Design of Stable Controller for Rössler Chaotic System

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Abstract: - This paper is based on the design and analysis of three different types of controllers for stabilizing the Rössler chaotic system. The controllers are based on Takagi-Sugeno (T-S) fuzzy model, Mamdani fuzzy model and Sliding mode control (SMC). The concept behind the design of each controller is to drive the highly oscillating chaotic dynamics to a stable steady state value. The control action for the fuzzy controllers are based on the design of fuzzy rules. In case of sliding mode control (SMC), the control action is attained by designing an asymptotically stable sliding surface in such a way that the system states reach the sliding surface in a finite time. MATLAB toolbox YALMIP is used for solving Linear Matrix Inequality (LMI) optimization problems in the case of T-S fuzzy model and SMC. The Mamdani fuzzy model is implemented using MATLAB fuzzy inference system toolbox. The effectiveness of the above three methods is validated by MATLAB simulation results. Lastly a comparative study based on the results of the above methods is presented for identifying the best method.

Key-Words: - Chaotic, T-S fuzzy model, SMC, YALMIP toolbox, LMI

1 Introduction

The theory behind the chaotic dynamics is that small changes in a system can cause large effects as time progresses. In the late 1800s Poincaré was the first person to examine the possibility of chaotic phenomena in a deterministic system that exhibited aperiodic behavior and found sensitive dependence on initial conditions in a three-body problem [1]. In 1963, the MIT meteorologist Edward Lorenz experimented on a simplified model of convecting fluids in the atmosphere to get an insight about the uncertainty in weather predictions [1]. Lorenz found that the system never reached a stable equilibrium state rather it was oscillating in an irregular and aperiodic manner. He started doing the simulation of his equations by taking two initial conditions very close to each other and found that as the time progresses the resulting behaviors were totally different. This led to the invention of the famous "Lorenz chaotic attractor" [1]. The theory of chaos has bloomed after the year 1970 and lot more researches [2-4] are being carried out in this field. However, obtaining analytical solution is very difficult.

In the recent years the subject of artificial intelligence has gained a rapid development in the field of research. The adaptive control, neural control, fuzzy control provides a good platform for the design of control system for solving complex nonlinear systems associated with various uncertainties. Neural network and fuzzy system have the advantage that the smooth nonlinear functions can be modelled in a precise set thereby reducing the error [5]. The concept of fuzzy logic was introduced in "Fuzzy Sets" which was published by Lofti A. Zadeh in the year 1965 [6]. In the year 1975, E. H. Mamdani and S. Assilian developed a controller capable to follow instructions and provide strategies based on verbal communication and not on experience [7]. The plant to be controlled is combination of a steam engine and boiler which is highly nonlinear. The model of the plant has two inputs: heat injected to the boiler and opening of the throttle at the input of the engine cylinder and two outputs: steam pressure in the boiler and engine speed [7]. In 1985, T. Takagi and M. Sugeno proposed a mathematical procedure to construct the fuzzy model of a particular system where fuzzy rules and reasoning were used [8]. The main advantage of the T-S fuzzy model is that for each rule the local dynamics of the system are expressed in a way that the system model becomes linear. For each of the fuzzy subsystem rule, a corresponding control rule is designed by using parallel distributed compensation (PDC) method.

In the year 1977, Vadim I. Utkin published a survey paper based on variable structure systems with sliding modes [9]. The motivation behind the design of variable structure systems (VSS) is that the control parameter has the ability to change its structure at any instant from one member of a set to another member of the same set of all possible continuous functions of the state. The switching logic is defined based on the selection of parameters from each structures. The SMC is the special form of variable structure control [10]. SMC is designed to drive the system and constrain the system states to remain within the neighbourhood of the switching function [10]. This approach has an advantage that the system becomes completely insensitive to parametric variations and thereby making the plant disturbance free.

This paper is divided into seven sections. Section (2) presents the Rössler system and uncontrolled behaviour of its state trajectories. In section (3), T-S fuzzy model is presented along with five subsections. The subsections include: parallel distributed compensation (PDC), stability analysis using LMI, fuzzy modeling, fuzzy control and simulation results and output. Section (4) presents Mamdani fuzzy model with five subsections. The subsections include: defining inputs and outputs, universe of discourse and membership functions, defining rule matrix table, defuzzification process, and simulation results and output. In section (5), sliding mode control is presented along with four subsections. The subsections include: system description, sliding surface design, controller design and simulation results and output. In section (6), a comparison table for selecting the best controller is presented. Finally in section (7), the conclusion part is presented.

2 Rössler System

The most widely known chaotic attractor is the Lorenz attractor which consists of two nonlinear terms. In the 1970s, the German scientist O.E. Rössler found a new way of making a system in three dimensions based on relaxation-type systems which consists of an autonomous oscillation in two variables which upon moulding with a third variable forms an S-shaped slow manifold in the phase space. The Rössler system is derived from a system by the combination of a two variable chemical oscillator (variable x_1 and x_2) and a single variable chemical hysteresis term x_3. The set of equations are given by (1) as mentioned in [11]-

$$\dot{x}_1(t) = k_1 + k_2 x_1(t) - \frac{(k_3 x_2(t) + k_4 x_3(t))x_1(t)}{(x_1(t) + K)}$$

$$\dot{x}_2(t) = k_5 x_1(t) - k_6 x_2(t)$$

$$\mu\dot{x}_3(t) = k_7 x_1(t) + k_8 x_3(t) - k_9 x_3^2(t) - \frac{k_{10} x_3(t)}{x_3(t) + K'} \quad (1)$$

where $x_1(t)$, $x_2(t)$, $x_3(t)$ denotes the concentration of substances A, B and C respectively. K and K' are Michaelis constants. On simplication of equation (1), the Rössler system equations and are given by equation (2) as mentioned in [12]-

$$\left.\begin{aligned} \dot{x_1}(t) &= -x_2(t) - x_3(t) \\ \dot{x_2}(t) &= x_1(t) + a x_2(t) \\ \dot{x_3}(t) &= b x_1(t) + (x_1(t) - c)x_3(t) \end{aligned}\right\} \quad (2)$$

where a, b and c are the system parameters. The Rössler equations consist of a single nonlinear term (i.e. $x_1(t)x_3(t)$). Considering two cases of parameter values. In the first case, the parameter values of the system is chosen as $a = 0.34, b = 0.4$ and $c = 14$. In the second case, the parameter values of the system are $a = 0.2, b = 0.2$ and $c = 5.7$. The parameter values for the above two cases gives rise to two different chaotic attractors. The Lyapunov exponents for the system by considering the first case are approximately found to be 0.055013, -0.056818 and -12.802656. The Lyapunov exponents for the second case are 0.006788, -0.258514 and -5.137789. The Lyapunov exponents are calculated by using Lyapunov Exponent Toolbox [13]. A positive Lyapunov exponent indicates chaotic phenomenon [14]. Since one of the Lyapunov exponent found to be positive hence, the Rössler system behaves as a chaotic system. The Rössler attractor formed by considering both the case are shown in Fig. 1 and Fig. 2 respectively.

