A Case Study on Squarification in Probabilistic Evolution Theory (PREVTH) for Henon-Heiles Systems

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Abstract: The Probabilistic Evolution Theory (PREVTH) has been effectively developed in recent few years to solve the explicit autonomous ODE(s) accompanied by certain initial conditions. The theory focuses on the cases where right hand side functions are conical in unknowns at the right hand side. If it is not so certain space extension technies are used to get conicality. [1–6]. Theory gives an analytical Kronecker power series solution almost for all practically encountered systems. The squarification reduces the enormous sparsity and gets very high efficiency. This work is designed for the application of PREVTH Squarification on Henon-Heiles systems as a case study.

Key–Words: Ordinary differential equation, probabilistic evolution theory, Henon-Heiles, squarification

1 Introduction

This section focuses on the initial value problem of a set of explicit first order autonomous ODE with right hand side functions given by

$$\dot{\mathbf{x}}(t) = \mathbf{F}_0 + \mathbf{F}_1\mathbf{x}(t) + \mathbf{F}_2\mathbf{x}(t)^{\otimes 2} \qquad (1)$$

where t represents the independent variable we can call “time”. Here, $\otimes$ on the exponent means Kronecker power. $\mathbf{x}$ is a n element vector which is composed of unknowns. After “Constancy Adding Space Extension (CASE)” which is detailed in [4–6], is applied and flexibilities are chosen appropriately, the solution of (1) is given below

$$\mathbf{x}(t) = e^{-\beta t}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{1-e^{-\beta t}}{\beta}\right)^{j}\mathbf{T}_j\mathbf{a}^{\otimes j+1} \qquad (2)$$

where $\mathbf{T}_j$s are certain rectangular matrices that are the type $n \times n^j$. This solution is written under the specifications

$$\mathbf{F}_0 \equiv \mathbf{0}, \quad \mathbf{F}_1 \equiv -\beta\mathbf{I}_n \qquad (3)$$

Here, β is arbitrarily inserted parameter whose value can be determined in accordance with certain needs. $\mathbf{T}_j$s rectangular matrices in (2) can be written that

$$\mathbf{T}_j \equiv \prod_{k=1}^{j}\mathbf{M}_k, \quad j = 0, 1, 2, ... \qquad (4)$$

where $\mathbf{M}$ matrices $n^k \times n^{k+1}$ dimensional rectangular matrices and are called “Monocular Matrices”. $\mathbf{M}_k$ is explicitly given below

$$\mathbf{M}_k \equiv \sum_{\ell=0}^{k-1}\mathbf{I}_n^{\otimes \ell}\otimes\mathbf{F}\otimes\mathbf{I}_n^{\otimes k-1-\ell}, \quad k = 1, 2, ... \qquad (5)$$

$\mathbf{T}_j$s can be called “Telescope Matrices”since they carry the matrices from n^j dimensional spaces to n dimensional spaces. Telescope matrices which are extremely sparse (having plenty of zero elements), can be put into a more concise structure by using Squarification. Thus, the disadvantages coming from the sparsity of the telescopic expansion in Kronecker powers of initial vector, disappear. The most promising one is the squarifictaion of the telescope matrices [5].

2 Squarification of Telescope Matrices

In (2), if $\mathbf{a}$ is an n element vector, $\mathbf{a}^{\otimes j+1}$ is a very higher dimensional vector with type of n^{j+1}. To get conciseness in calculations for the product of the $n \times n^j$ type telescope $\mathbf{T}_j$ matrix and $\mathbf{a}^{\otimes j+1}$ vector with n^{j+1} element, we can foresee that

$$\mathbf{T}_j\mathbf{a}^{\otimes j+1} = \mathbf{S}_j(\mathbf{a})\mathbf{a}, \quad j = 0, 1, 2, ... \qquad (6)$$

where $\mathbf{S}_j$s are $n \times n$ type square matrices. These can be called “Squarified Telescope Matrices”or briefly “SquTelMats”.

In (4), when j becomes 0, the product over $\mathbf{M}$ matrices is assumed to be identity matrix. In that case, it is obtained that

$$\mathbf{T}_0\mathbf{a} = \mathbf{a} = \mathbf{S}_0(\mathbf{a})\mathbf{a} \implies \mathbf{S}_0(\mathbf{a}) \equiv \mathbf{I}_n \qquad (7)$$

I_n is an identity matrix with the type $n \times n$.

When taking j equal to 1 from (4), it is obtained that

$$\mathbf{T}_1\mathbf{a}^{\otimes 2} = \mathbf{M}_1\mathbf{a}^{\otimes 2} = \mathbf{F}\mathbf{a}^{\otimes 2} \qquad (8)$$

where $\mathbf{F}$ is $n \times n^2$ type generator matrix.

Let us now partition $\mathbf{F}$ which is of $n \times n^2$ type to $n \times n$ type square blocks as follows

$$\mathbf{F} \equiv \left[\begin{array}{ccc} \mathbf{F}^{(1)} & \cdots & \mathbf{F}^{(n)} \end{array} \right] \qquad (9)$$

From this equation, we can write

$$\mathbf{F} = \sum_{i=1}^{n} \mathbf{e}_i^T \otimes \mathbf{F}^{(i)} \qquad (10)$$

Here $\mathbf{e}$ is an n element unit vector. Thus multiplying $\mathbf{F}$ and $\mathbf{a}^{\otimes 2}$, we can write that

$$\mathbf{F}\mathbf{a}^{\otimes 2} = \sum_{i=1}^{n} \left(\mathbf{e}_i^T \otimes \mathbf{F}^{(i)}\right)(\mathbf{a} \otimes \mathbf{a}) \qquad (11)$$

Using the distributive properties of Kronecker product, from (11) we can conclude that

$$\left(\mathbf{e}_i^T \otimes \mathbf{F}^{(i)}\right)(\mathbf{a} \otimes \mathbf{a}) = \left(\mathbf{e}_i^T\mathbf{a}\right) \otimes \left(\mathbf{F}^{(i)}\mathbf{a}\right) \qquad (12)$$

