A Variety of Wave Amplitudes for the Conformable Fractional (2+1)-dimensional Ito Equation

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Abstract: The conformable fractional derivative and adequate fractional complex transform are implemented to discuss the fractional higher-dimensional Ito equation analytically. The Jacobi elliptic function method and Riccati equation mapping method are successfully used for this purpose. New exact solutions in terms of linear, rational, periodic and hyperbolic functions for the wave amplitude are derived. The obtained solutions are entirely new and can be considered as a generalization of the existing results in the ordinary derivative case. Numerical simulations of some obtained solutions with special choices of free constants and various fractional orders are displayed.

KeyWords: Conformable fractional derivative; Jacobi elliptic function method; Riccati equation mapping method; Ito equation; Nonlinear dynamics; Exact solution.

1. Introduction

Fractional differential equations (FDEs) were introduced to generalize the differential equations with integer orders. Over the last two decades, and due to the significant role played in mathematical modeling with applications in science, engineering, finance and information technology [1-3], FDEs have attracted the mathematicians' interests. The development of software symbolic computations helps researchers accomplish these tasks. As there is no one method can treat the various kinds of nonlinear FDEs, wide range of efficient schemes have been proposed, modified, and expanded for seeking numeric, semi-analytic and exact closed-form solutions for such problems to understand qualitative and measurable features of complex phenomena. Among these methods, we mention the bifurcation method [4], Hirota bilinear method [5], (G'/G) method and its modification [6], Adomian decomposition method and its extensions [7-10], auxiliary equation method[11], exponential-rational function method [12], F-expansion method [13], He's variational iteration method [14], inverse scattering method [15], differential transform method and its reduction [16-18], homogeneous balance method [19], Lie symmetry method [20], first integral method [21], residual power series method [22-24], generalized Riccati equation mapping method [25], $\exp\left(-\phi(\xi)\right)$ method [26], Jacobi elliptic function expansion method [27], functional variable method and generalized Kudryashov method [28-29], simplest equation method and its modification [30-31], and the sub-equation method [32].

In 1980, Ito constructed the (2+1)-dimensional integro-differential equation of the form

$$u_{tt}+u_{xxxt}+3\left(2u_x u_t+u\,u_{xt}\right)+3u_{xx}\left(\int u_t dx\right)+\alpha u_{yt}+\beta u_{xt}=0, \quad (1)$$

as a general form of the bilinear KdV equation [33]. In Eq.(1), the unknown function $u(x,y,t)$ represents the relevant wave amplitude. α and β are known real parameters.

Recently, many authors have interested in studying the (2+1)-dimensional Ito equation Eq.(1); Wazwaz [34] applied the tanh–coth method to derive single soliton and periodic solutions. Also, N-solitons were derived by combining Hereman's method and Hirota's method. The extended homoclinic test technique and the bilinear method was performed to obtain single, two-solitons, periodic and doubly-periodic wave solutions [35]. Hyperbolic and periodic solutions were obtained using the extended F-expansion method [36]. The $\left(G'/G\right)$ method were used to carry out one-soliton solutions [37]. Adem [38] deduced multiple wave solutions by using the multiple exp-function algorithm. The Wronskian determinant technique was employed by Yildirim and Yasa [39]. Lump and stripe solutions, and the diversity of interactions basing on the Hirota bilinear form were investigated by Ma et. al [40-42].

In this work, the focus is to construct new analytic solutions for the fractional version of Eq.(1)

$$\frac{\partial^{2\gamma}u}{\partial t^{2\gamma}}+\frac{\partial^{4\gamma}u}{\partial x^{3\gamma}\partial t^{\gamma}}+6\frac{\partial^{\gamma}u}{\partial x^{\gamma}}\frac{\partial^{\gamma}u}{\partial t^{\gamma}}+3u\frac{\partial^{2\gamma}u}{\partial x^{\gamma}\partial t^{\gamma}}+3\frac{\partial^{2\gamma}u}{\partial x^{2\gamma}}I_x^{\gamma}\left(\frac{\partial^{\gamma}u}{\partial t^{\gamma}}\right)+\alpha\frac{\partial^{\gamma}u}{\partial y^{\gamma}\partial t^{\gamma}}+\beta\frac{\partial^{2\gamma}u}{\partial x^{\gamma}\partial t^{\gamma}}=0, \quad (2)$$

where $\partial^{n\gamma}$ is the $n^{th}-$order $\gamma-$fractional derivative operator, I^{γ} is the $\gamma-$fractional anti-derivative operator, $t,x,y\in(0,\infty)$, and $0<\gamma\le 1$.

