Stochastic Stability and permanence for delay of an epidemic model with incidence rate

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Abstract: **-** In this work, we consider a nonlinear epidemic model with temporary immunity and a saturated incidence rate. N (t) at time t, this population is divide into eight sub-classes, with N(t) = S(t) + I(t) +I_1(t)+I_2(t)+O(t)+A(t)+ Q_1(t)+ Q_2(t). S(t), I(t), I_1(t), I_2(t), O(t), A(t), Q_1(t) and Q_2(t), denote the sizes of the population susceptible to disease, infectious members, HIV infected members that do not know they are infected, HIV members that know they are infected, members suffering from other opportunistic infections, AIDS members, and quarantine members. With the possibility of infection through temporary immunity, respectively. The stability of a disease-free status equilibrium and the existence of endemic equilibrium determined by the ratio called the basic reproductive number.
The model has been studied the permanence of the epidemic and Stochastic stability of the free disease equilibrium under certain conditions.
Key Words: **-** Basic reproduction number, endemic equilibrium, epidemic model stability, stochastic stability, saturated incidence.

1 Introduction

This paper considers the following nonlinear epidemic model with temporary immunity:

$$\begin{cases} \dot{S}(t) = \lambda + \nu - \upsilon - \mu + d\ \mathrm{S}(\mathrm{t}) - \beta \frac{\mathrm{S(t)I(t)}}{1+\mathrm{a}_1 S(\mathrm{t}) + \mathrm{a}_2 I(\mathrm{t})} \\ +b_2 \mathrm{e}^{-\mu_6 \tau} \mathrm{Q}_2(\mathrm{t}-\tau), \\ \dot{I}(t) = \beta \frac{\mathrm{S(t)I(t)}}{1+\mathrm{a}_1 S(\mathrm{t}) + \mathrm{a}_2 I(\mathrm{t})} - \alpha_1 + \alpha_2 + \delta_1 + \sigma\ I(\mathrm{t}) \\ -\ \mu_0 + d\ I(\mathrm{t}) + b_1 Q_1(t), \\ \dot{I}_1(t) = \alpha_1 \mathrm{I(t)} - \theta \mathrm{I}_1(\mathrm{t}) - \gamma_1 \mathrm{I}_1(\mathrm{t}) - \mu_1 + d\ \mathrm{I}_1(\mathrm{t}), \\ \dot{I}_2(t) = \alpha_2 \mathrm{I(t)} + \theta \mathrm{I}_1(\mathrm{t}) - \delta_2 \mathrm{I}_2(\mathrm{t}) - \gamma_2 \mathrm{I}_2(\mathrm{t}) - (\mu_2 + d)\mathrm{I}_2(\mathrm{t}), \\ \dot{O}(t) = \sigma \mathrm{I(t)} - (\mu_3 + d)\mathrm{O}(\mathrm{t}) + \rho O(t), \\ \dot{A}(t) = \sum_{i=1}^{2} \gamma_i \mathrm{I}_i(\mathrm{t}) - (\mu_4 + d)\mathrm{A}(\mathrm{t}) - \delta_3 A(\mathrm{t}), \\ \dot{Q}_1(t) = \delta_1 \mathrm{I(t)} + \delta_2 \mathrm{I}_2(\mathrm{t}) - (\mu_5 + d)\mathrm{Q}_1(t) - \mathrm{b}_1 Q_1(t) + \delta_3 \mathrm{A}(t), \\ \dot{Q}_2(t) = \rho O(t) - (\mu_6 + d)\mathrm{Q}_2(t) - b_2 \mathrm{e}^{-\mu_6 \tau} \mathrm{Q}_2(\mathrm{t}-\tau). \end{cases} \quad (1)$$

Consider a population of size N (t) at time t, this population is divide into for eight sub-classes, with

N(t) = S(t) + I(t) +I_1(t)+I_2(t)+O(t)+A(t)+ Q_1(t)+ Q_2(t),

S(t), I(t), I_1(t), I_2(t), O(t), A(t), Q_1(t) and Q_2(t), denote the sizes of the population susceptible to disease, infectious members, HIV infected members that do not know they are infected, HIV members that know they are infected, members suffering from other opportunistic infections, AIDS members, and quarantine members.

With the possibility of infection through temporary immunity, respectively.

The positive constants μ, μ_0, μ_1, μ_2, μ_3, μ_4, μ_5 and μ_6 represent the death rates of susceptible, infectious, HIV infected members that do not know they are infected, HIV members that know they are infects, members suffering from other opportunistic infections, AIDS, and quarantine members.

Biologically, it is natural to assume that:

$\mu \leq \min\{\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\}$.

The positive constant d is natural mortality rate.

The positive constant β is the average numbers of contacts infective for S and I. The positive constant α_1, α_2, are the average numbers of contacts. The positive constant λ represent the incidence rate of the population.

The positive constants γ_1, γ_2, are the numbers of transfer or conversion of infected those who know and do not know who are HIV-positive become AIDS patients.