The Kronecker product of scalar and matrix or vector equals to the product of scalar and matrix or vector. In that case

$$\left(\mathbf{e}_i^T \otimes \mathbf{F}^{(i)}\right)(\mathbf{a} \otimes \mathbf{a}) = a_i\mathbf{F}^{(i)}\mathbf{a}, \quad i = 1, 2, 3, ... \qquad (13)$$

Therefore we can write

$$\mathbf{F}\mathbf{a}^{\otimes 2} = \left(\sum_{i=1}^{n} a_i\mathbf{F}^{(i)}\right)\mathbf{a} \qquad (14)$$

Using the Kronecker product of two different vectors, we can conclude that

$$\mathbf{F}(\mathbf{a} \otimes \mathbf{b}) = \left(\sum_{i=1}^{n} a_i\mathbf{F}^{(i)}\right)\mathbf{b} \qquad (15)$$

The matrix $\mathbf{F}$ is squarified by the vector $\mathbf{a}$ as follows

$$\lfloor \mathbf{F}, \mathbf{a} \rceil = \sum_{i=1}^{n} a_i\mathbf{F}^{(i)} \qquad (16)$$

Here $\lfloor$ symbol stands for taking the base from left hand side and $\rceil$ symbol stands for pushing this base from right hand side.

The joint work of Melike Ebru Kırkın and Coşar Gözükırmızı describes reductive cases between its third and fifth sections inclusive. [2].

3 The Case Of Symmetric Commutative Blocks

When all blocks of $\mathbf{F}$ are symmetric commutative and eigenvectors sets of these are same, we conclude that

$$\mathbf{F} \equiv \left[\begin{array}{ccc} \mathbf{\Phi}_1 & \ldots & \mathbf{\Phi}_n \end{array} \right], \quad \mathbf{\Phi}j\mathbf{\Phi}_k - \mathbf{\Phi}_k\mathbf{\Phi}j = \mathbf{0}$$
$$\mathbf{\Phi}_j = \mathbf{\Phi}_j^T, \quad j, k = 1, 2, \ldots, n \qquad (17)$$

In that case, the spectral composition of commutative matrices dictates us that

$$\mathbf{\Phi}_j = \varphi_{j,1}\mathbf{u}_1\mathbf{u}_1^T + \cdots + \varphi_{j,n}\mathbf{u}_n\mathbf{u}_n^T\,,$$
$$j = 1, 2, ..., n \qquad (18)$$

Therefore, in (15), substituting $\mathbf{\Phi}$ for $\mathbf{F}^{(i)}$, the image of Kronecker product of two vectors under $\mathbf{F}$ is concluded that

$$\mathbf{F}(\mathbf{a} \otimes \mathbf{b}) = \sum_{j=1}^{n} a_j\mathbf{\Phi}_j\mathbf{b}\,. \qquad (19)$$

From (18), substituting $\sum_{k=1}^{n} \varphi_{j,k}\mathbf{u}_k\mathbf{u}_k^T$ for $\mathbf{\Phi}_j$

$$\mathbf{F}(\mathbf{a} \otimes \mathbf{b}) = \sum_{j=1}^{n} a_j \sum_{k=1}^{n} \varphi_{j,k}\mathbf{u}_k\mathbf{u}_k^T\mathbf{b} \qquad (20)$$

where a linear combination of the eigenvalues of each $\mathbf{\Phi}$s appear. This equation can be rewritten

$$\mathbf{F}(\mathbf{a} \otimes \mathbf{b}) = \sum_{k=1}^{n} \left(\sum_{j=1}^{n} a_j\varphi_{j,k}\right)\left(\mathbf{u}_k^T\mathbf{b}\right)\mathbf{u}_k \qquad (21)$$

If we consider the vector $\boldsymbol{\varphi}_k$whose ascending index elements are $\varphi_{1,k}, \varphi_{2,k}, \cdots \varphi_{n,k}$ then

$$\mathbf{F}(\mathbf{a} \otimes \mathbf{b}) = \sum_{k=1}^{n} \left(\mathbf{a}^T\boldsymbol{\varphi}_k\right)\left(\mathbf{u}_k^T\mathbf{b}\right)\mathbf{u}_k \qquad (22)$$

where the orthogonality between $\mathbf{a}$ and φ is apparent and $\mathbf{b}$ is orthogonal to all.

3.1 Squtelmats For Symmetric Commutative Blocks

Consider S4, where there are 24 additive expressions, for squtelmats to symmetric block case which involves the following expression. We can focus on this

$$\mathbf{F}(\mathbf{I}_n \otimes \mathbf{F})(\mathbf{I}_n^{\otimes 2} \otimes \mathbf{F})(\mathbf{I}_n^{\otimes 3} \otimes \mathbf{F})\mathbf{a}^{\otimes 5} \qquad (23)$$

This multiplication should be from right to left. From (23), the first multiplication can be treated as follows

$$(\mathbf{I}_n^{\otimes 3} \otimes \mathbf{F})\mathbf{a}^{\otimes 5} = \mathbf{a}^{\otimes 3} \otimes \mathbf{F}\mathbf{a}^{\otimes 2} = \mathbf{a}^{\otimes 3} \otimes \mathbf{F}(\mathbf{a} \otimes \mathbf{a})$$
$$= \left[\mathbf{a}^{\otimes 3} \otimes \left(\sum_{k=1}^{n} \left(\mathbf{a}^T\varphi_k\right)\left(\mathbf{u}_k^T\mathbf{a}\right)\mathbf{u}_k\right)\right] \qquad (24)$$