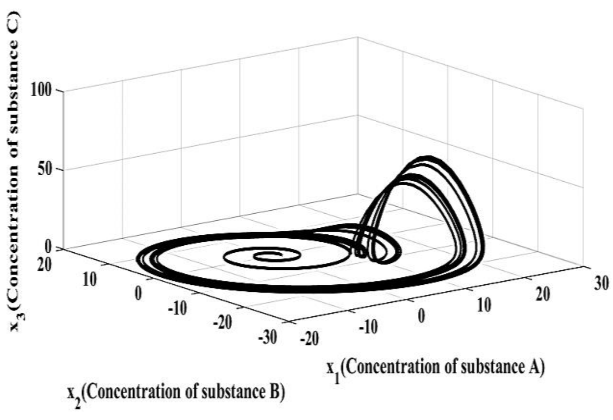

Fig. 1 Rössler attractor for first case with an initial condition [1, -1, 0]

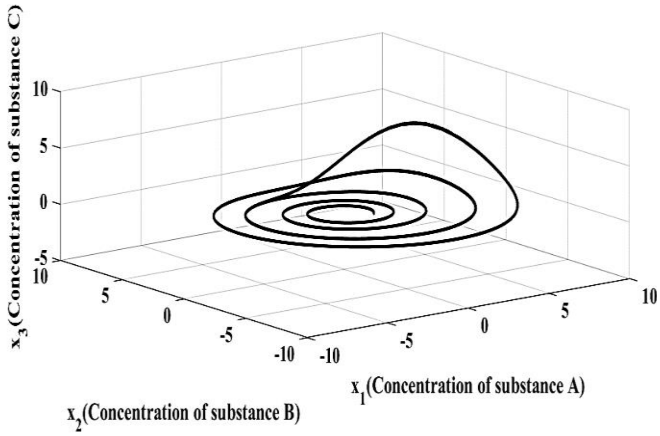

Fig. 2 Rössler attractor for second case with an initial condition [1, -1, 0]

For the first case, the behavioural patterns of the uncontrolled state trajectories of Rössler system with initial condition as [1, -1, 0] are shown in Figure 3a), 3(b) and 3(c) respectively.

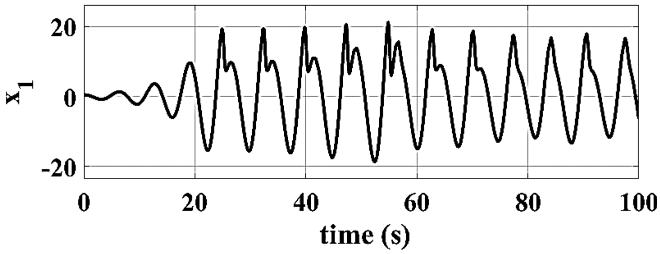

Fig. 3(a) Uncontrolled state trajectory of x_1(concentration of substance A) vs $time$ (seconds)

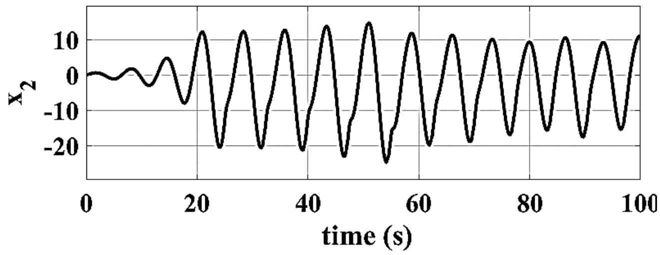

Fig. 3(b) Uncontrolled state trajectory of x_2(concentration of substance B) vs $time$ (seconds)

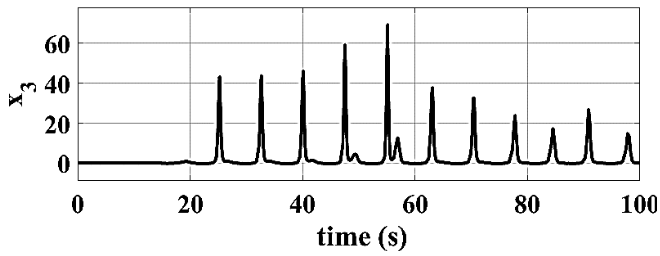

Fig. 3(c) Uncontrolled state trajectory of x_3(concentration of substance C) vs $time$ (seconds)

For the second case, the behavioural patterns of the uncontrolled state trajectories of Rössler system with initial condition as [1, -1, 0] are shown in Figure 4(a), 4(b) and 4(c) respectively.

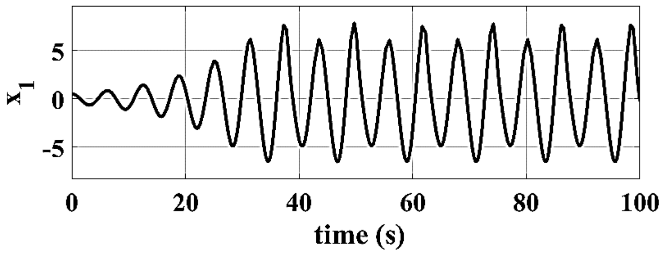

Fig. 4(a) Uncontrolled state trajectory of x_1(concentration of substance A) vs $time$ (seconds)

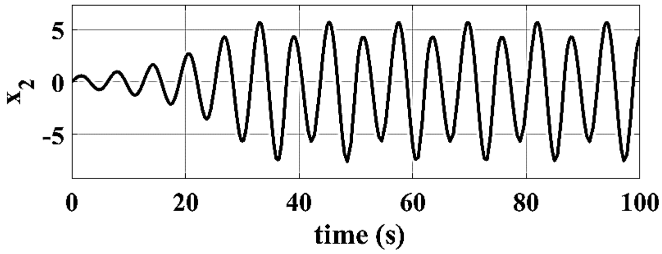

Fig. 4(b) Uncontrolled state trajectory of x_2(concentration of substance B) vs $time$ (seconds)

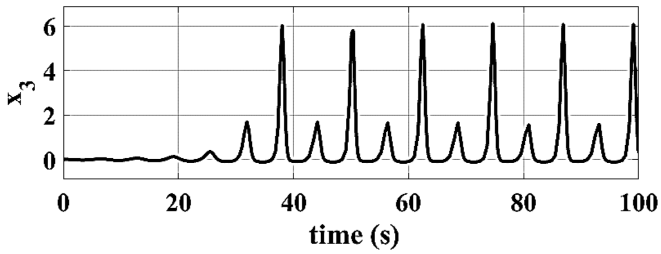

Fig. 4(c) Uncontrolled state trajectory of x_3(concentration of substance C) vs $time$ (seconds)

In section (3), the design of T-S fuzzy model is illustrated for modelling of Rössler system.