Parallel to the increasing of interest in fractional calculus, various definitions for fractional derivative and corresponding anti-derivative were suggested [2-3]. Among these efforts, the Caputo and Riemann-Liouville derivatives with physical meanings. These definitions found the acceptance by researchers and were extensively used in the field of FDEs, despite they do not meet some basic formulas like the constants' derivative, product, quotient or chain rules, in addition to the disability of achieve the exact solutions for many problems. In general, it is an open problem in fractional calculus. Recently, Khalil et al. [43] have proposed a new definition of fractional derivative, known as conformable fractional derivative (CFD). The CFD overcomes the shortage of others. It satisfies the most basic properties of derivative with integer order. The ability of converting the conformable fractional partial differential equations into integer-order differential equations gives the CFD the advantage to process the nonlinear fractional partial differential equations analytically. Abdeljawad [44] and Atangana [45] have discussed the CFD and concluded some useful properties. To overcome the setback of getting physical interpretation, a generalization of the conformable derivative is investigated by Zhao and Luo [46]. Because of the efficiency and applicability of the CFD, many researchers employed it to tackle the partial fractional differential equations (PFDEs). See [6, 10, 27-28, 32, 47-51] and bibliography included therein.

The rest of the paper is organized as follows: the needed basics of the conformable fractional calculus theory are presented in the next section. Mathematical analysis of the Jacobi elliptic function method and the Riccati equation mapping method with application to Eq.(1) are included in Section 3. Some obtained solutions are illustrated for various fractional orders in the conclusion and discussion section; Section 4.

2. Conformable Fractional Derivative and Integration

In this part, the fundamental concepts and facts of the conformable factional calculus, which will be used here, are listed.

Definition 1. [43] Let $u(t)$ be a function defined for $t>0$. The conformable fractional derivative for $u(t)$ of order γ, $0<\gamma\le 1$, is defined as

$$D_t^{\gamma}u=\lim_{h\to 0}\frac{u\left(t+h\,t^{1-\gamma}\right)-u(t)}{h}. \tag{3}$$

Theorem 1. [43, 44] Suppose that $u(t)$ and $v(t)$ are two γ-differentiable functions on some interval $J\subseteq(0,\infty)$, $\gamma\in(0,1]$, a, b and k are real numbers. Then

1. The conformable differential operator is linear,
2. $D_t^{\gamma}k=0$,
3. $D_t^{\gamma}t^{k}=k\,t^{k-\gamma}$,
4. $D_t^{\gamma}(uv)=v\,D_t^{\gamma}u+u\,D_t^{\gamma}v$,
5. $D_t^{\gamma}\left(\dfrac{u}{v}\right)=\dfrac{v\,D_t^{\gamma}u-u\,D_t^{\gamma}v}{v^{2}}$,
6. $D_t^{\gamma}u\left(v(t)\right)=t^{1-\gamma}v'(t)u'\left(v(t)\right)$,
7. $D_t^{\gamma}u=t^{1-\gamma}u'$,

are satisfied for all $t\in J$.

Definition 2. [43] Let $u(t)$ be a function defined on $[t_0,t)$, $t>t_0>0$, and $0<\gamma\le 1$. The conformable $\gamma-$fractional integral of $u(t)$ on the given interval is defined by

$$I_t^{\gamma}u=\int_{t_0}^{t}t^{\gamma-1}u(t)dt. \tag{4}$$

Theorem 2. [42] Suppose that $u(t)$ is a continuous function on some interval $J\subseteq(0,\infty)$ and $\gamma\in(0,1]$. Then

$$D_t^{\gamma}\left(I_t^{\gamma}u\right)=u\left(t\right), \qquad \text{for all} \qquad t\in J. \tag{5}$$

3. Analytic treatment of conformable fractional Ito equation

To investigate Eq.(2), and by putting $u=\dfrac{\partial^{\gamma}v}{\partial x^{\gamma}}$, Eq.(2) will be

$$\frac{\partial^{3\gamma}v}{\partial x^{\gamma}\partial t^{2\gamma}}+\frac{\partial^{5\gamma}v}{\partial x^{4\gamma}\partial t^{\gamma}}+6\frac{\partial^{2\gamma}v}{\partial x^{2\gamma}}\frac{\partial^{2\gamma}v}{\partial x^{\gamma}\partial t^{\gamma}}+3\frac{\partial^{\gamma}v}{\partial x^{\gamma}}\frac{\partial^{3\gamma}v}{\partial x^{2\gamma}\partial t^{\gamma}}+3\frac{\partial^{3\gamma}v}{\partial x^{3\gamma}}\frac{\partial^{\gamma}v}{\partial t^{\gamma}}+\alpha\frac{\partial^{3\gamma}v}{\partial x^{\gamma}\partial y^{\gamma}\partial t^{\gamma}}+\beta\frac{\partial^{3\gamma}v}{\partial x^{2\gamma}\partial t^{\gamma}}=0 \tag{6}$$

Assume that the exact solution of Eq.(6) has the form

$$v\left(x,y,t\right)=v\left(\xi\right), \tag{7}$$

where $\xi=\dfrac{1}{\gamma}\left(\kappa x^{\gamma}+\ell y^{\gamma}-\eta t^{\gamma}\right)$ is the wave variable, κ, ℓ, and the wave frequency η are constants to be determined. Under this consideration, Eq.(6) will be carried into the following nonlinear ordinary differential equation

$$\left(\eta-\alpha\ell-\beta\kappa\right)v^{(3)}-\kappa^3 v^{(5)}-3\kappa^2\left(\left(v'\right)^2\right)''=0. \tag{8}$$