If the identity matrix and the vector are in the same type, then their Kronecker product is equal to itself of the vector. Here, $\mathbf{F}\left(\mathbf{a}\otimes\mathbf{a}\right)$ can be written using (22). When the result is inserted to its place in (22), we can multiply out the next term

$$\begin{aligned}&\left(\mathbf{I}_n^{\otimes 2}\otimes\mathbf{F}\right)\left[\mathbf{a}^{\otimes 3}\otimes\left(\sum_{k=1}^{n}\left(\mathbf{a}^T\varphi_k\right)\left(\mathbf{u}_k^T\mathbf{a}\right)\mathbf{u}_k\right)\right]\\&=\mathbf{a}^{\otimes 2}\otimes\mathbf{F}\left(\mathbf{a}\otimes\left(\sum_{k=1}^{n}\left(\mathbf{a}^T\varphi_k\right)\left(\mathbf{u}_k^T\mathbf{a}\right)\mathbf{u}_k\right)\right)\end{aligned}\tag{25}$$

where, we have used the fact that $\mathbf{a}^{\otimes 3}$ can be considered as the Kronecker product of $\mathbf{a}^{\otimes 2}$ and $\mathbf{a}$. Thus by also using the Kronecker product of $\mathbf{a}^{\otimes 2}$ with the consistent identity matrix we can arrive at

$$\begin{aligned}&\left(\mathbf{I}_n^{\otimes 2}\otimes\mathbf{F}\right)\left[\mathbf{a}^{\otimes 3}\otimes\left(\sum_{k=1}^{n}\left(\mathbf{a}^T\varphi_k\right)\left(\mathbf{u}_k^T\mathbf{a}\right)\mathbf{u}_k\right)\right]\\&=\mathbf{a}^{\otimes 2}\otimes\sum_{k_2=1}^{n}\left(\mathbf{a}^T\varphi_{k_2}\right)\\&\left(\mathbf{u}_{k_2}^T\left[\sum_{k_1=1}^{n}\left(\mathbf{a}^T\varphi_{k_1}\right)\left(\mathbf{u}_{k_1}^T\mathbf{a}\right)\mathbf{u}_{k_1}\right]\right)\mathbf{u}_{k_2}\end{aligned}\tag{26}$$

Using orthonormality we can get

$$\begin{aligned}&\left(\mathbf{I}_n^{\otimes 2}\otimes\mathbf{F}\right)\left[\mathbf{a}^{\otimes 3}\otimes\left(\sum_{k=1}^{n}\left(\mathbf{a}^T\varphi_k\right)\left(\mathbf{u}_k^T\mathbf{a}\right)\mathbf{u}_k\right)\right]\\&=\mathbf{a}^{\otimes 2}\otimes\sum_{k_1=1}^{n}\left(\mathbf{a}^T\varphi_{k_1}\right)^2\left(\mathbf{u}_{k_1}^T\mathbf{a}\right)\mathbf{u}_{k_1}\end{aligned}\tag{27}$$

Multiplying out the next term with this expression we write

$$\begin{aligned}&\left(\mathbf{I}_n\otimes\mathbf{F}\right)\left(\mathbf{a}^{\otimes 2}\otimes\sum_{k_1=1}^{n}\left(\mathbf{a}^T\varphi_{k_1}\right)^2\left(\mathbf{u}_{k_1}^T\mathbf{a}\right)\mathbf{u}_{k_1}\right)\\&=\mathbf{a}\otimes\mathbf{F}\left(\mathbf{a}\otimes\sum_{k_1=1}^{n}\left(\mathbf{a}^T\varphi_{k_1}\right)^2\left(\mathbf{u}_{k_1}^T\mathbf{a}\right)\mathbf{u}_{k_1}\right)\\&=\mathbf{a}\otimes\sum_{k_1=1}^{n}\left(\mathbf{a}^T\varphi_{k_1}\right)^3\left(\mathbf{u}_{k_1}^T\mathbf{a}\right)\mathbf{u}_{k_1}\end{aligned}\tag{28}$$

and finally

$$\begin{aligned}&\mathbf{F}(\mathbf{I}_n\otimes\mathbf{F})(\mathbf{I}_n^{\otimes 2}\otimes\mathbf{F})(\mathbf{I}_n^{\otimes 3}\otimes\mathbf{F})\mathbf{a}^{\otimes 5}\\&=\sum_{k_1=1}^{n}\left(\mathbf{a}^T\varphi_{k_1}\right)^4\left(\mathbf{u}_{k_1}^T\mathbf{a}\right)\mathbf{u}_{k_1}\end{aligned}\tag{29}$$

where we have obtained one finite sum forming a linear combination of the eigenvectors.

Consider the another term of $\mathbf{S}_4$

$$\mathbf{F}(\mathbf{I}_n\otimes\mathbf{F})(\mathbf{I}_n^{\otimes 2}\otimes\mathbf{F})(\mathbf{I}_n^{\otimes 2}\otimes\mathbf{F}\otimes\mathbf{I}_n)\mathbf{a}^{\otimes 5}\,.\tag{30}$$

This structure is more complicated. When the similar operations are applied, we can obtain that

$$\begin{aligned}&\mathbf{F}(\mathbf{I}_n\otimes\mathbf{F})(\mathbf{I}_n^{\otimes 2}\otimes\mathbf{F})(\mathbf{I}_n^{\otimes 2}\otimes\mathbf{F}\otimes\mathbf{I}_n)\mathbf{a}^{\otimes 5}\\&=\sum_{k_2=1}^{n}\sum_{k_1=1}^{n}\left(\mathbf{a}^T\varphi_{k_2}\right)\left(\mathbf{a}^T\varphi_{k_1}\right)\left(\mathbf{u}_{k_1}^T\mathbf{a}\right)\\&\left(\mathbf{u}_{k_1}^T\varphi_{k_2}\right)\left(\mathbf{u}_{k_2}^T\mathbf{a}\right)\mathbf{u}_{k_2}\end{aligned}\tag{31}$$

This squarification problem is important to simplify to problem of the finding linear combination coefficients. Moreover working with scalar with coefficients are simplier than working with scalar matrices.