3 Takagi-Sugeno (T-S) Fuzzy Model

The n[th] rule of the T-S model is as follows [15]:

Rule n:

IF $z_1(t)$ *is* M_{n1} *and*................ $z_p(t)$ *is* $M_{np}(t)$,

THEN $\dot{x}(t) = A_n x(t) + B_n u(t)$,

$$y(t) = C_n x(t), \quad \text{for } n = 1,2, \dots, r \qquad (3)$$

where M_{nj} represents fuzzy set, r indicates total number of rules; $x(t) \in R^n$ indicates state vector, $u(t) \in R^m$ indicates input vector, $y(t) \in R^q$ indicates output vector, $A_i \in R^{n\times n}$, $B_i \in R^{n\times m}$ and $C_i \in R^{q\times n}$; $z_1(t), \dots, z_p(t)$ represents the premise variables that are the function of state variables. Each linear consequent term represented by $A_n x(t) + B_n u(t)$ are called "fuzzy subsystem". The resulting fuzzy systems are given by:

$$\dot{x}(t) = \frac{\sum_{n=1}^{r} w_n(z(t))\{A_n x(t) + B_n u(t)\}}{\sum_{n=1}^{r} w_n(z(t))}$$

$$= \sum_{n=1}^{r} h_n\big(z(t)\big)\{A_n x(t) + B_n u(t)\}, \qquad (4)$$

$$y(t) = \frac{\sum_{n=1}^{r} w_n\big(z(t)\big) C_n x(t)}{\sum_{n=1}^{r} w_n(z(t))}$$

$$= \sum_{n=1}^{r} h_n\big(z(t)\big) C_n x(t), \qquad (5)$$

where $z(t) = z_1(t), \dots, z_p(t)$,

$$w_n\big(z(t)\big) = \prod_{j=1}^{p} M_{nj}\left(z_j(t)\right),$$

$$h_n\big(z(t)\big) = \frac{w_n\big(z(t)\big)}{\sum_{n=1}^{r} w_n\big(z(t)\big)}, \qquad (6)$$

The term $M_{nj}(z_j(t))$ represents membership function.

Since $\sum_{n=1}^{r} w_n\big(z(t)\big) > 0$ and $w_n\big(z(t)\big) \geq 0$,

$$\text{for n} = 1,2, \dots, r$$

We have $\sum_{n=1}^{r} h_n\big(z(t)\big) = 1$ *and* $h_n\big(z(t)\big) \geq 0$,

$$\text{for n} = 1,2, \dots, r$$

In the subsection *3.1*, the design of the fuzzy controller is done by using PDC method.

3.1 Parallel Distributed Compensation (PDC)

The n[th] rule of the PDC is as follows [15]:

Rule n:

IF

$z_1(t)$ is M_{n1} and................ $z_p(t)$ is $M_{np}(t)$,

THEN

$$u(t) = -K_n x(t), \quad n = 1,2, \dots, r \qquad (7)$$

Then the resulting controller is

$$u(t) = -\frac{\sum_{n=1}^{r} w_n\big(z(t)\big) K_n x(t)}{\sum_{n=1}^{r} w_n\big(z(t)\big)}$$

$$= -\sum_{n=1}^{r} h_n\big(z(t)\big) K_n x(t) \qquad (8)$$

Substituting equation (8) in equation (4) thereby obtaining the closed loop system as

$$\dot{x}(t) = \sum_{n=1}^{r} h_n\big(z(t)\big)\{A_n - B_n K_n\} x(t), \qquad (9)$$

After the design of the controller, the subsection *3.2* is about obtaining the optimized value of state feedback gain matrices for the design of the stable fuzzy controller.

3.2 Stability Analysis using LMI

For the design of stable decay rate fuzzy controller, the inequalities in equation (10), (15) and (16) are to be satisfied [15]:

$$P > 0 \qquad (10)$$

$$(A_n - B_n K_n)^T P + P(A_n - B_n K_n) < 0,$$

$$\text{for n} = j = 1,2, \dots, r \qquad (11)$$

and $\quad {G_{nj}}^T P + P G_{nj} \leq 0, \quad \text{for n} < j \leq r$

$$\text{such that } \mathrm{h_n} \cap \mathrm{h_j} \neq \emptyset \qquad (12)$$

where

$$G_{nj} = \frac{\big(A_n - B_n K_j\big) + \big(A_j - B_j K_n\big)}{2}$$

for a stable fuzzy controller design, pre-multiply and post-multiply the equations (11) and (12) by P^{-1} and define $X = P^{-1}$ (such that $X > 0$) and $M_n = K_n X$ yields:

$$XA_n{}^T + A_n X - M_n{}^T B_n^T - B_n M_n < 0,$$

$$\text{for n} = j = 1,2,\dots,r \tag{13}$$

and

$$XA_n{}^T + A_n X + XA_j{}^T + A_j X - \left(B_n M_j + B_j M_n\right) - \left(M_n{}^T B_j^T + M_j{}^T B_n^T\right) < 0,$$

$$\text{for n} < j \le r \tag{14}$$

For guaranteed exponential decay of states $x(t) \to 0$, the condition is $\alpha V(x) + \dot{V}(x) \le 0$. So, equation (13) and (14) are modified to [16]:

$$X > 0,$$

$$XA_n{}^T + A_n X - M_n{}^T B_n^T - B_n M_n + \alpha X < 0,$$

$$\text{for n} = j = 1,2,\dots,r \tag{15}$$

$$XA_n{}^T + A_n X + XA_j{}^T + A_j X - \left(B_n M_j + B_j M_n\right) - \left(M_n{}^T B_j^T + M_j{}^T B_n^T\right) + \alpha X \le 0,$$

$$\text{for n} < j \le r \tag{16}$$

where $\alpha > 0$, $X = P^{-1}$ and $M_n = K_n X$

The next two subsections *3.3 and 3.4* is all about the fuzzy modelling and fuzzy control of the Rössler system.

