*Based on equation (59) and (60), we can write:*

$$\Delta V(k) = \varepsilon^T(k)\Psi\varepsilon(k) \tag{61}$$

*with*

$$\varepsilon(k) = \begin{pmatrix} e(k) \\ e(k-h_1(k)) \end{pmatrix} \tag{62}$$

$$\Psi = \begin{pmatrix} \chi & h_1 H^T P H_d \\ & h_1 H_d^T P H_d - Q - R \end{pmatrix} < 0 \tag{63}$$

*where* $\chi = h_1 H^T P H - h_1 P + Q + R$

*So* $\Delta V(k) < 0$ *is equivalent to* $\Psi < 0$.

*To avoid the quadratic from present in equations (63), we propose to rewrite this equation as following form:*

$$\Psi = U - \Omega^T \Xi^{-1} \Omega \tag{64}$$

*where:*

$$U = \begin{bmatrix} Q - R - h_1 P & 0 \\ 0 & -Q - R \end{bmatrix} \tag{65}$$

$$\Omega = \begin{bmatrix} \sqrt{h_1} H & \sqrt{h_1} H_d \end{bmatrix} \tag{66}$$

$$\Xi = -P^{-1} \tag{67}$$

*According to the Schur lemma [14],* $\Psi < 0$ *and* $\Xi < 0$ *if and only if:*

$$\Lambda = \begin{bmatrix} \Xi & \Omega \\ \Omega^T & U \end{bmatrix} < 0 \tag{68}$$

*By applying a congruence transformation [14] to* $\Lambda$ *such as:*

$$\Pi^T \Lambda \Pi < 0 \tag{69}$$

*with*

$$\Pi = \begin{bmatrix} P & 0 & 0 \\ * & I & 0 \\ * & * & I \end{bmatrix} < 0 \tag{70}$$

*Replacing* $\Lambda$ *and* $\Pi$ *by their expressions (68) and (69) and using (48)-(53), Theorem 2 holds.*

# 5 Numerical Example

## 5.1 The model matrices

$$E = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, A = \begin{pmatrix} -5 & 0 \\ 0 & -3 \end{pmatrix}, A_d = \begin{pmatrix} -1 & 0 \\ 0 & -0.5 \end{pmatrix},$$

$$B = \begin{pmatrix} 2 & 0 \\ -1 & 3.2 \end{pmatrix}, B_d = \begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & -1 \\ 0.5 & 3 \end{pmatrix},$$

$$F = \begin{pmatrix} 4 & 1 \\ 0 & -2 \end{pmatrix}, D_1 = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, D_2 = D_{d1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$D_{d2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad G = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 2 & -1 \end{pmatrix}$$

The state delay has a sinusoid form such $h_1(k) = 3sin(0.5k)$, then we can easily deduce that $h_1 = 3s$. The input delay $h_2$ is such $h_2 = 1s$.

## 5.2 Observer matrices

$$H = \begin{pmatrix} -3.22 & 22.17 \\ 1.73 & -11.94 \end{pmatrix}, \quad H_d = \begin{pmatrix} -0.57 & 2.8 \\ 0.3 & -1.51 \end{pmatrix},$$

$$L_1 = \begin{pmatrix} 8.66 \\ -4.66 \end{pmatrix}, P_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, J_d = \begin{pmatrix} 4.33 & -4.33 \\ -2.33 & 2.33 \end{pmatrix},$$

$$L_2 = \begin{pmatrix} 5.05 \\ -2.72 \end{pmatrix}, J = \begin{pmatrix} 10.83 & -6.93 \\ -5.83 & 3.73 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} -0.33 \\ 0.33 \end{pmatrix}, L_d = \begin{pmatrix} 0.36 \\ -1.19 \end{pmatrix}, N_2 = N_{d1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$N_1 = 10^{-15}\begin{pmatrix} -1.7 \\ -0.97 \end{pmatrix}, N_{d2} = 10^{-15}\begin{pmatrix} 5.66 \\ -3.4 \end{pmatrix}$$

## 5.3 Figures and interpretations

Fig.1 and Fig.2 explicit, respectively, the used known and unknown input vectors in the numerical example. As shown both inputs have two components.

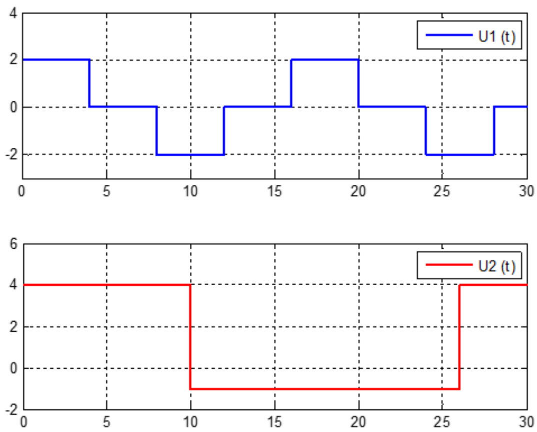


Figure 1: The known input vector

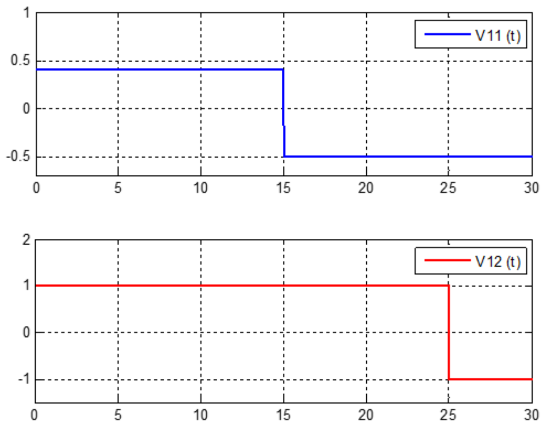


Figure 2: The used unknown input

Fig.3 draws both real and estimated functional state vectors. We can remark a quick convergence during the permanent phase and a short transitory phase $t \in [0, 5]$.