4 The Case Of Equal Blocks

Under consideration of equality blocks of $\mathbf{F}$, we can obtain the squarification results, as follows

$$\begin{aligned}\lfloor\mathbf{F},\mathbf{a}\rceil&=\sum_{i=1}^{n}a_i\,\mathbf{\Phi}=\sum_{i=1}^{n}\mathbf{e}_i^T\mathbf{a}\mathbf{\Phi},\\\lfloor\mathbf{F},\mathbf{a}\rceil^j&=\left(\sum_{i=1}^{n}a_i\right)^j\mathbf{\Phi}^j,\\\lfloor\mathbf{F},\lfloor\mathbf{F},\mathbf{a}\rceil\mathbf{a}\rceil&=\sum_{i_2=1}^{n}\mathbf{e}_{i_2}^T\left(\sum_{i=1}^{n}\mathbf{e}_i^T\mathbf{a}\mathbf{\Phi}\right)\mathbf{a}\mathbf{\Phi}\\&=\sum_{i_2=1}^{n}\sum_{i_1=1}^{n}\mathbf{e}_{i_2}^T\left(\mathbf{e}_{i_1}^T\mathbf{a}\mathbf{\Phi}\right)\mathbf{a}\mathbf{\Phi}\end{aligned}\tag{32}$$

$$\begin{aligned}&\lfloor\mathbf{F},\lfloor\mathbf{F},\lfloor\mathbf{F},\mathbf{a}\rceil\mathbf{a}\rceil\mathbf{a}\rceil=\\&\sum_{i_3=1}^{n}\sum_{i_2=1}^{n}\sum_{i_1=1}^{n}\mathbf{e}_{i_3}^T\left(\mathbf{e}_{i_2}^T\left(\mathbf{e}_{i_1}^T\mathbf{a}\mathbf{\Phi}\right)\mathbf{a}\mathbf{\Phi}\right)\mathbf{a}\mathbf{\Phi}\end{aligned}\tag{33}$$

These reductions are derived from commutativity and symmetry together.

5 The Case Of Equal Identity Matrix Blocks

Under consideration of equality identity matrix blocks of $\mathbf{F}$, the squarifications can be concluded that

$$\lfloor\mathbf{F},\mathbf{a}\rceil=\left(\sum_{i=1}^{n}\mathbf{e}_i^T\mathbf{a}\right)\mathbf{I}=\sum_{i=1}^{n}a_i\,\mathbf{I}\,,$$

$$\lfloor\mathbf{F},\mathbf{a}\rceil^j=\left(\sum_{i=1}^{n}a_i\right)^j\mathbf{I}\,,$$

$$\lfloor \mathbf{F}, \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \rceil = \sum_{i_2=1}^{n} \sum_{i_1=1}^{n} \mathbf{e}_{i_2}^T \left(\mathbf{e}_{i_1}^T \mathbf{a} \right) \mathbf{a}$$
$$= \left(\sum_{i_2=1}^{n} \sum_{i_1=1}^{n} a_{i_1} a_{i_2} \right) \mathbf{I} \qquad (34)$$

$$\lfloor \mathbf{F}, \lfloor \mathbf{F}, \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \rceil \mathbf{a} \rceil$$
$$= \sum_{i_3=1}^{n} \sum_{i_2=1}^{n} \sum_{i_1=1}^{n} \mathbf{e}_{i_3}^T \left(\mathbf{e}_{i_2}^T \left(\mathbf{e}_{i_1}^T \mathbf{a} \right) \mathbf{a} \right) \mathbf{a}$$
$$= \left(\sum_{i_3=1}^{n} \sum_{i_2=1}^{n} \sum_{i_1=1}^{n} a_{i_1} a_{i_2} a_{i_3} \right) \mathbf{I} \qquad (35)$$

It is used since a squarification produces a square matrix by definition.

5.1 Squtelmats For Equal Identity Matrix Blocks

Therefore some squtelmats can be given as follows

$$\mathbf{S}_0 = \mathbf{I},$$
$$\mathbf{S}_1 = \left(\sum_{i=1}^{n} a_i \right) \mathbf{I},$$
$$\mathbf{S}_2 = 2 \left(\sum_{i=1}^{n} a_i \right)^2 \mathbf{I},$$
$$\mathbf{S}_3 = 6 \left(\sum_{i=1}^{n} a_i \right)^3 \mathbf{I} \qquad (36)$$

6 Recursion Of Squarification

$$\mathbf{F}(\mathbf{a} \otimes \mathbf{b}) = \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{b} \qquad (37)$$

$$\begin{aligned} \mathbf{F}(\mathbf{F} \otimes \mathbf{I})\mathbf{a}^{\otimes 3} &= \mathbf{F}(\mathbf{F}\mathbf{a}^{\otimes 2} \otimes \mathbf{a}) \\ &= \mathbf{F}(\lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \otimes \mathbf{a}) \\ &= \lfloor \mathbf{F} \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \rceil \mathbf{a} \end{aligned} \qquad (38)$$

$$\begin{aligned} \mathbf{F}(\mathbf{I} \otimes \mathbf{F})\mathbf{a}^{\otimes 3} &= \mathbf{F}(\mathbf{a} \otimes \mathbf{F}\mathbf{a}^{\otimes 2}) \\ &= \mathbf{F}(\mathbf{a} \otimes \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a}) \\ &= \lfloor \mathbf{F}, \mathbf{a} \rceil \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \end{aligned} \qquad (39)$$

$$\begin{aligned} \mathbf{S}_2 &= \lfloor \mathbf{F} \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \rceil \mathbf{a} + \lfloor \mathbf{F}, \mathbf{a} \rceil \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \\ &= \lfloor \mathbf{F}, \mathbf{a} \rceil^2 + \lfloor \mathbf{F} \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \rceil \mathbf{a} \\ &= \mathbf{S}_1(\mathbf{a})^2 + \lfloor \mathbf{F} \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \rceil \mathbf{a} \end{aligned} \qquad (40)$$

$$\begin{aligned} & \mathbf{F}(\mathbf{I} \otimes \mathbf{F})(\mathbf{I} \otimes \mathbf{I} \otimes \mathbf{F})\mathbf{a}^{\otimes 4} \\ = \; & \mathbf{F}(\mathbf{I} \otimes \mathbf{F})(\mathbf{a} \otimes \mathbf{a} \otimes \mathbf{F}\mathbf{a}^{\otimes 2}) \\ = \; & \mathbf{F}(\mathbf{I} \otimes \mathbf{F})(\mathbf{a} \otimes \mathbf{a} \otimes \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a}) \\ = \; & \mathbf{F}(\mathbf{a} \otimes \mathbf{F}(\mathbf{a} \otimes \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a})) \\ = \; & \mathbf{F}(\mathbf{a} \otimes \lfloor \mathbf{F}, \mathbf{a} \rceil \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a}) \\ = \; & \lfloor \mathbf{F}, \mathbf{a} \rceil \lfloor \mathbf{F}, \mathbf{a} \rceil^2 \mathbf{a} \\ = \; & \lfloor \mathbf{F}, \mathbf{a} \rceil^3 \mathbf{a} \end{aligned} \qquad (41)$$