3.3 Fuzzy Modelling

Considering the Rössler chaotic system in equation (2) having an input term $u(t)$ is given as:

$$\dot{x_1}(t) = -x_2(t) - x_3(t)$$

$$\dot{x_2}(t) = x_1(t) + a x_2(t)$$

$$\dot{x_3}(t) = b x_1(t) + (x_1(t) - c) x_3(t) + u(t)$$

where a, b and c are the system parameters and $u(t)$ is the control input to be designed. The single nonlinear term is $x_1(t)x_3(t)$. If somehow the nonlinearity can be resolved, the system becomes linear. To obtain the linear system, assume that $x_1(t) \in [-d, +d]$ and $d > 0$. Then the fuzzy model that exactly representing the Rössler's system under $x_1(t) \in [-d, +d]$ as:

Rule 1:

$IF\ x_1(t)$ is M_1,

$$THEN\ \dot{x}(t) = A_1 x(t) + Bu(t) \tag{17}$$

Rule 2:

$IF\ x_1(t)$ is M_2,

$$THEN\ \dot{x}(t) = A_2 x(t) + Bu(t) \tag{18}$$

Here, $B_n = B$ that is a common input matrix is taken and $x(t) = [x_1(t) \quad x_2(t) \quad x_3(t)]^T$.

$$A_1 = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ b & 0 & -d-c \end{bmatrix}, A_2 \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ b & 0 & d-c \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Also $M_1\left(x_1(t)\right) = \frac{1}{2}\left(1 - \frac{x_1(t)}{d}\right)$, $M_2\left(x_1(t)\right) = \frac{1}{2}\left(1 + \frac{x_1(t)}{d}\right)$,

where $d = 3$. The plot of triangular membership function is shown in Figure 5.

3.4 Fuzzy Control

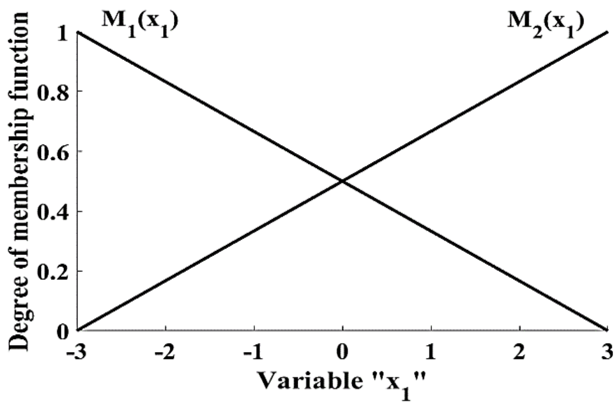

Fig. 5 Plot of Triangular Membership functions of $M_1\left(x_1(t)\right)$ and $M_2\left(x_1(t)\right)$

Rule 1:

$IF\ x_1(t)$ is M_1,

$$THEN\ u(t) = -K_1 x(t) \tag{19}$$

Rule 2:

$IF\ x_1(t)$ is M_2,

$$THEN\ u(t) = -K_2 x(t) \tag{20}$$

Since the fuzzy model of the Rössler system has only two rules, the PDC also have only two rules. So, the resulting PDC fuzzy controller is

$$u(t) = -\frac{\sum_{n=1}^{2} w_n\big(z(t)\big)K_n x(t)}{\sum_{n=1}^{2} w_n\big(z(t)\big)}$$

$$= -\sum_{n=1}^{2} h_n\big(z(t)\big)K_n x(t), \tag{21}$$

Substituting equation (21) in equation (4) thereby obtaining the closed loop system as:

$$\dot{x}(t) = \sum_{n=1}^{2} h_n\big(z(t)\big)\{A_n - BK_n\}x(t) \tag{22}$$

The design of fuzzy modelling & control is done in MATLAB Simulink and suitable simulation results & output are obtained which are shown in subsection *3.5*.

3.5 Simulation Results and Output

For the first case, solving the set of LMIs in equations (15) and (16) taking $\alpha = 2$ using MATLAB YALMIP toolbox [17], the optimized values of symmetric positive definite matrix and state feedback gain matrices are given as-

$$P = \begin{bmatrix} 31.2427 & 39.8487 & -6.6334 \\ 39.8487 & 60.5482 & -7.7181 \\ -6.6334 & -7.7181 & 1.8910 \end{bmatrix}$$

$$K_1 = [-28.2032 \quad -31.5142 \quad -9.4327]$$

$$K_2 = [-28.2032 \quad -31.5142 \quad -3.4327]$$

By substituting the above K_1 and K_2 values in the MATLAB Simulink, the controlled state trajectories are shown in Figure 6(a), 6(b) and 6(c) respectively. The control input is added after $t > 10$ sec for showing the oscillating nature up to 10 sec.

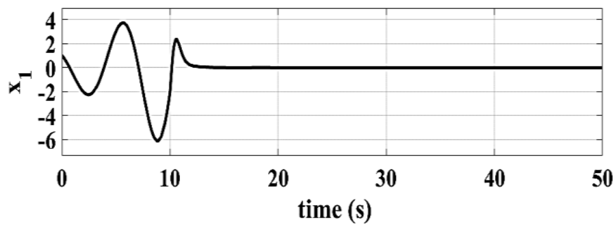

Fig. 6(a) Controlled state trajectory of x_1(concentration of substance A) vs $time$(seconds)

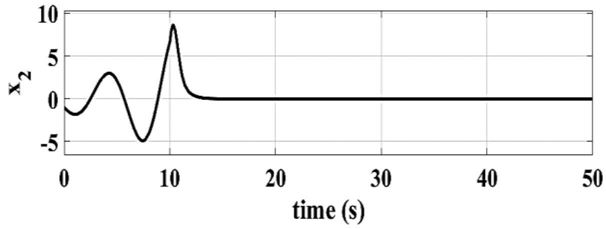

Fig. 6(b) Controlled state trajectory of x_2(concentration of substance B) vs $time$ (seconds)

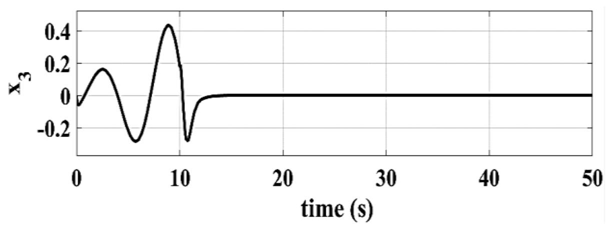

Fig. 6(c) Controlled state trajectory of x_3(concentration of substance C) vs $time$(seconds)

For the second case, solving the set of LMIs in equations (15) and (16) taking $\alpha = 2$ using MATLAB YALMIP toolbox [17], the optimized values of symmetric positive definite matrix and state feedback gain matrices are given as-

$$P = \begin{bmatrix} 27.4868 & 31.3198 & -6.2322 \\ 31.3198 & 44.0859 & -6.4242 \\ -6.2322 & -6.4242 & 1.8997 \end{bmatrix}$$

$$K_1 = [-24.7149 \quad -23.8654 \quad -1.5691]$$

$$K_2 = [-24.7149 \quad -23.8654 \quad 4.4309]$$

By substituting the above K_1 and K_2 values in the MATLAB Simulink, the controlled state trajectories are shown in Figure 7(a), 7(b) and 7(c) respectively. The control input is added after $t > 10$ sec for showing the oscillating nature up to 10 sec.

