+2bd+d^2+4a+4c}}{2}$,

$\lambda_5, \lambda_6 = \frac{b+3d\pm\sqrt{b^2+6bd+9d^2+4a+12c}}{2}$,

Then, there exist $\overline{K}$ and K (defined by a, b, c, d), which assign the eigenvalues as negative as possible.

We will try to reach consensus with three different particular feedbacks.

i) For $a = -0.01$, $b = -0.05$, $c = -0.05$, $d = -0.02$ the eigenvalues are

$\lambda_1, \lambda_2 = -0.0250{+}0.0968i, -0.0250{-}0.0968i,$

$\lambda_3, \lambda_4 = -0.0350{+}0.2424i, -0.0350{-}0.2424i,$

$\lambda_5, \lambda_6 = -0.0550{+}0.3962i, -0.0550{-}0.3962i,$

so, the system has been stabilized.

For initial condition $\widehat{X}(0) = (0, 2, 2, 3, -1, -2)^t$, the trajectory of each of the systems $\widehat{\mathcal{X}}_1 = \overline{A}\widehat{\mathcal{X}}_1$, $\widehat{\mathcal{X}}_2 = (\overline{A} + BK)\widehat{\mathcal{X}}_2$, $\widehat{\mathcal{X}}_3 = (\overline{A} + 3BK)\widehat{\mathcal{X}}_3$ are showed in figure 1.

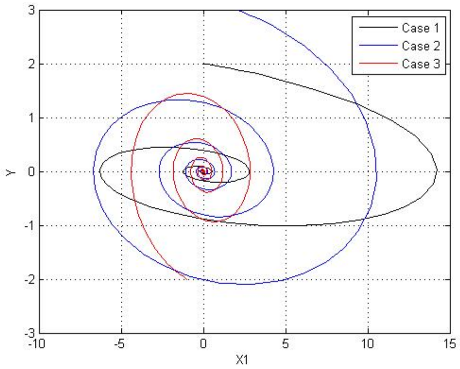

Figure 1. Trajectories 1

The graphic shows that the three trajectories arrive at a common point.

ii) For $a = -0.1$, $b = -0.5$, $c = -0.5$, $d = -0.2$ the eigenvalues are

$\lambda_1, \lambda_2 = -0.2500{+}0.1936i, -0.2500{-}0.1936i,$

$\lambda_3, \lambda_4 = -0.3500{+}0.6910i, -0.3500{-}0.6910i,$

$\lambda_5, \lambda_6 = -0.5500{+}1.1391i, -0.5500{-}1.1391i,$

so, the system has been stabilized.

For the same initial condition than the first case, i.e. $\widehat{X}(0) = (0, 2, 2, 3, -1, -2)^t$, the trajectory of each of the systems $\widehat{\mathcal{X}}_1 = \overline{A}\widehat{\mathcal{X}}_1$, $\widehat{\mathcal{X}}_2 = (\overline{A} + BK)\widehat{\mathcal{X}}_2$, $\widehat{\mathcal{X}}_3 = (\overline{A} + 3BK)\widehat{\mathcal{X}}_3$ are showed in figure 2.

It is noted that in this second case, the eigenvalues have a negative real part smaller than the first case, then consensus is reached faster.

iii) If we consider $a = -1$, $b = -5$, $c = -5$, and $d = -2$, the eigenvalues are:

$\lambda_1, \lambda_2 = -0.2087, -4.7913$

$\lambda_3, \lambda_4 = 1, -6$

$\lambda_5, \lambda_6 = -9.2749, -1.7251,$

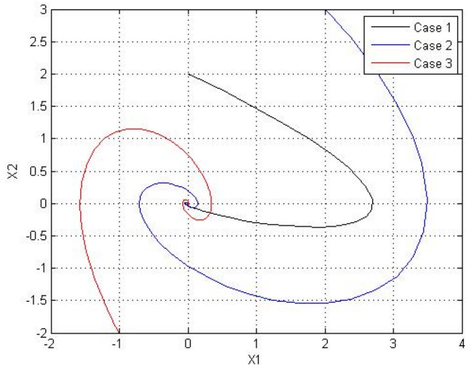

Figure 2. Trajectories 2

and the system is also stabilized.