$$\begin{aligned} & \mathbf{F}(\mathbf{I} \otimes \mathbf{F})(\mathbf{I} \otimes \mathbf{F} \otimes \mathbf{I})\mathbf{a}^{\otimes 4} \\ = \; & \mathbf{F}(\mathbf{I} \otimes \mathbf{F})(\mathbf{a} \otimes \mathbf{F}\mathbf{a}^{\otimes 2} \otimes \mathbf{a}) \\ = \; & \mathbf{F}(\mathbf{I} \otimes \mathbf{F})(\mathbf{a} \otimes \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \otimes \mathbf{a}) \\ = \; & \mathbf{F}(\mathbf{a} \otimes \mathbf{F}(\lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \otimes \mathbf{a})) \\ = \; & \mathbf{F}(\mathbf{a} \otimes \lfloor \mathbf{F} \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \rceil \mathbf{a}) \\ = \; & \lfloor \mathbf{F}, \mathbf{a} \rceil \lfloor \mathbf{F}, \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \rceil \mathbf{a} \end{aligned} \qquad (42)$$

$$\begin{aligned} & \mathbf{F}(\mathbf{F} \otimes \mathbf{I})(\mathbf{I} \otimes \mathbf{I} \otimes \mathbf{F})\mathbf{a}^{\otimes 4} \\ = \; & \mathbf{F}(\mathbf{F} \otimes \mathbf{I})(\mathbf{a} \otimes \mathbf{a} \otimes \mathbf{F}\mathbf{a}^{\otimes 2}) \\ = \; & \mathbf{F}(\mathbf{F} \otimes \mathbf{I})(\mathbf{a} \otimes \mathbf{a} \otimes \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a}) \\ = \; & \mathbf{F}(\mathbf{F}\mathbf{a}^{\otimes 2} \otimes \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a}) \\ = \; & \mathbf{F}(\lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \otimes \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a}) \\ = \; & \lfloor \mathbf{F}, \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \rceil \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \end{aligned} \qquad (43)$$

$$\begin{aligned} & \mathbf{F}(\mathbf{F} \otimes \mathbf{I})(\mathbf{I} \otimes \mathbf{F} \otimes \mathbf{I})\mathbf{a}^{\otimes 4} \\ = \; & \mathbf{F}(\mathbf{F} \otimes \mathbf{I})(\mathbf{a} \otimes \mathbf{F}\mathbf{a}^{\otimes 2} \otimes \mathbf{a}) \\ = \; & \mathbf{F}(\mathbf{F} \otimes \mathbf{I})(\mathbf{a} \otimes \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \otimes \mathbf{a}) \\ = \; & \mathbf{F}(\mathbf{F}(\mathbf{a} \otimes \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a}) \otimes \mathbf{a}) \\ = \; & \mathbf{F}(\lfloor \mathbf{F}, \mathbf{a} \rceil \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \otimes \mathbf{a}) \\ = \; & \lfloor \mathbf{F}, \lfloor \mathbf{F}, \mathbf{a} \rceil \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \rceil \mathbf{a} \end{aligned} \qquad (44)$$

$$\begin{aligned} & \mathbf{F}(\mathbf{F} \otimes \mathbf{I})(\mathbf{F} \otimes \mathbf{I} \otimes \mathbf{I})\mathbf{a}^{\otimes 4} \\ = \; & \mathbf{F}(\mathbf{F} \otimes \mathbf{I})(\mathbf{F}\mathbf{a}^{\otimes 2} \otimes \mathbf{a} \otimes \mathbf{a}) \\ = \; & \mathbf{F}(\mathbf{F} \otimes \mathbf{I})(\lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \otimes \mathbf{a} \otimes \mathbf{a}) \\ = \; & \mathbf{F}(\mathbf{F}(\lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \otimes \mathbf{a}) \otimes \mathbf{a}) \\ = \; & \mathbf{F}(\lfloor \mathbf{F}, \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \rceil \mathbf{a} \otimes \mathbf{a}) \\ = \; & \lfloor \mathbf{F}, \lfloor \mathbf{F}, \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \rceil \mathbf{a} \rceil \mathbf{a} \end{aligned} \qquad (45)$$

$$\begin{aligned} & \mathbf{F}(\mathbf{I} \otimes \mathbf{F})(\mathbf{F} \otimes \mathbf{I} \otimes \mathbf{I})\mathbf{a}^{\otimes 4} \\ = \; & \mathbf{F}(\mathbf{I} \otimes \mathbf{F})(\mathbf{F}\mathbf{a}^{\otimes 2} \otimes \mathbf{a} \otimes \mathbf{a}) \\ = \; & \mathbf{F}(\mathbf{I} \otimes \mathbf{F})(\lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \otimes \mathbf{a} \otimes \mathbf{a}) \\ = \; & \mathbf{F}(\lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \otimes \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a}) \\ = \; & \lfloor \mathbf{F}, \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \rceil \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \end{aligned} \qquad (46)$$

$$\begin{aligned} \mathbf{S}_3 \; = \; & \lfloor \mathbf{F}, \mathbf{a} \rceil^3 \mathbf{a} \\ + \; & \lfloor \mathbf{F}, \mathbf{a} \rceil \lfloor \mathbf{F}, \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \rceil \mathbf{a} \end{aligned}$$