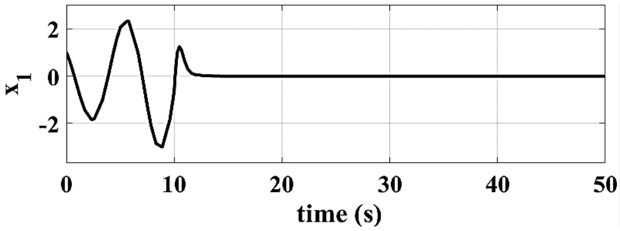

Fig. 7(a) Controlled state trajectory of x_1(concentration of substance A) vs $time$(seconds)

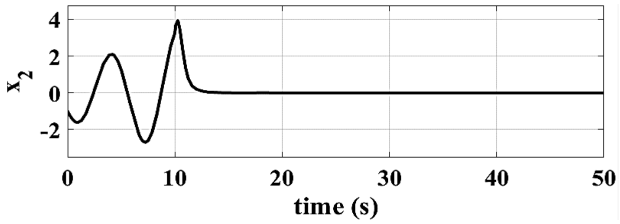

Fig. 7(b) Controlled state trajectory of x_2(concentration of substance B) vs $time$ (seconds)

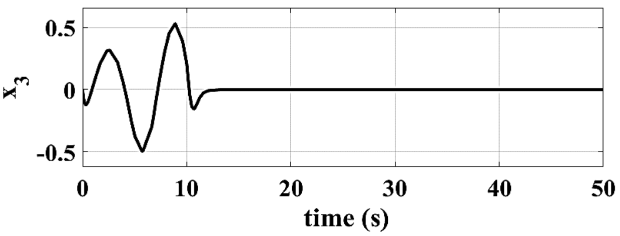

Fig. 7(c) Controlled state trajectory of x_3(concentration of substance C) vs $time$(seconds)

4 Mamdani Fuzzy Model

The Mamdani fuzzy controller is implemented using MATLAB fuzzy inference system toolbox. This method is implemented using IF-THEN rules. Consider the Rössler system in equation (2) along with the output $y = x_1(t)$ [18].

$$\dot{x}_1(t) = -x_2(t) - x_3(t)$$

$$\dot{x_2}(t) = x_1(t) + ax_2(t) \qquad (23)$$

$$\dot{x}_3(t) = bx_1(t) + (x_1(t) - c)x_3(t)$$

$$y = x_1(t)$$

Figure 8 represents the block diagram for Mamdani fuzzy model associated with Rössler system.

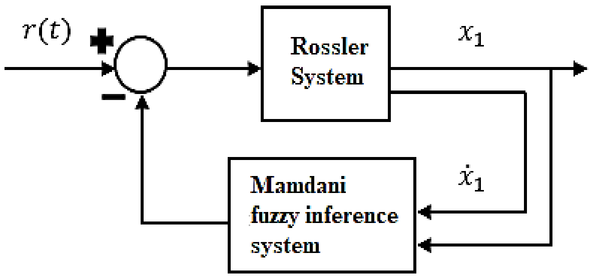

Fig. 8 Mamdani fuzzy model of Rössler system

The steps involved in the fuzzy inference system to obtain the crisp value at the output is: defining the inputs and outputs, deciding the universe of discourse and membership functions, defining the rules and combining them to obtain the overall control response, defuzzification of the overall fuzzified control response.

The next subsections *4.1*, *4.2*, *4.3* and *4.4* are illustrated on the basis of the above-mentioned steps.

4.1 Defining Inputs and Outputs

The Mamdani fuzzy inference system has two inputs: x_1 and $\dot{x}_1$ and one output: u. The reference input to the closed loop system is taken as: step input.

4.2 Universe of Discourse and Membership Functions

Table 1. Variables and Membership Functions

Input and Output Variables	Universe of Discourse	Type of Membership functions
x_1	-30 to +30	Triangular
$\dot{x}_1$	-300 to +300	Triangular
u	-30 to +30	Triangular

The formula for triangular membership function is given by [5]-

$$f(x,a,b,c)=\begin{cases}0 & x\le a\\ \dfrac{x-a}{b-a} & a\le x\le b\\ \dfrac{c-x}{c-b} & b\le x\le c\\ 0 & x\ge c\end{cases} \quad (24)$$

The membership functions of x_1, $\dot{x}_1$ and u are shown in Figure 9, Figure 10 and Figure 11 respectively.