3.1. Using the Jacobi elliptic function method

Continuing the process started before, where $\phi(\xi)$ is the Jacobi elliptic function which satisfies

Integrating Eq.(8) twice with respect to ξ and setting the integration constants to be zeros gives the missing-v equation

$$\left(\eta-\alpha\ell-\beta\kappa\right)v'-\kappa^3 v^{(3)}-3\kappa^2\left(v'\right)^2. \tag{9}$$

Reducing the order of Eq.(9) by assuming $w(\xi)=v'(\xi)$ implies

$$\left(\eta-\alpha\ell-\beta\kappa\right)w-\kappa^3 w''-3\kappa^2 w^2=0. \tag{10}$$

Making Balance between w'' and w^2 in Eq.(10) results $m=2$. In what follow, the Jacobi-elliptic function method [27-28] and the Riccati equation mapping method [23-26] will be employed to construct new abundant exact travelling wave solutions for conformable fractional Ito equation Eq.(2) by treating Eq.(10). In both, the solution of Eq.(10) is expresses as

$$w\left(\xi\right)=A_2\,\phi\left(\xi\right)^2+A_1\phi\left(\xi\right)+A_0, \qquad A_2\neq 0, \tag{11}$$

where $\phi(\xi)$ is a function that satisfies some solvable nonlinear ordinary differential equation, and A_i 's ($i=0,1,2$) are parameters to be determined

$$\phi'\left(\xi\right)=\sqrt{P\,\phi^4\left(\xi\right)+Q\,\phi^2\left(\xi\right)+R}, \tag{12}$$

where P, Q and R are constants within certain values to be given. Substituting Eq.(11) into

Eq.(10) by making use of Eq.(12), and setting the coefficients of ϕ^i, $i=0,1,\ldots,4$, to be zeros, result the following set of simultaneous algebraic equations in terms of A_0, A_1, A_2, κ, ℓ and η:

$$\left(\eta-\alpha\ell-\beta\kappa\right)A_0-\left(3\kappa^2+2RA_2\right)A_0^2=0, \tag{13}$$

$$\left(\eta-\alpha\ell-\kappa\left(Q\kappa^2+6A_0\kappa+\beta\right)\right)A_1=0, \tag{14}$$

$$3\kappa^2A_1^2-A_2\left(\kappa^2\left(6A_0-4Q\kappa\right)+\eta-\alpha\ell-\kappa\beta\right)=0, \tag{15}$$

$$2A_1\kappa^2\left(3A_2+P\kappa\right)=0, \tag{16}$$

$$3A_2\kappa^2\left(A_2+2P\kappa\right)=0. \tag{17}$$

Excluding the trivial solution, the solvable system Eqs.(13)-(17) results: $A_2=-2P\kappa$, $A_1=0$, $A_0=\frac{1}{6\kappa^2}\left(\eta-\alpha\ell-\beta\kappa-4Q\kappa^3\right)$, and $\eta=\alpha\ell+\beta\kappa\pm4\kappa^2|\kappa|\sqrt{Q^2-3PR}$. Accordingly, and to avoid the duplicate obtained solutions, Eq. (6) will get the traveling-wave solutions as follows:

Case 1. For $P=R=1$, $Q=-2$, and for $P=R=-1$, $Q=2$, the solutions of Eq.(12) are $\phi=cd$ and $\phi=dn$ respectively. Hence

$$v_{01}\left(x,y,t\right)=\pm\frac{4}{3}\kappa\xi. \tag{18}$$

Case 2. For $P=R=1$, $Q=-2$, and for $P=-Q=1$, $R=0$, the solutions of Eq.(12) are $\phi=sn$ and $\phi=cn$ respectively. Hence

$$v_{02}\left(x,y,t\right)=2\kappa\tanh\left(\xi\right), \tag{19}$$

$$v_{03}\left(x,y,t\right)=-\frac{4}{3}\kappa\xi+2\kappa\tanh\left(\xi\right). \tag{20}$$

Case 3. For $P=-Q=1$, $R=0$, the solution of Eq.(12) is $\phi=dc$. Hence

$$v_{04}\left(x,y,t\right)=-2\kappa\tan\left(\xi\right), \tag{21}$$

$$v_{05}\left(x,y,t\right)=\frac{4}{3}\kappa\xi-2\kappa\tan\left(\xi\right). \tag{22}$$

Case 4. For $P=\frac{1}{4}$, $Q=-1$, $R=0$, and for $P=-Q=1$, $R=0$, the solutions of Eq.(12) are $\phi=ns$ and $\phi=ns\pm ds$ respectively. Hence

$$v_{06}\left(x,y,t\right)=2\kappa\cot\left(\xi\right), \tag{23}$$

$$v_{07}\left(x,y,t\right)=\frac{4}{3}\kappa\xi+2\kappa\cot\left(\xi\right). \tag{24}$$