$$
\begin{aligned}
&+ \quad \lfloor \mathbf{F}, \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \rceil \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \\
&+ \quad \lfloor \mathbf{F}, \lfloor \mathbf{F}, \mathbf{a} \rceil \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \rceil \mathbf{a} \\
&+ \quad \lfloor \mathbf{F}, \lfloor \mathbf{F}, \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \rceil \mathbf{a} \rceil \mathbf{a} \\
&+ \quad \lfloor \mathbf{F}, \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \rceil \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \\
&= \quad 3\mathbf{S}_1(\mathbf{a})\mathbf{S}_2(\mathbf{a}) - 2\mathbf{S}_1(\mathbf{a}) \\
&+ \quad \lfloor \mathbf{F}_1 \mathbf{S}_2(a) \rceil \qquad (47)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{S}_4 \quad &= \quad \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{S}_3 + 3\lfloor \mathbf{F}, \mathbf{S}_2 \mathbf{a} \rceil \lfloor \mathbf{F}, \mathbf{a} \rceil \\
&+ \quad 3\lfloor \mathbf{F}, \lfloor \mathbf{F}, \mathbf{a} \rceil \mathbf{a} \rceil \mathbf{S}_2 + \lfloor \mathbf{F}, \mathbf{S}_3 \mathbf{a} \rceil \qquad (48)
\end{aligned}
$$

$$
\mathbf{S}_j = \sum_{k=0}^{j-1} \binom{j-1}{k} \lfloor \mathbf{F}, \mathbf{S}_k \mathbf{a} \rceil \mathbf{S}_{j-1-k}, \qquad j = 1, 2, \ldots \qquad (49)
$$

7 Implementation For Henon Heiles Systems

The Henon Heiles System is described with the four equations which are given as follows

$$\dot{x} = p_x \qquad (50)$$

$$\dot{p_x} = -x - 2\lambda xy \qquad (51)$$

$$\dot{y} = p_y \qquad (52)$$

$$\dot{p}_y = -y - \lambda(x^2 - y^2) \qquad (53)$$

We can write these four equations as second degree multinomial ODE sets with right hand side.

$$\dot{x}_1 = x_2 \qquad (54)$$

$$\dot{x}_2 = -x_1 - 2x_1 x_3 \qquad (55)$$

$$\dot{x}_3 = x_4 \qquad (56)$$

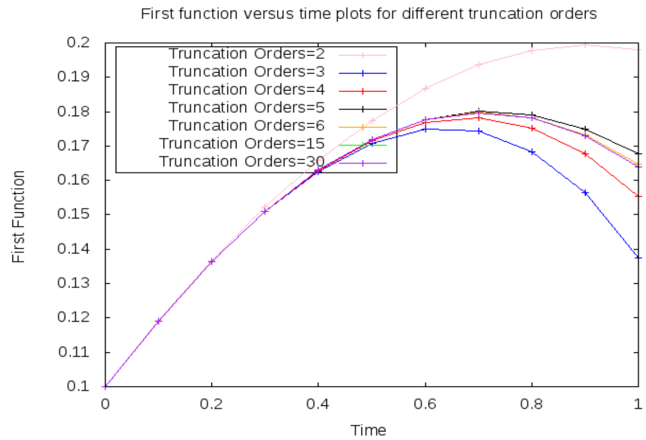
$$\dot{x}_4 = -x_3 - x_1^2 + x_3^2 \qquad (57)$$

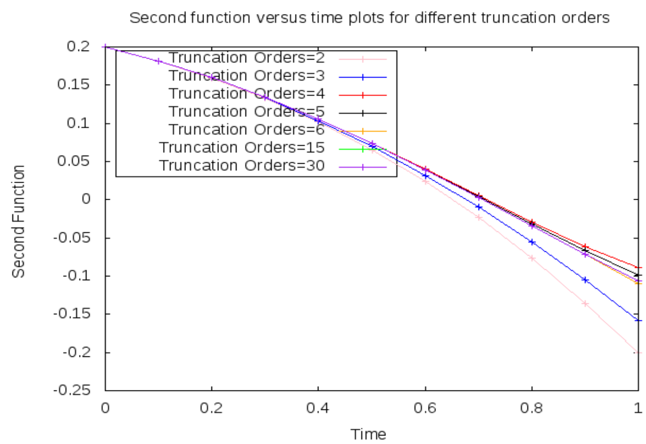
We can construct algorithms by Mupad to easily find the solution for Henon Heiles System. In this algorithm, it is used the recursion of Squarification in (49).

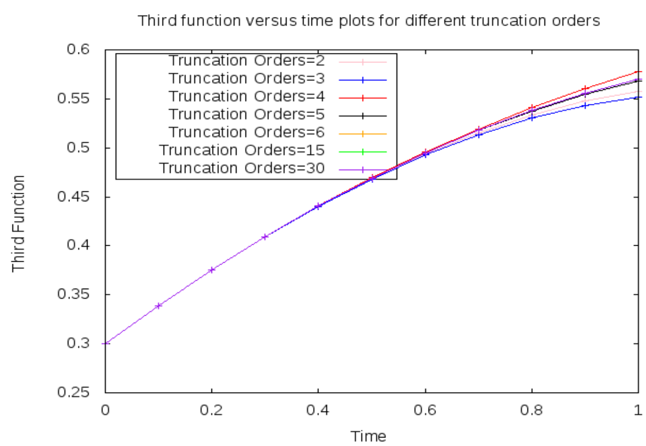
The initial conditions are assumed that

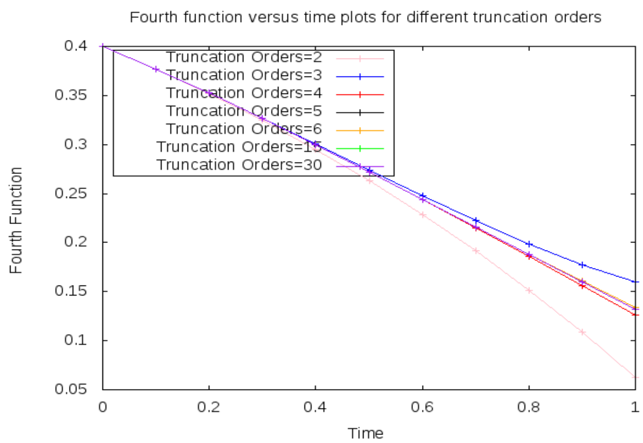
$$\mathbf{x}(0) = \mathbf{a} \equiv [0.1 \quad 0.2 \quad 0.3 \quad 0.4]^T \qquad (58)$$