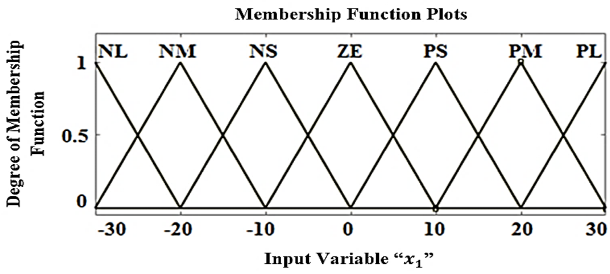

Fig. 9 Membership function plot of x_1

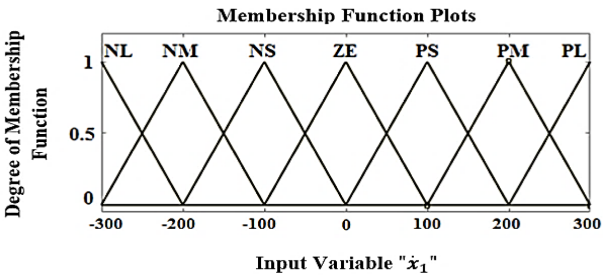

Fig. 10 Membership function plot of $\dot{x}_1$

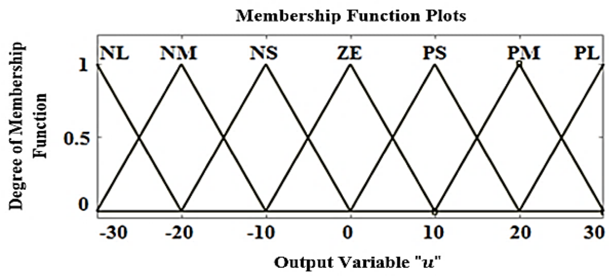

Fig. 11 Membership function plot of u

4.3 Defining Rule Matrix Table

Table 2. Fuzzy Rule Matrix

$\dot{x}_1$ \ x_1	NL	NM	NS	ZE	PS	PM	PL
NL	NL	NM	NM	NM	NS	NS	ZE
NM	NM	NM	NS	NS	NS	ZE	PS
NS	NM	NM	NS	NS	ZE	PS	PS
ZE	NM	NS	NS	ZE	PS	PS	PM
PS	NS	NS	ZE	PS	PS	PM	PM
PM	NS	ZE	PS	PS	PS	PM	PM
PL	ZE	PS	PS	PM	PM	PM	PL

where

NL = Negative Large

NM = Negative Medium

NS = Negative Small

ZE = Zero

PS = Positive Small

PM = Positive Medium

PL= Positive Large

The rule matrix table is expressed as:

Rule 1: IF x_1 is NL and IF $\dot{x}_1$ is NL - THEN u is NL

Rule 2: IF x_1 is NM and IF $\dot{x}_1$ is NL - THEN u is NM

.

.

.

Rule 48: IF x_1 is PM and IF $\dot{x}_1$ is PL - THEN u is PM

Rule 49: IF x_1 is PL and IF $\dot{x}_1$ is PL - THEN u is PL

Based on the above rule matrix table, the fuzzy surface plot is shown in Figure 12.

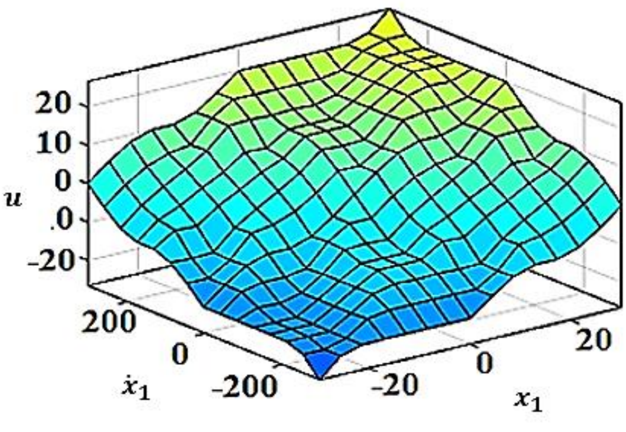

Fig. 12 Fuzzy Surface plot

4.4 Defuzzification Process

The Defuzzification method is chosen to be Centroid (also known as Center of gravity) method. The formula for centroid method is given by-

$$X^* = \sum_{x=1}^{n} \frac{\mu(x).x}{\sum_{x=1}^{n} \mu(x)}, \tag{25}$$

where x is the fuzzy variable and $\mu(x)$ is the membership function and X^* is the defuzzified value.

4.5 Simulation Results and Output

The control input is added after $t > 10$ sec for showing the oscillating nature up to 10 sec. The controlled state trajectories of x_1, x_2 and x_3 are shown in Figure 13(a), Figure 13(b) and Figure 13(c) respectively for the first case.

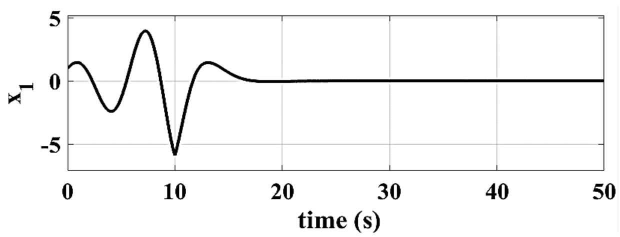

Fig. 13(a) Controlled state trajectory of x_1(concentration of substance A) vs $time$(seconds)

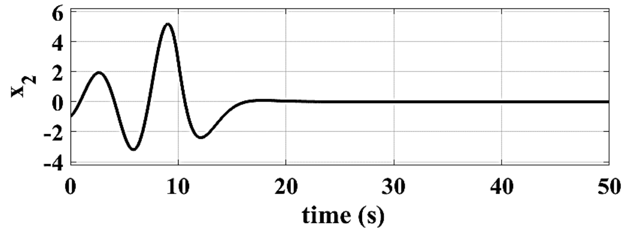

Fig. 13(b) Controlled state trajectory of x_2(concentration of substance B) vs $time$ (seconds)

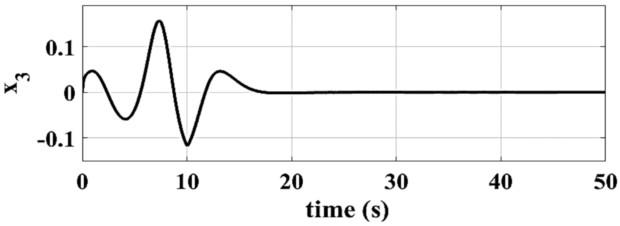

Fig. 13(c) Controlled state trajectory of x_3(concentration of substance C) vs $time$ (seconds)

The control input is added after $t > 10$ sec for showing the oscillating nature up to 10 sec. The controlled state trajectories of x_1, x_2 and x_3 are shown in Figure 14(a), Figure 14(b) and Figure 14(c) respectively for the second case.

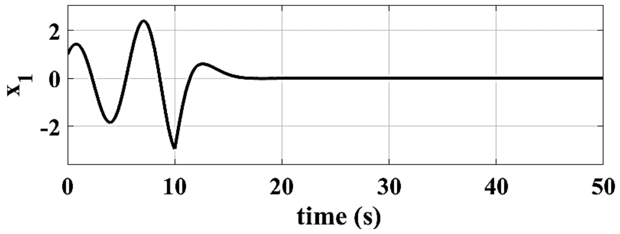

Fig. 14(a) Controlled state trajectory of x_1(concentration of substance A) vs $time$(seconds)

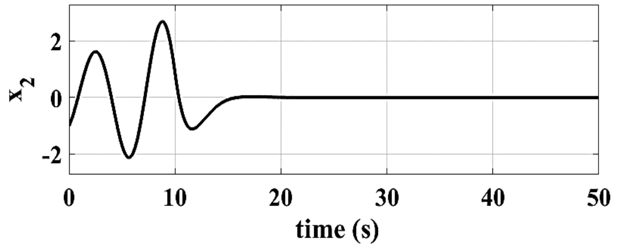

Fig. 14(b) Controlled state trajectory of x_2(concentration of substance B) vs $time$ (seconds)