Case 5. For $P=R=1$, $Q=-2$, the solution of Eq.(12) is $\phi=ns$. Hence

$$v_{08}(x,y,t)=2\kappa\coth(\xi), \tag{25}$$

$$v_{09}(x,y,t)=-\frac{4}{3}\kappa\xi+2\kappa\coth(\xi). \tag{26}$$

Case 6. For $P=R=\frac{1}{4}$, $Q=\frac{1}{2}$, the solutions of Eq.(12) are $\phi=ns\pm cs$ and $\phi=nc\pm sc$. Hence

$$v_{10}(x,y,t)=-\kappa\tan\left(\frac{1}{2}\xi\right), \tag{27}$$

$$v_{11}(x,y,t)=\frac{1}{3}\kappa\xi-\kappa\tan\left(\frac{1}{2}\xi\right). \tag{28}$$

$$v_{12}(x,y,t)=\kappa\cot\left(\frac{1}{2}\xi\right), \tag{29}$$

$$v_{13}(x,y,t)=\frac{1}{3}\kappa\xi+\kappa\cot\left(\frac{1}{2}\xi\right). \tag{30}$$

$$v_{14}(x,y,t)=-2\kappa\frac{\sin\left(\frac{1}{2}\xi\right)}{\cos\left(\frac{1}{2}\xi\right)\pm\sin\left(\frac{1}{2}\xi\right)}, \tag{31}$$

$$v_{15}(x,y,t)=\frac{1}{3}\kappa\xi-2\kappa\frac{\sin\left(\frac{1}{2}\xi\right)}{\cos\left(\frac{1}{2}\xi\right)\pm\sin\left(\frac{1}{2}\xi\right)}. \tag{32}$$

Case 7. For $P=R=\frac{1}{4}$, $Q=-\frac{1}{2}$, the solutions of Eq.(12) are $\phi=ns\pm cs$, $\phi=\frac{sn}{1\pm cn}$ and $\phi=sn\pm i\,cn$. Hence

$$v_{16}(x,y,t)=\kappa\tanh\left(\frac{1}{2}\xi\right). \tag{33}$$

$$v_{17}(x,y,t)=-\frac{1}{3}\kappa\xi+\kappa\tanh\left(\frac{1}{2}\xi\right). \tag{34}$$

$$v_{18}(x,y,t)=\kappa\coth\left(\frac{1}{2}\xi\right). \tag{35}$$

$$v_{19}(x,y,t)=-\frac{1}{3}\kappa\xi+\kappa\coth\left(\frac{1}{2}\xi\right). \tag{36}$$

$$v_{20}(x,y,t)=2\kappa\frac{\sinh\left(\frac{1}{2}\xi\right)}{\cosh\left(\frac{1}{2}\xi\right)+i\sinh\left(\frac{1}{2}\xi\right)}. \tag{37}$$

$$v_{21}(x,y,t)=-\frac{1}{3}\kappa\xi+2\kappa\frac{\sinh\left(\frac{1}{2}\xi\right)}{\cosh\left(\frac{1}{2}\xi\right)+i\sinh\left(\frac{1}{2}\xi\right)}. \tag{38}$$

$$v_{22}(x,y,t)=\kappa\tanh(\xi)-i\,\kappa\,\text{sech}(\xi), \tag{39}$$

$$v_{23}(x,y,t)=-\frac{1}{3}\kappa\xi+\kappa\tanh(\xi)-i\,\kappa\,\text{sech}(\xi). \tag{40}$$

Where $\xi = \frac{1}{\gamma}\left(\kappa x^{\gamma} + \ell y^{\gamma} - \eta t^{\gamma}\right)$. $\kappa \neq 0$, ℓ are arbitrary constants, and $\gamma \in (0,1]$.

3.2.Using the Riccati equation mapping method

As in the previous subsection, and by using the Riccati equation mapping scheme [25-28], $\phi(\xi)$ in Eq.(11) is assumed to satisfy the Riccati equation

$$\phi'(\xi) = p\,\phi^2(\xi) + q\,\phi(\xi) + r\,. \tag{41}$$

As before, and with arbitrary constants p, q and r subject to some restrictions, substituting Eq.(11) with Eq.(41) into the relevant equation Eq.(10), and vanishing the coefficients of ϕ^i, $i = 0,1,\ldots,4$, we get

$$r\,\kappa^3\left(qA_1 + 2pA_2\right) - \left(\eta - \alpha\,\ell - \beta\,\kappa\right)A_0 + 3\kappa^2 A_0^2 = 0\,, \tag{42}$$

$$6\kappa^2\left(A_0A_1 + p\,q\,\kappa A_2\right) - \left(\eta - \alpha\,\ell - \kappa\left(\left(q^2 + 2p\,r\right)\kappa^2 + \beta\right)\right)A_1 = 0\,, \tag{43}$$

$$3\left(A_1 + p\,q\,\kappa\right)\kappa^2 A_1 - \left(\eta - \alpha\,\ell - \kappa\beta - \kappa^2\left(6A_0 + \kappa\left(4q^2 + 8p\,r\right)\right)\right)A_2 = 0\,, \tag{44}$$

$$2\kappa^2\left(3A_1A_2 + p\left(pA_1 + 5qA_2\right)\kappa\right) = 0\,, \tag{45}$$

$$3A_2\kappa^2\left(A_2 + 2p^2\kappa\right) = 0\,. \tag{46}$$

Solving the system in Eqs.(42)-(46), with eliminating the trivial solution, the following two sets of solutions are obtained:

Set 1:

$A_2 = -2p^2\kappa$, $A_1 = -2pq\,\kappa$, $A_0 = -2p\,r\,\kappa$, $\eta = \alpha\ell + \beta\kappa + \theta\kappa^3$, and $\theta = q^2 - 4p\,r$.