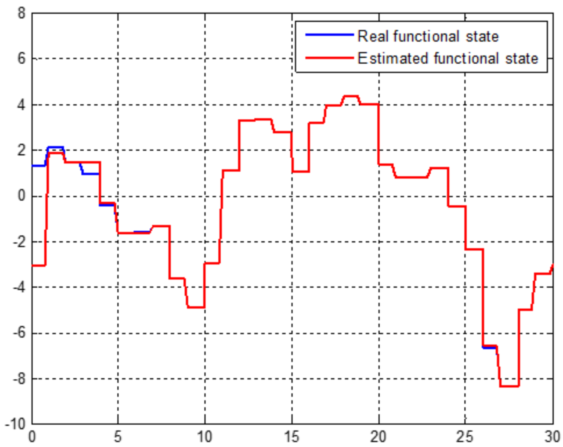


Figure 3: The real and estimated functional state vectors

Fig.4 draws both real and estimated functional unknown input vectors. we can check the quick convergence dynamic of the proposed observer.

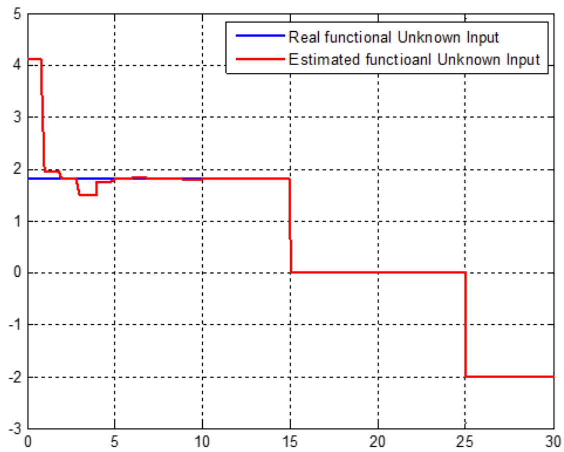


Figure 4: The real and estimated functional state vectors

Fig.5 and Fig.6 show the estimation error of the functional state vector and the unknown input which converge asymptotically to 0 and confirm the effectiveness of the proposed approach.

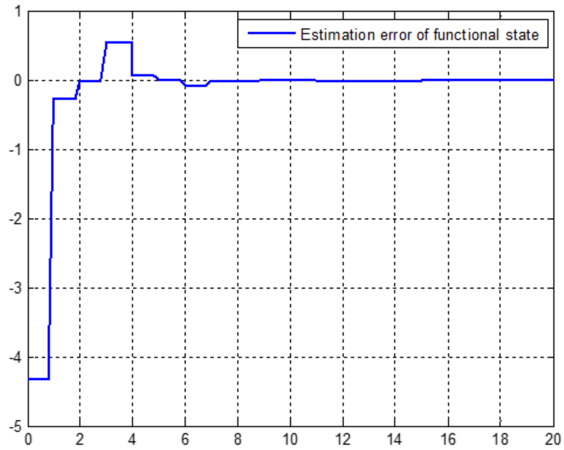


Figure 5: The state estimation error

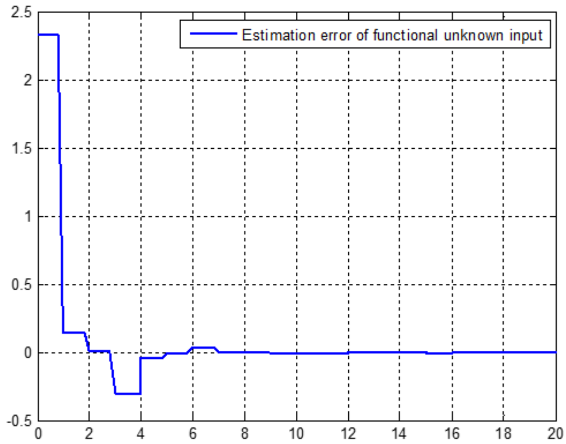


Figure 6: The unknown input estimation error

## 6 Conclusion

In this paper, authors have presented an observer scheme for singular bilinear systems with variable state delay. A constant delay has been also considered in the input vector in both regular and bilinear form. The observer proves its effectiveness tested on a numerical example and reconstruct both functional state vector and a part of the unknown input vector. The proposed approach is based on the Lyapunov Krasovskii stability theory so the observer gain is calculated by using a Lyapunov functional and ensures an unbiasedness dynamic. The estimation error is independent from the presented unknown input and the considered input delay.

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