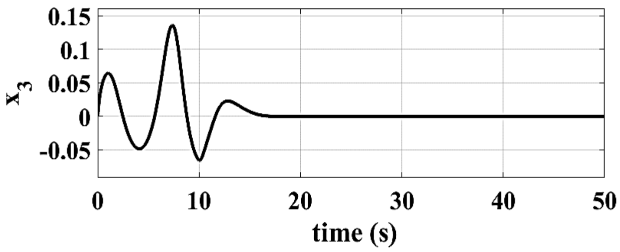

Fig. 14(c) Controlled state trajectory of x_3(concentration of substance C) vs $time$ (seconds)

5 Sliding Mode Control

For the design of a robust controller for an uncertain system, SMC is frequently used [19]. The main advantages of the SMC are: easily realized, response is faster, transient response is better and not sensitive to plant variations or external disturbances. The design of the SMC consists of two steps: first is the design of an asymptotically stable sliding surface and second is the design of a suitable reaching law such that the system states reaches the sliding surface in finite time.

The next subsection *5.1*, is about the description of Rössler system and representation of the system in matrix form.

5.1 System Description

Consider the Rössler system with two input terms u_1 and u_2 in the second and third equation of equation (2).

$$\dot{x_1}(t) = -x_2(t) - x_3(t)$$

$$\dot{x_2}(t) = x_1(t) + ax_2(t) + u_1 \qquad (26)$$

$$\dot{x_3}(t) = bx_1(t) + (x_1(t) - c)x_3(t) + u_2$$

where $a = 0.34, b = 0.4, c = 14$ are the system parameters and $u = (u_1, u_2)$ are the control inputs to be designed. The Rössler system can be represented as [19]:

$$\dot{x} = Ax + f(x) + Bu \qquad (27)$$

where

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ b & 0 & -c \end{bmatrix}, f(x) = \begin{bmatrix} 0 \\ 0 \\ x_1(t)x_3(t) \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The control inputs in a SMC is designed as:

$$u = u_{eq} + u_s \qquad (28)$$

where u_{eq} is called the equivalent control and u_s is called the switching control.

The next subsection *5.2*, is about the design of the sliding surface.

5.2 Sliding Surface Design

The basic aim in this section is the design the sliding surface for the nonlinear Rössler system in equation (26). The presence of the nonlinear term $f(x)$ in the system makes the design of the sliding surface difficult. Therefore, a dynamic compensator is introduced. The equation of the dynamic compensator is given by [19]-

$$\dot{z} = Kx - z \qquad (29)$$

where z represents the compensator states in R^2, K is the suitable gain matrix to be designed in $R^{2\times3}$ using LMI. The sliding surface is designed as

$$s = Cx + z \qquad (30)$$

where $C = [C_1, I]$ in $R^{2\times3}$, C_1 in $R^{2\times1}$ and I in $R^{2\times2}$. Differentiating equation (30) with respect to time yields

$$\dot{s} = C\dot{x} + \dot{z} = CAx + Cf(x) + CBu + Kx - z \quad (31)$$

It can be easily inferred that $CB = I$. For the design of the equivalent control u_{eq} make $\dot{s} = 0$. So u_{eq} is given by-

$$u_{eq} = -CAx - Cf(x) - Kx + z \quad (32)$$

Substituting equation (32) in equation (27) yields

$$\dot{x} = Ax + f(x) + B(-CAx - Cf(x) - Kx + z)$$
$$= (A - BCA - BK)x + Bz \quad (33)$$

On the sliding surface $s = 0$ yields $z = -Cx$. Hence, equation (33) is expressed as

$$\dot{x} = (A - BCA - BK - BC)x \quad (34)$$

Based on the Lyapunov's stability criterion, the closed loop system in equation (34) is asymptotically stable if there exists a scalar energy function $V(x) > 0$ and $\dot{V}(x) < 0$. Consider $V(x) = \frac{1}{2}x^T x$ as the Lyapunov function candidate of the system in equation (34). The derivative of $V(x)$ with respect to time is given by-

$$\dot{V}(x) = \frac{1}{2}x^T \Psi x$$

where

$$\Psi = A - BCA - BK - BC + A^T - A^T C^T B^T - K^T B^T - C^T B^T \quad (35)$$

The controller design is illustrated in the next subsection *5.3*.

5.3 Controller Design

The switching control u_s is obtained by considering the exponential reaching law [19]. The exponential reaching law is given as-

$$\dot{s} = -\mu s, \quad \mu > 0 \quad (36)$$

Here $\mu = 1.5$. The solution to equation (36) is $s = s(0)e^{-\mu t}$ [10] implies that the states of the system $x(t)$ approaches the switching manifolds faster when s is large. For achieving asymptotic stability of u_s, the Lyapunov function is chosen as-

$$V(x) = \frac{1}{2}s^T s \quad (37)$$

The derivative of $V(x)$ with respect to time is given as-

$$\dot{V}(x) = \frac{1}{2}(s^T\dot{s} + \dot{s}^T s) = -\mu\|s\| \quad (38)$$

By making $\dot{s} = 0$, the switching control is given as-

$$u_s = -\mu s \quad (39)$$

Substituting equation (32) and equation (39) in equation (28), the overall SMC is given as-

$$u = (-CAx - Cf(x) - Kx + z) + (-\mu s) \quad (40)$$

In subsection *5.4*, simulation results & output are shown.

5.4 Simulation Results and Output

For the first case, solving the LMI in equation (35) using MATLAB YALMIP toolbox [17] the optimized values of C and K are obtained as-

$$C = \begin{bmatrix} -0.9201 & 1 & 0 \\ 0.9201 & 0 & 1 \end{bmatrix}$$

$$K = \begin{bmatrix} -0.0799 & -0.4245 & 0 \\ -1.9201 & 0 & 1.4158 \end{bmatrix}$$

The control input is added after $t > 10$ sec for showing the oscillating nature up to 10 sec. The controlled state trajectories of x_1, x_2 and x_3 are shown in Figure 15(a), Figure 15(b) and Figure 15(c) respectively.