Set 2:

$A_2 = -2p^2\kappa$, $A_1 = -2pq\,\kappa$, $A_0 = -\frac{1}{3}\left(q^2 + 2p\,r\right)\kappa$, $\eta = \alpha\ell + \beta\kappa - \theta\kappa^3$, and $\theta = q^2 - 4p\,r$.

For **Set 1**, the soliton and soliton-like solutions of Eq. (6) are classified as follows:

Case 1. When $\theta > 0$, and $p\,q \neq 0$ (or $p\,r \neq 0$), we get

$$v_{01}(x,y,t) = \sqrt{\theta}\,\kappa\tanh\left(\frac{1}{2}\sqrt{\theta}\xi\right), \tag{47}$$

$$v_{02}(x,y,t) = \sqrt{\theta}\,\kappa\coth\left(\frac{1}{2}\sqrt{\theta}\xi\right), \tag{48}$$

$$v_{03}(x,y,t) = \frac{2\sqrt{\theta}\,\kappa\sinh\left(\frac{1}{2}\sqrt{\theta}\xi\right)}{\cosh\left(\frac{1}{2}\sqrt{\theta}\xi\right) \pm i\,\sinh\left(\frac{1}{2}\sqrt{\theta}\xi\right)}, \tag{49}$$

$$v_{04}(x,y,t)=\sqrt{\theta}\,\kappa\frac{A\cosh\left(\sqrt{\theta}\xi\right)\pm\sqrt{A^2+B^2}}{A\sinh\left(\sqrt{\theta}\xi\right)+B}, \tag{50}$$

$$v_{05}(x,y,t)=\kappa\frac{2pr\sqrt{\theta}\sinh\left(\sqrt{\theta}\xi\right)-q\theta}{2pr\left(1+\cosh\left(\sqrt{\theta}\xi\right)\right)+\theta}, \tag{51}$$

$$v_{06}(x,y,t)=-\kappa\frac{2pr\sqrt{\theta}\sinh\left(\sqrt{\theta}\xi\right)+q\theta}{2pr\left(1-\cosh\left(\sqrt{\theta}\xi\right)\right)+\theta}. \tag{52}$$

Case 2. When $\theta<0$, and $pq\neq 0$ (or $pr\neq 0$), we get

$$v_{07}(x,y,t)=-\sqrt{-\theta}\,\kappa\tan\left(\frac{1}{2}\sqrt{-\theta}\xi\right), \tag{53}$$

$$v_{08}(x,y,t)=\sqrt{-\theta}\,\kappa\cot\left(\frac{1}{2}\sqrt{-\theta}\xi\right), \tag{54}$$

$$v_{09}(x,y,t)=\frac{-2\sqrt{-\theta}\,\kappa\sin\left(\frac{1}{2}\sqrt{-\theta}\xi\right)}{\cos\left(\frac{1}{2}\sqrt{-\theta}\xi\right)\pm\sin\left(\frac{1}{2}\sqrt{-\theta}\xi\right)}, \tag{55}$$

$$v_{10}(x,y,t)=\sqrt{-\theta}\,\kappa\frac{A\cos\left(\sqrt{-\theta}\xi\right)\pm\sqrt{A^2-B^2}}{A\sin\left(\sqrt{-\theta}\xi\right)+B}, \tag{56}$$

$$v_{11}(x,y,t)=-\kappa\frac{2pr\sqrt{-\theta}\sin\left(\sqrt{-\theta}\xi\right)+q\theta}{2pr\left(1+\cos\left(\sqrt{-\theta}\xi\right)\right)+\theta}, \tag{57}$$

$$v_{12}(x,y,t)=\kappa\frac{2pr\sqrt{-\theta}\sin\left(\sqrt{-\theta}\xi\right)-q\theta}{2pr\left(1-\cos\left(\sqrt{-\theta}\xi\right)\right)+\theta}. \tag{58}$$

Case 3. When $r=0$ and $pq\neq 0$, we get

$$v_{13}(x,y,t)=\frac{2\xi_0 q\kappa}{\xi_0+\cosh(q\xi)-\sinh(q\xi)}, \tag{59}$$

$$v_{14}(x,y,t)=\frac{-2\xi_0 q\kappa}{\xi_0+\cosh(q\xi)+\sinh(q\xi)}, \tag{60}$$

Case 4. When $q=r=0$ and $p\neq 0$, we get

$$v_{15}(x,y,t)=\frac{2p\kappa}{\xi_0+p\xi}. \tag{61}$$

Where $\xi=\frac{1}{\gamma}\left(\kappa x^{\gamma}+\ell y^{\gamma}-\eta t^{\gamma}\right)$. $\kappa\neq 0$, ℓ, ξ_0 are arbitrary constants , and $\gamma\in(0,1]$.

