On Buffon needle problem for an irregular lattice

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Abstract: In the previous papers [1] and [6] the authors introduced in the Buffon-Laplace type problems so-called obstacles. They considered two lattices and considering a classic Buffon type problem introducing in the first moment the maximum value of probability, i.e. reducing the probability interval and in the second considering an irregular lattice. In [5] Caristi and Ferrara considered also a Buffon type problem considering the possibles deformations of the lattice and in [2] Caristi, Puglisi and Stoka considered another particular regular lattices with eight sides. Fengfan and Deyi [4] study similar problem using two concepts, the generalized support function and restricted chord function, both referring to the convex set, which were introduced by Delin in [3]. In this paper, we consider another particular irregular lattice (see fig. 1) and considering the formula of the kinematic measure of Poincaré [7] and the result of Stoka [9] we study a Buffon problem for this irregular lattice. We determine the probability of intersection of a body test needle of length l, $l < \frac{a}{3}$.

Key–Words: Geometric probability, integral geometry, Buffon problem, lattice of regions, kinematic measure
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1 Preliminaires

In this section we present some results and considerations that will be needed in the rest of the paper.

Consider the irreguar lattice $\Re$ with a fundamental region C_0 composed of the union by four trinagles and an exagon (fig. 1) with $a \leq b$:

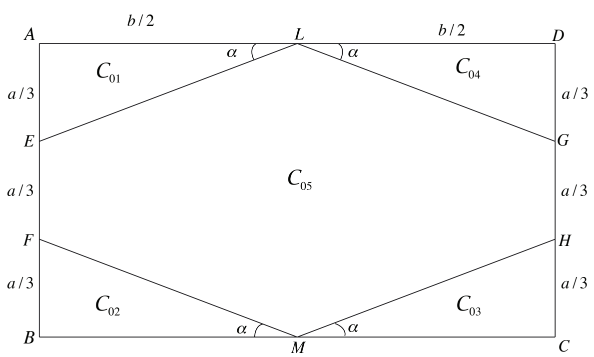

fig.1

We know that, any congruent polygon can be inlaid in a plane. In this way we obtain a lattice that covers the plane. A set of points in the plane is called a domain if it is open and connected. A set of points is called a region if it is the union of a domain with some, or all of its boundary points. From the lattice of fundamental regions in the plane, we understand a sequence of congruent regions that represent the Santalò conditions [8]:

With the notations of this figure we have

$$b = \frac{2a}{3} ctg\alpha, \qquad |GL| = |HM| = |LE| =$$

$$|MF| = \frac{a}{3\sin\alpha},$$

$$areaC_0 = \frac{2a^2}{3}, \qquad Arctg\frac{2}{3} \leq \alpha \leq \frac{\pi}{4}.$$

We want to compute the probability that a segment s with random position and of constant length l, $l < \frac{a}{3}$ intersects a side of lattice $\Re$, i.e. the probability P_{int} that a segment s intersects a side of the fundamental cell C_0.

The position of the segment s is determinated by its middle point and by the angle φ that s formed with the line AD o BC.

To compute the probability P_{int} we consider the limiting positions of segment s, for a specified value of φ, in the cells C_{0i}, $(i = 1, 2, 3)$(fig.2).

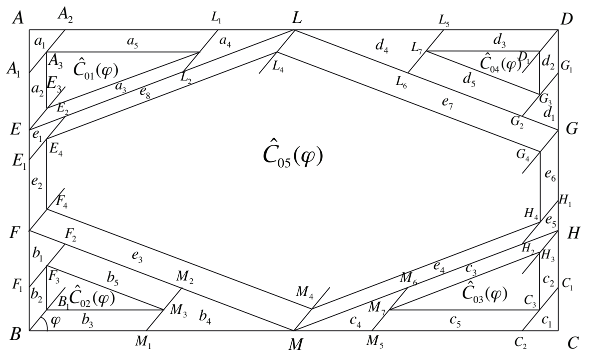

fig.2

By denoting $M_i\,(i=1,...,5)$ as the set of segments s which have their center in C_{0i} and N_i the set of segments s all contained in the cell C_{0i} we have [9]:

$$P_{int} = 1 - \frac{\sum_{i=1}^{5}\mu(N_i)}{\sum_{i=1}^{5}\mu(M_i)}, \tag{1}$$

where μ is the Lebesgue measure in the Euclidean plane.

To compute the above measure $\mu(M_i)$ and $\mu(N_i)$ we use the Poincaré kinematic measure [7] $dk = dx \wedge dy \wedge d\varphi$, where x, y are the coordinates of the middle point of s and φ is the fixed angle.