Using the truncuations of 2,3,4,5,6,15 and 30, we can find the graphics of the functions in 0 to 1 time slot

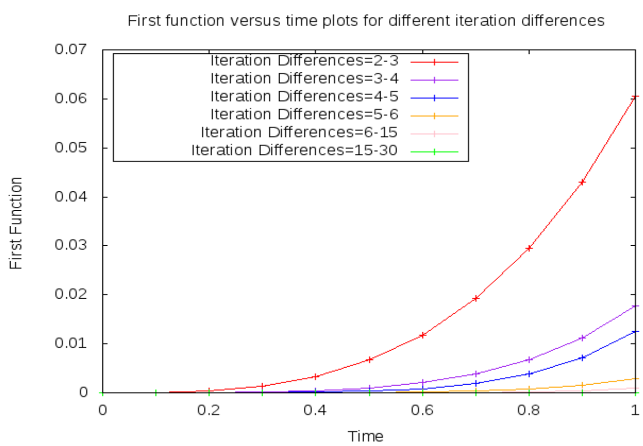


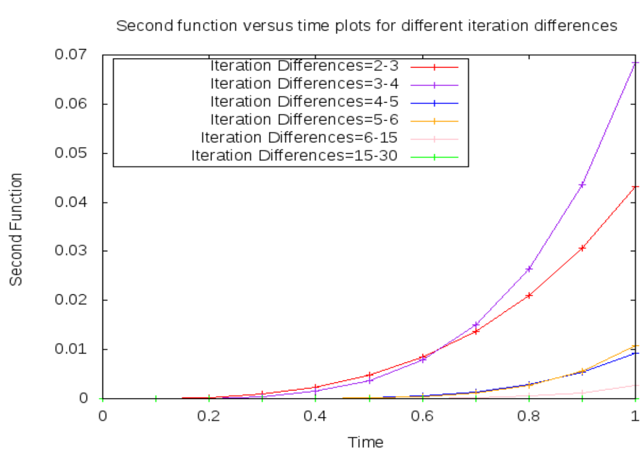


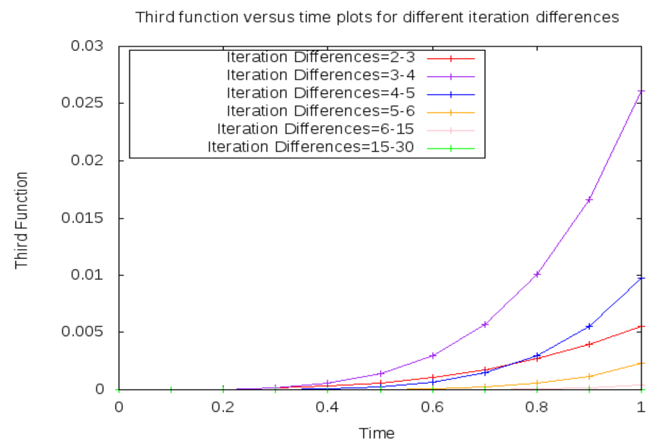


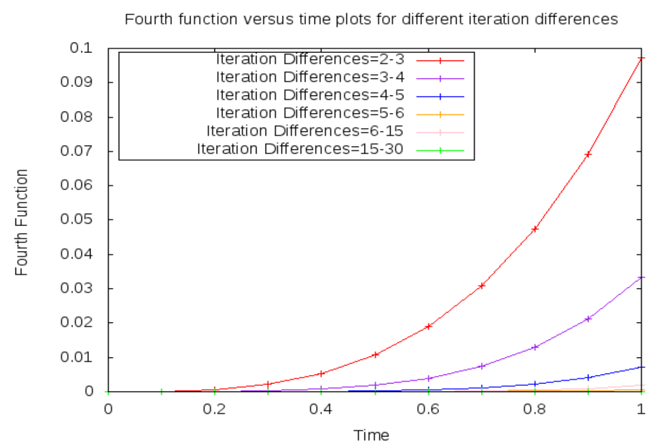


According to the initial conditions which are given by (58), the absoute error as follows