For **Set 2**, the solutions of Eq. (6) are listed as follows:

Case 1. When $\theta>0$, and $pq\neq 0$ (or $pr\neq 0$), we get

$$v_{16}(x,y,t)=-\kappa\left(\frac{1}{3}\theta\xi-\sqrt{\theta}\,\tanh\left(\frac{1}{2}\sqrt{\theta}\xi\right)\right), \tag{62}$$

$$v_{17}(x,y,t)=-\kappa\left(\frac{1}{3}\theta\xi-\sqrt{\theta}\,\kappa\coth\left(\frac{1}{2}\sqrt{\theta}\xi\right)\right), \tag{63}$$

$$v_{18}(x,y,t)=-\kappa\left(\frac{1}{3}\theta\xi-\frac{2\sqrt{\theta}\,\sinh\left(\frac{1}{2}\sqrt{\theta}\xi\right)}{\cosh\left(\frac{1}{2}\sqrt{\theta}\xi\right)\pm i\,\sinh\left(\frac{1}{2}\sqrt{\theta}\xi\right)}\right), \quad (64)$$

$$v_{19}(x,y,t)=-\kappa\left(\frac{1}{3}\theta\xi-\sqrt{\theta}\,\frac{A\cosh\left(\sqrt{\theta}\xi\right)\pm\sqrt{A^2+B^2}}{A\sinh\left(\sqrt{\theta}\xi\right)+B}\right), \quad (65)$$

$$v_{20}(x,y,t)=-\kappa\left(\frac{1}{3}\theta\xi-\frac{2pr\sqrt{\theta}\sinh\left(\sqrt{\theta}\xi\right)-q\theta}{2pr\left(1+\cosh\left(\sqrt{\theta}\xi\right)\right)+\theta}\right), \quad (66)$$

$$v_{21}(x,y,t)=-\kappa\left(\frac{1}{3}\theta\xi+\frac{2pr\sqrt{\theta}\sinh\left(\sqrt{\theta}\xi\right)+q\theta}{2pr\left(1-\cosh\left(\sqrt{\theta}\xi\right)\right)+\theta}\right). \quad (67)$$

Case 2. For $\theta<0$, and $pq\neq 0$ (or $pr\neq 0$), we get

$$v_{22}(x,y,t)=-\kappa\left(\frac{1}{3}\theta\xi+\sqrt{-\theta}\,\tan\left(\frac{1}{2}\sqrt{-\theta}\xi\right)\right), \quad (68)$$

$$v_{23}(x,y,t)=-\kappa\left(\frac{1}{3}\theta\xi-\sqrt{-\theta}\,\cot\left(\frac{1}{2}\sqrt{-\theta}\xi\right)\right), \quad (69)$$

$$v_{24}(x,y,t)=-\kappa\left(\frac{1}{3}\theta\xi+\frac{2\sqrt{-\theta}\,\kappa\sin\left(\frac{1}{2}\sqrt{-\theta}\xi\right)}{\cos\left(\frac{1}{2}\sqrt{-\theta}\xi\right)\pm\sin\left(\frac{1}{2}\sqrt{-\theta}\xi\right)}\right), \quad (70)$$

$$v_{25}(x,y,t)=-\kappa\left(\frac{1}{3}\theta\xi-\sqrt{-\theta}\,\frac{A\cos\left(\sqrt{-\theta}\xi\right)\pm\sqrt{A^2-B^2}}{A\sin\left(\sqrt{-\theta}\xi\right)+B}\right), \quad (71)$$

$$v_{26}(x,y,t)=-\kappa\left(\frac{1}{3}\theta\xi+\frac{2pr\sqrt{-\theta}\sin\left(\sqrt{-\theta}\xi\right)+q\theta}{2pr\left(1+\cos\left(\sqrt{-\theta}\xi\right)\right)+\theta}\right), \quad (72)$$

$$v_{27}(x,y,t)=-\kappa\left(\frac{1}{3}\theta\xi-\frac{2pr\sqrt{-\theta}\sin\left(\sqrt{-\theta}\xi\right)-q\theta}{2pr\left(1-\cos\left(\sqrt{-\theta}\xi\right)\right)+\theta}\right). \quad (73)$$

Case 3. For $r=0$ and $pq\neq 0$, we get

$$v_{28}(x,y,t)=-q\kappa\left(\frac{1}{3}q\xi-\frac{4\xi_0\sinh\left(\frac{1}{2}q\xi\right)}{(\xi_0+1)\left((\xi_0+1)\cosh\left(\frac{1}{2}q\xi\right)\pm(\xi_0-1)\sinh\left(\frac{1}{2}q\xi\right)\right)}\right). \quad (74)$$

Where $\xi=\frac{1}{\gamma}\left(\kappa x^{\gamma}+\ell y^{\gamma}-\eta t^{\gamma}\right)$. $\kappa\neq 0$, ℓ, $\xi_0\neq 1$ are arbitrary constants , and $\gamma\in(0,1]$.