2 Main results

Considering that $l < \frac{a}{3}$ we can prove

Theorem. The probability that a random segment s of constant length $l < \frac{a}{3}$ intersects a side of lattice $\Re$ is:

$$P_{int} = \frac{3tg\alpha}{(\pi-2\alpha)a^2}\left\{\frac{al}{3}(4 - 4\sin\alpha + ctg\alpha + 5ctg\alpha\cos\alpha) + \frac{l^2}{4}[3 + 2\sin 2\alpha - 5\cos 2\alpha + (1 - tg\alpha + ctg\alpha)(\pi - 2\alpha)]\right\}. \tag{2}$$

Proof. Taking into account the symmetries of the lattice and the different values of φ we have:

$$area\widehat{C}_{01}(\varphi) = areaC_{01} - \sum_{i=1}^{5} areaa_i(\varphi),$$

$$area\widehat{C}_{02}(\varphi) = areaC_{02} - \sum_{i=1}^{5} areab_i(\varphi)$$

$$area\widehat{C}_{03}(\varphi) = areaC_{03} - \sum_{i=1}^{5} areac_i(\varphi).$$

$$area\widehat{C}_{04}(\varphi) = areaC_{04} - \sum_{i=1}^{5} aread_i(\varphi)$$

$$area\widehat{C}_{05}(\varphi) = areaC_{05} - \sum_{i=1}^{5} areae_i(\varphi)$$

We obtain that:

$$\mu(M_i) = \int_{\alpha}^{\frac{\pi}{2}} d\varphi \int\int_{\{(x,y)\epsilon C_{0i}\}} dxdy = \int_{\alpha}^{\frac{\pi}{2}} (areaC_{0i})\,d\varphi = \left(\frac{\pi}{2}-\alpha\right) areaC_{0i},$$

$$(i = 1, ..., 5).$$

then

$$\sum_{i=1}^{5}\mu(M_i) = \left(\frac{\pi}{2}-\alpha\right)\sum_{i=1}^{5} areaC_{0i} = \left(\frac{\pi}{2}-\alpha\right) areaC_0 = \frac{(\pi-2\alpha)\,ctg\alpha}{3}a^2. \tag{3}$$

In same way to compute $\mu(N_i)$ we have that:

$$A_1(\varphi) = A_3(\varphi) = \sum_{i=1}^{5} areaa_i(\varphi) = \frac{al}{6}[ctg\alpha\cos\varphi + (ctg\varphi + 1)\sin\varphi] - \frac{l^2}{4}[(1 + ctg\alpha)\sin 2\varphi + 1 - \cos 2\varphi],$$

$$A_2(\varphi) = A_4(\varphi) = \sum_{i=1}^{5} areab_i(\varphi) = \frac{al}{3}(\cos\varphi + ctg\alpha\sin\varphi) - \frac{l^2}{4}[2\sin 2\varphi + (tg\alpha - ctg\alpha)\cos 2\varphi + tg\alpha + ctg\alpha],$$

and

$$A_5(\varphi) = \sum_{i=1}^{8} areae_i(\varphi) = \frac{al}{3}(\cos\varphi + ctg\alpha\sin\varphi) - \frac{l^2}{4}[\sin 2\varphi - tg\alpha\cos 2\varphi - tg\alpha].$$

Then we obtain that:

$$\mu\left(N_i\right)=\int_\alpha^{\frac{\pi}{2}} d\varphi \int\int_{\left\{(x,y)\epsilon\widehat{C}_{0i}(\varphi)\right\}} dxdy =$$

$$\int_\alpha^{\frac{\pi}{2}}\left[area\widehat{C}_{0i}\left(\varphi\right)\right]d\varphi=\int_\alpha^{\frac{\pi}{2}}\left[areaC_{0i}-A_i\left(\varphi\right)\right]d\varphi=$$

$$\left(\frac{\pi}{2}-\alpha\right)areaC_{0i}-\int_\alpha^{\frac{\pi}{2}}\left[A_i\left(\varphi\right)\right]d\varphi.$$

and

$$\sum_{i=1}^{3}\mu\left(N_i\right)=\frac{\left(\pi-2\alpha\right)ctg\alpha}{3}a^2- \int_\alpha^{\frac{\pi}{2}}\left[\sum_{i=1}^{3}A_1\left(\varphi\right)\right]d\varphi. \quad (4)$$

In the end, from (1), (3) and (4) we obtain the probability (2).

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