In this case, the trajectories are showed in figure 3.

In this third case the eigenvalues have the smaller real part than the second and first case and the consensus is reached much faster than the first and second case.

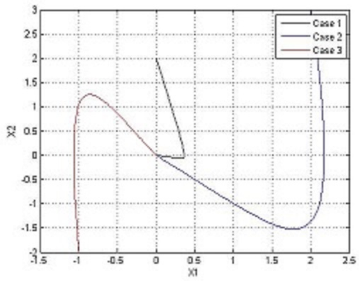

Figure 3. Trajectories 3

4 Dynamic of multi-agent having no identical dynamical mode

Now, we are going to introduce in a similar way than the case where the multianet have identical mode, we consider a multi-agent where the dynamic of each agent is given by the following dynamical systems:

$$\left.\begin{array}{ll} \dot{x}^1 & = A_1x^1 + B_1u^1 \\ \vdots & \\ \dot{x}^k & = A_kx^k + B_ku^k \end{array}\right\} \tag{8}$$

$x^i \in \mathbb{R}^n$, $u^i \in \mathbb{R}^m$, $1 \leq i \leq k$. Where matrices A_i and B_i are not necessarily equal.

The communication topology among agents is defined by means the undirected graph $\mathcal{G}$ with

i) Vertex set: $\mathcal{V} = \{1, \ldots, k\}$

ii) Edge set: $\mathcal{E} = \{(i,j) \mid i,j \in \mathcal{V}\} \subset \mathcal{V} \times \mathcal{V}$.

an in a similar way as before, we say that the consensus is achieved using local information if there exists a state feedback

$$u^i = K_i \sum_{j \in \mathcal{N}_i} (x^i - x^j),\ 1 \leq i \leq k$$

such that

$$\lim_{t \to \infty} \|x^i - x^j\| = 0,\ 1 \leq i, j \leq k.$$

The closed-loop system obtained under this feedback is as follows

$$\dot{\mathcal{X}} = \mathcal{A}\mathcal{X} + \mathcal{B}\mathcal{K}\mathcal{Z}$$

where

$$\mathcal{X} = \begin{pmatrix} x^1 \\ \vdots \\ x^k \end{pmatrix}, \quad \dot{\mathcal{X}} = \begin{pmatrix} \dot{x}^1 \\ \vdots \\ \dot{x}^k \end{pmatrix}$$

$$\begin{array}{ll} \mathcal{A} & = \text{diagonal}\,(A_1, \ldots, A_k) \\ \mathcal{B} & = \text{diagonal}\,(B_1, \ldots, B_k) \\ \mathcal{K} & = \text{diagonal}\,(K_1, \ldots, K_k) \end{array}$$

$$\mathcal{Z} = \begin{pmatrix} \sum_{j \in \mathcal{N}_1} x^1 - x^j \\ \vdots \\ \sum_{j \in \mathcal{N}_k} x^k - x^j \end{pmatrix}.$$

Calling

$$\overline{\mathcal{B}\mathcal{K}} = \mathcal{B} \cdot \mathcal{K}$$

and observing that

$$\mathcal{Z} = (\mathcal{L} \otimes I_n)\mathcal{X}$$

we deduce the following proposition

Proposition 23 *The closed-loop system can be deduced in terms of matrices $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{K}$ in the following manner.*

$$\dot{\mathcal{X}} = (\mathcal{A} + \overline{\mathcal{B}\mathcal{K}}(\mathcal{L} \otimes I_n))\mathcal{X} \tag{9}$$

We are interested in K_i such that the consensus is achieved.