4. Discussion and Conclusion

A variety of closed-form travelling-wave solutions for the conformable fractional (2+1)-dimensional non-local Ito equation are investigated by means of the Jacobi elliptic function method and the Riccati equation mapping method. The two methods reduce the size of computational work, and cover many other methods like the functional variable method, the generalized Kudryashov method, the simple equation method and its extensions, the sub-equation method, and many others mentioned in the literature. Several types of complex and real travelling-wave solutions are formally extracted. The obtained solutions include regular as well as singular periodic, kink, and solitary wave solutions. Some of these solutions are displayed in Figures 1-2 for distinct values of the fraction γ. Depending on the choice of free parameters in the obtained results, different physical structures could be obtained. Such solutions will be helpful to understand the physical behavior of models in applied sciences.

As no researchers make consideration to solve Eq.(6) (or equivalently Eq.(2)), the solutions achieved throughout this paper are firstly presented and not published before to the best of our knowledge. The solutions are all verified by putting them back into the original equations with the aid of the *Mathematica* symbolic computation package 11. To completely determine the solutions of Eq.(2), one can easily apply the formula in Eq.(5) with respect to space variable x. In general, the two methods are somewhat similar, simple, applicable and inclusive to tackle several types of nonlinear evolution equations with integer and fraction derivatives.