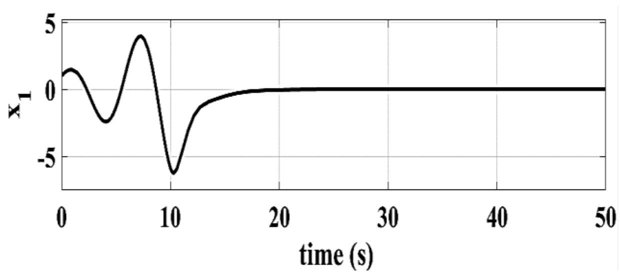

Fig. 15(a) Controlled state trajectory of x_1(concentration of substance A) vs $time$ (seconds)

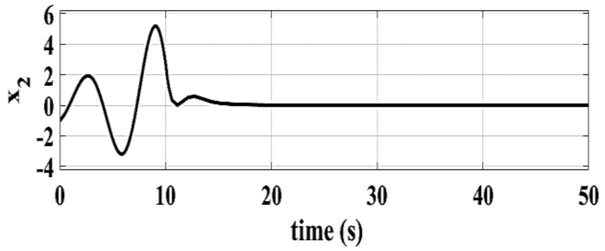

Fig. 15(b) Controlled state trajectory of x_2(concentration of substance B) vs $time$ (seconds)

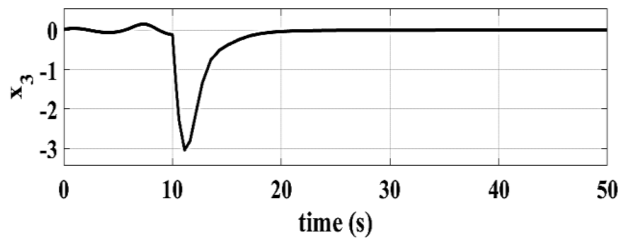

Fig. 15(c) Controlled state trajectory of x_3(concentration of substance C) vs $time$ (seconds)

For the second case, solving the LMI in equation (35) using MATLAB YALMIP toolbox [17] the optimized values of C and K are obtained as-

$$C = \begin{bmatrix} -0.9201 & 1 & 0 \\ 0.9201 & 0 & 1 \end{bmatrix}$$

$$K = \begin{bmatrix} -0.0799 & -0.4245 & 0 \\ -1.9201 & 0 & 1.4158 \end{bmatrix}$$

The control input is added after $t > 10\,\text{sec}$ for showing the oscillating nature up to 10 sec. The controlled state trajectories of x_1, x_2 and x_3 are shown in Figure 16(a), Figure 16(b) and Figure 16(c) respectively.

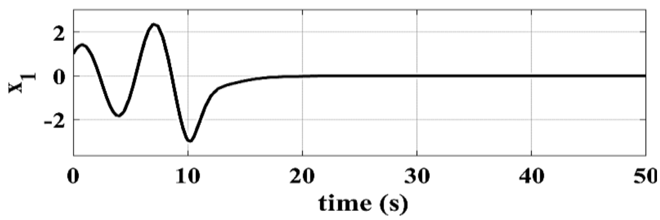

Fig. 16(a) Controlled state trajectory of x_1(concentration of substance A) vs $time$ (seconds)

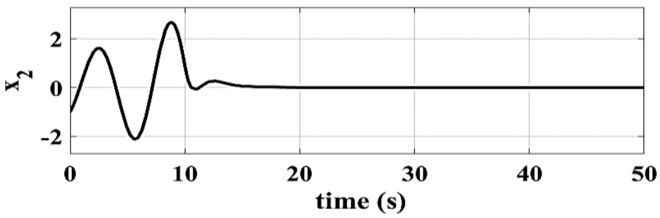

Fig. 16(b) Controlled state trajectory of x_2(concentration of substance B) vs $time$ (seconds)

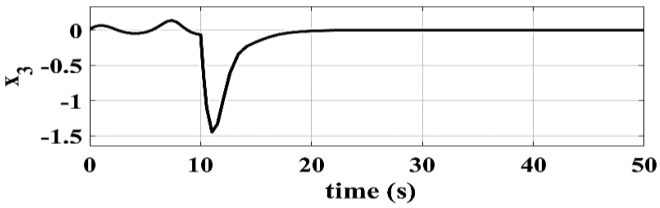

Fig. 16(c) Controlled state trajectory of x_3(concentration of substance C) vs $time$ (seconds)

6 Comparison of T-S Fuzzy Controller with Mamdani Fuzzy Controller with Sliding Mode Controller

For the first case, Table 3 indicates the comparison chart among the T-S fuzzy controller, Mamdani fuzzy controller and the sliding mode controller.

Table 3. Comparing T-S fuzzy controller, Mamdani fuzzy controller, Sliding Mode Controller

	T-S Fuzzy Control	Mamdani Fuzzy Control	Sliding Mode Control
Maximum value of x_1 (concentration of substance A)	3.741	3.976	3.941
Maximum value of x_2 (concentration of substance B)	8.631	5.194	5.164
Maximum value of x_3 (concentration of substance C)	0.437	0.157	0.162

Transient period of x_1 (concentration of substance A) (in seconds)	15.643	25	24.668
Transient period of x_2 (concentration of substance B) (in seconds)	15.643	25	24.668
Transient period of x_3 (concentration of substance C) (in seconds)	15.643	25	24.668

For the second case, Table 4 indicates the comparison chart among the T-S fuzzy controller, Mamdani fuzzy controller and the sliding mode controller.

Table 4. Comparing T-S fuzzy controller, Mamdani fuzzy controller, Sliding Mode Controller

	T-S Fuzzy Control	Mamdani Fuzzy Control	Sliding Mode Control
Maximum value of x_1 (concentration of substance A)	2.335	2.4	2.370
Maximum value of x_2 (concentration of substance B)	3.933	2.690	2.698
Maximum value of x_3 (concentration of substance C)	0.527	0.132	0.135
Transient period of x_1 (concentration of substance A) (in seconds)	15.643	25	24.668
Transient period of x_2 (concentration of substance B) (in seconds)	15.643	25	24.668
Transient period of x_3 (concentration of substance C) (in seconds)	15.643	25	24.668

7 Conclusion

It is clearly observed from the MATLAB simulation that the T-S fuzzy controller, Mamdani fuzzy controller and sliding mode controller successfully drives the highly chaotic system states to a stable steady state value. So far as settling time is concerned, the T-S fuzzy controller has better performance as compared to the other two controllers as it is evident from Table 3 and Table 4. The T-S fuzzy controller has a faster settling time but it has the highest magnitude of oscillations during the transient period. From both the tables, it can be easily seen that during the transient periods, the sliding mode controller has less magnitude of oscillation compared to the other two controllers. The Mamdani fuzzy controller takes maximum time to settle down and its magnitude of oscillations is slightly greater than that of SMC. Among the three controllers, the SMC has the best overall performance in reference to transient period and magnitude of oscillations with two input terms.

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