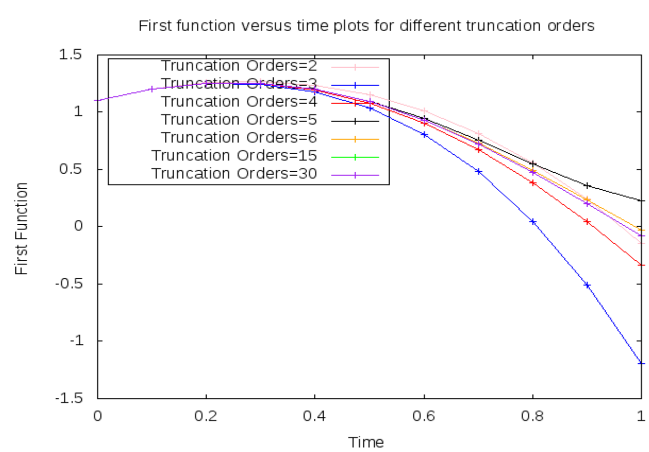


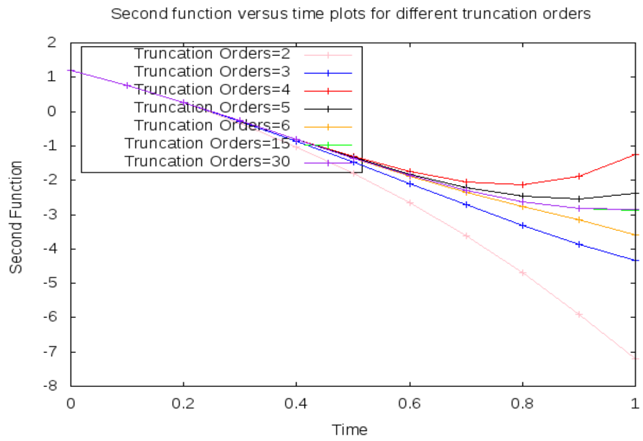


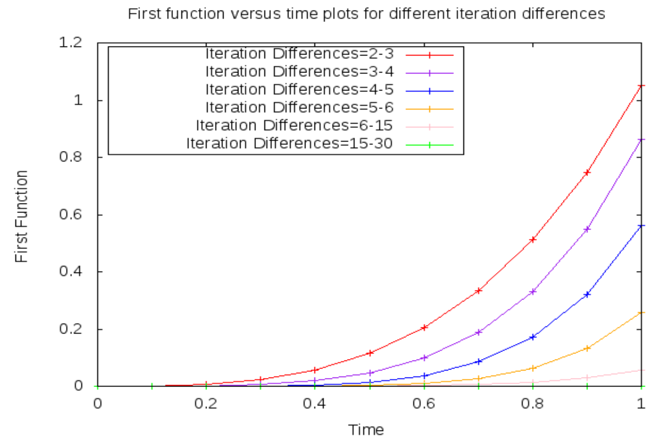


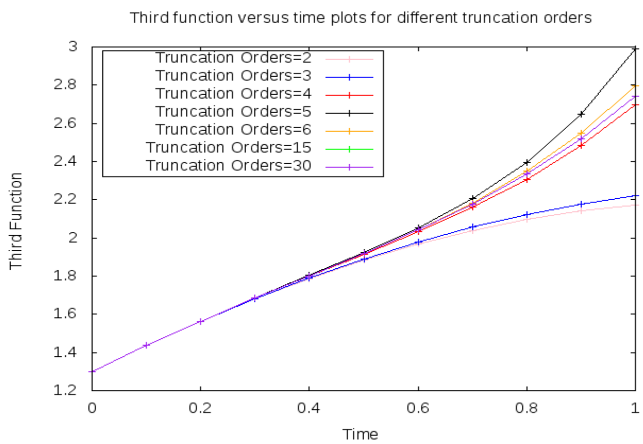
$$\mathbf{x}(0) = \mathbf{a} \equiv [1.1 \quad 1.2 \quad 1.3 \quad 1.4]^T \qquad (59)$$

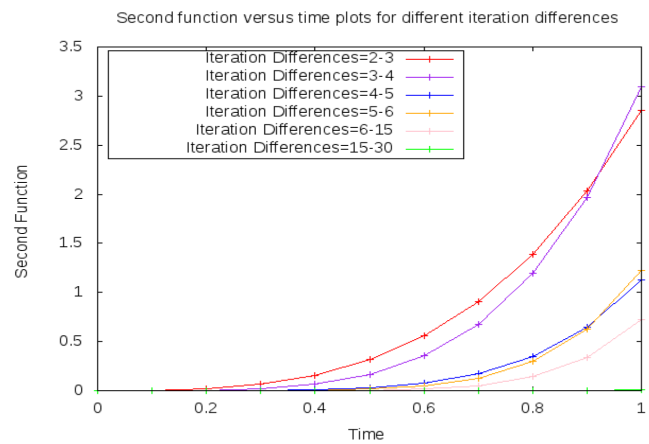
Using the truncuations of 2,3,4,5,6,15 and 30, we can find the graphics of the functions in 0 to 1 time slot

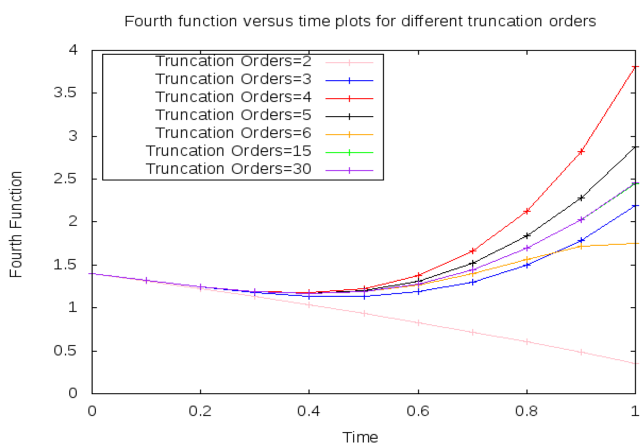


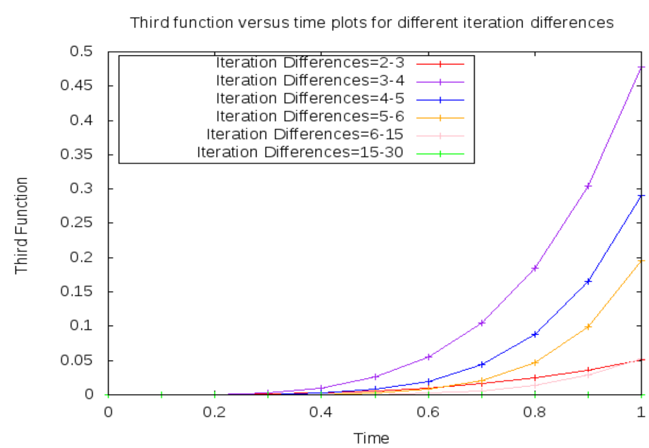




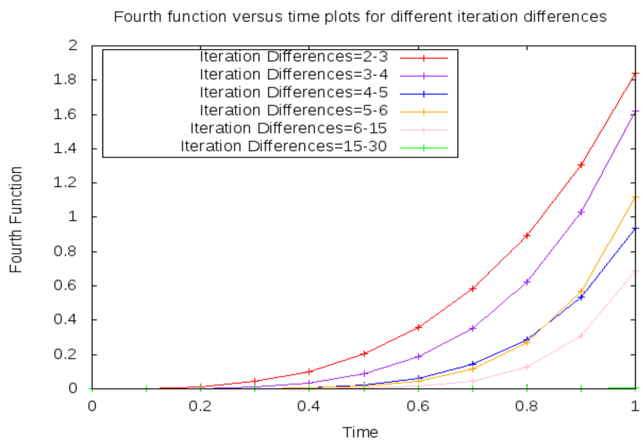








According to the initial conditions which are given by (59), the absoute error as follows



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