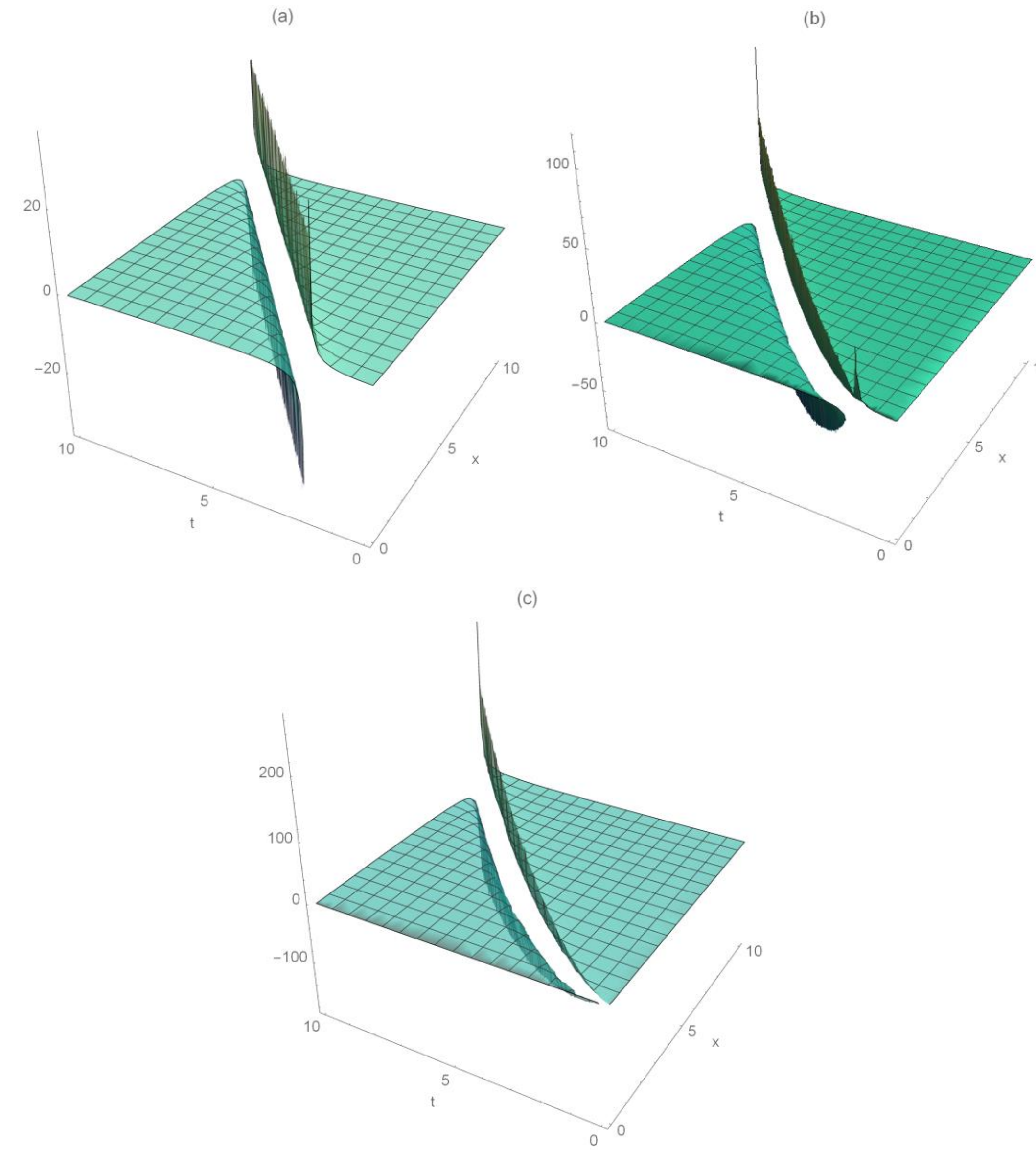

Figure 1. 3D singular kink profile of Eq.(50) in the $xt-$plane with (a) $\gamma=1$, (b) $\gamma=0.6$, and (c) $\gamma=0.2$, for $p=q=-r=\alpha=\beta=\kappa=\ell=A=0.5$, and $B=1$.

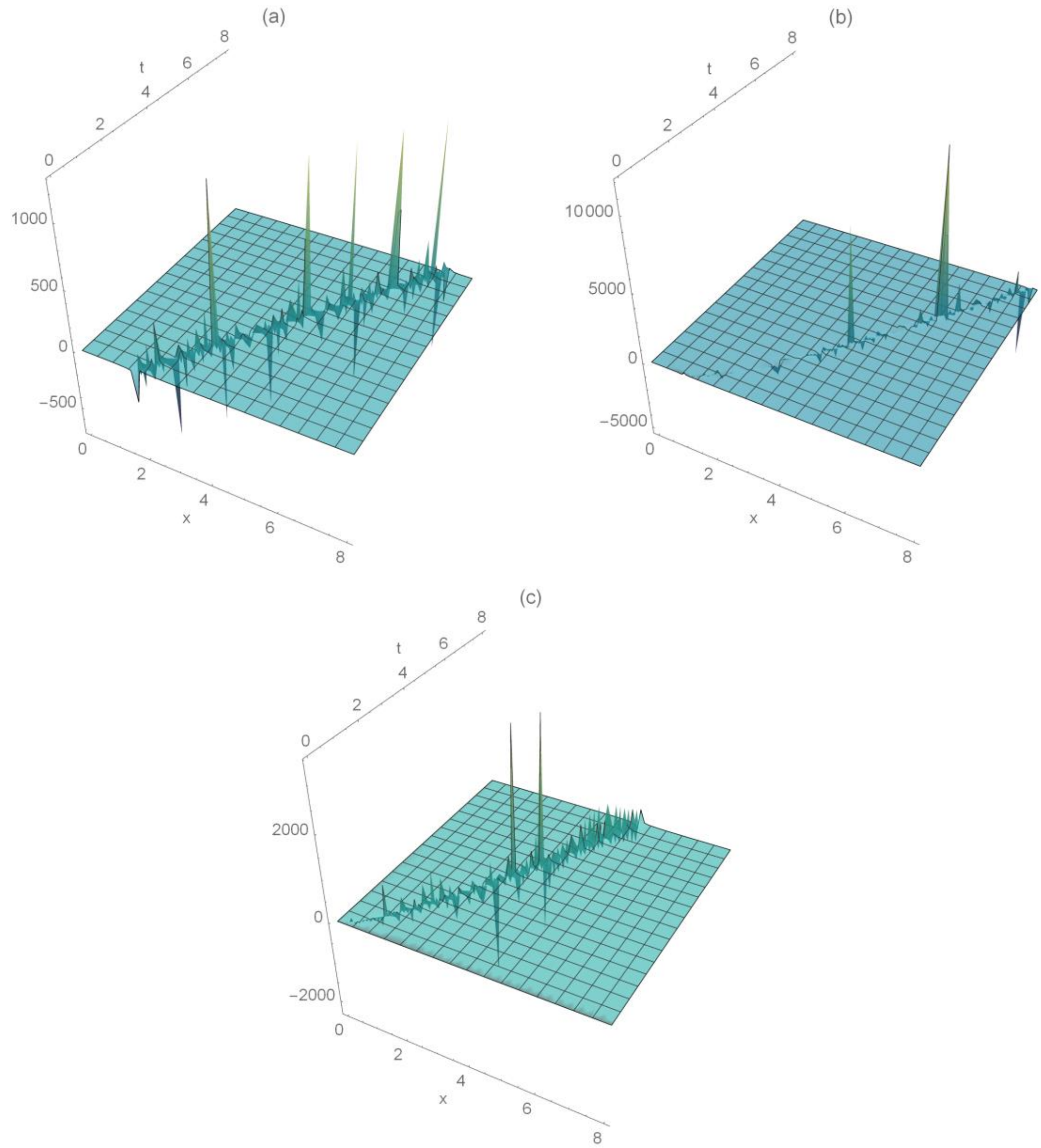

Figure 2. 3D singular periodic profile of Eq.(72) in the $xt-$plane with (a) $\gamma=1$, (b) $\gamma=0.6$, and (c) $\gamma=0.2$, for $p=q=-r=\alpha=\beta=\kappa=\ell=0.5$, and $A=B=1$.

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