Proposition 24 *We consider the system 8 which a connected adjacent topology. If the system 9 is stable the consensus problem has a solution.*

Corollary 25 *If the matrices A_i are stable. Then the consensus is achieved.*

Remark 26 *The system 9 can be written as*

$$\dot{\mathcal{X}} = \mathcal{A}\mathcal{X} + \mathcal{B}\mathcal{U} \quad \textit{with} \quad \mathcal{U} = \mathcal{K}(\mathcal{L} \otimes I_n)\mathcal{X}.$$

So,

Proposition 27 *A necessary (but not sufficient) condition for consensus to be reached is that the system*

$$\dot{\mathcal{X}} = \mathcal{A}\mathcal{X} + \mathcal{B}\mathcal{U} \tag{10}$$

is stabilizable.

Corollary 28 *A necessary condition for consensus to be reached is that the systems*

$$\dot{x}^i = A_i x^i + B_i u^i,\ \forall i = 1, \ldots, k$$

are stabilizable.

Remark 29 *The feedback $\mathcal{K}$ obtained from the feedbacks stabilizing the systems $\dot{x}^i = A_i x^i + B_i u^i$ does not necessarily stabilize the system $\dot{\mathcal{X}} = (\mathcal{A} + \overline{\mathcal{B}\mathcal{K}}(\mathcal{L} \otimes I_n))\mathcal{X}$.*

Example 2.

We consider the following two one-dimensional systems

$$\begin{array}{ll} \dot{x}^1 & = u^1 \\ \dot{x}^2 & = x^2 + u^2 \end{array}$$

The communication topology is defined by the undirected graph $\mathcal{V} = \{1, 2\}$, $\mathcal{E} = \{(1, 2)\} \subset \mathcal{V} \times \mathcal{V}$. So, the Laplacian is $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

Taking as $\mathcal{K} = \begin{pmatrix} 1 & -1 \\ 6 & -6 \end{pmatrix}$ we have

$$\mathcal{A} + \mathcal{B}\mathcal{K} = \begin{pmatrix} 1 & -1 \\ 6 & -5 \end{pmatrix}$$

with eigenvalues -0.2679, and -3.7321, then the system is stable.

But taking $k_1 = -1$ and $k_2 = -2$, clearly these feedbacks stabilize the systems, but taking as $\mathcal{K} = \begin{pmatrix} k_1 & \\ & k_2 \end{pmatrix} = \begin{pmatrix} -1 & \\ & -2 \end{pmatrix}$ we have

$$\mathcal{A} + \overline{\mathcal{B}\mathcal{K}}(\mathcal{L} \otimes I_n)) = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

with eigenvalues 0.4142, and -2.4142, then the system is not stable.

That is to say, we need to stabilize the system 10, must be stabilized with a feedback in the form $\mathcal{K}(\mathcal{L} \otimes I_n)$.

In our particular example, if we consider $k_1 = -2$ and $k_2 = 0$ the eigenvalues of $(\mathcal{A} + \overline{\mathcal{BK}}(\mathcal{L} \otimes I_n))$ are -2 and -1 and the system is stable. But, in this case, the system $\dot{x}_2 = x_2 + u_2$ with $k_2 = 0$ is not stable. Finally, if we consider $k_1 = -5$ and $k_2 = -3$, the systems $\dot{x}^i = (A_i + B_iK_i)x^i$ and $(\mathcal{A} + \overline{\mathcal{BK}}(\mathcal{L} \otimes I_n))$ are stable.

So, to solve the problem we need to obtain $\mathcal{K}$ in such a way that $\dot{x}^i = (A_i + B_iK_i)x^i$ and $(\mathcal{A} + \overline{\mathcal{BK}}(\mathcal{L} \otimes I_n))$ are stable.

5 Conclusions

In this paper the consensus problem for multi-agent singular systems, for the case where all agents have an identical linear dynamic mode, and finally we make a brief introduction to the case where the agents are of the same order but do not have the same linear dynamic. The solution of the consensus problem depends on the controllability of the singular system, then a rank criterion for controllability of singular system is introduced, thereby the work is more self-contained and understandable.

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