The constant ν is the parameter of emigration, υ is the parameter of the immigration. θ, the parameter of unaware infective to become aware infective by screening.

$\delta_1, \delta_2, \delta_3$, the numbers of transfer to conversion of I, I_2, A to Q_1. The positive constants σ the numbers of transfer or conversion of infected to members suffering from other opportunistic infections.

The term ρ, the numbers of transfer to conversion of, O to Q_2.

The term b_1 indicate that an individual has quarantined in a pool recovery before becoming infected and the term

$b_2 e^{-\mu_6 \tau} Q_2(t-\tau)$ indicate that an individual has quarantined in a pool recovery before becoming susceptible, where τ is the length of immunity period,

The formulation of the incidence rate

$\frac{S(t)I(t)}{1+a_1 S(t)+a_2 I(t)}$, with a_1 and a_2 are the constants parameters.

The initial condition of (1) given as:

$$\begin{aligned} &S(\eta)=\Phi_1(\eta),\ I(\eta)=\Phi_2(\eta), O(\eta)=\Phi_3(\eta), I_2(\eta)=\Phi_4(\eta),\\ &O(\eta)=\Phi_5(\eta), A(\eta)=\Phi_6(\eta), Q_1(\eta)=\Phi_7(\eta), Q_2(\eta)=\Phi_8(\eta).\\ &-\tau \le \eta \le 0 \end{aligned} \quad (2)$$

Where, $\Phi_i = (\Phi_1,\Phi_2,\Phi_3,\Phi_4,\Phi_5,\Phi_6,\Phi_7,\Phi_8)^T \in \mathbb{C}$ such that

$$\begin{aligned} &S(\eta)=\Phi_1(\eta)=\Phi_1(0)\ge 0,\ I(\eta)=\Phi_2(\eta)=\Phi_2(0)\ge 0,\\ &I_1(\eta)=\Phi_3(\eta)=\Phi_3(0)\ge 0, I_2(\eta)=\Phi_4(\eta)=\Phi_4(0)\ge 0,\\ &O(\eta)=\Phi_5(\eta)=\Phi_5(0)\ge 0, A(\eta)=\Phi_6(\eta)=\Phi_6(0)\ge 0,\\ &Q_1(\eta)=\Phi_7(\eta)=\Phi_7(0)\ge 0, Q_2(\eta)=\Phi_8(\eta)=\Phi_8(0)\ge 0.\\ &-\tau \le \eta \le 0, \end{aligned}$$

Let C denote the Banach space C ([-τ, 0], $\mathbb{R}^8$) of continuous functions mapping the interval [-τ, 0] into $\mathbb{R}^8$. With a biological meaning, we further assume that:

$\Phi_i(\eta)=\Phi_i(0)\ge 0$, for i = 1, 2, 3, 4, 5, 6, 7, 8.

The region

$$\Omega=\left\{ \begin{array}{l} (S(t),I(t),I_1(t),I_2(t),O(t),A(t),Q_1(t),Q_2(t)) \in \mathbb{R}^8_+,\\ S(t)+I(t)+I_1(t)+I_2(t)+O(t)+A(t)+Q_1(t)+Q_2(t) \le N < \frac{\lambda+\nu-\upsilon}{\mu+d} \end{array} \right\}$$

Is positively invariant.

Hence, system (1), can be rewritten as

$$\left\{ \begin{array}{l} \dot{S}(t)=\lambda+\nu-\upsilon-(\mu+d)S(t)-\beta\frac{S(t)I(t)}{1+a_1S(t)+a_2I(t)}\\ +b_2 e^{-\mu_6\tau}Q_2(t-\tau),\\ \dot{I}(t)=\beta\frac{S(t)I(t)}{1+a_1S(t)+a_2I(t)}+b_1Q_1(t)-c_0I(t),\\ \dot{I}_1(t)=\alpha_1 I(t)-c_1 I_1(t),\\ \dot{I}_2(t)=\alpha_2 I(t)+\theta I_1(t)-c_2 I_2(t),\\ \dot{O}(t)=\sigma I(t)-c_3 O(t),\\ \dot{A}(t)=\sum_{i=1}^{2}\gamma_i I_i(t)-c_4 A(t),\\ \dot{Q}_1(t)=\delta_1 I(t)+\delta_2 I_2(t)+\delta_3 A(t)-c_5 Q_1(t),\\ \dot{Q}_2(t)=\rho O(t)-(c_6+d)Q_2(t)-b_2 e^{-\mu_6\tau}Q_2(t-\tau). \end{array} \right. \quad (3)$$

With

$$\begin{aligned} c_0&=\alpha_1+\alpha_2+\delta_1+\sigma+\mu_0+d,\\ c_1&=\theta+\gamma_1+\mu_1+d,\\ c_2&=\gamma_2+\delta_2+\mu_2+d,\\ c_3&=\mu_3+d+\rho,\\ c_4&=\delta_3+\mu_4+d,\\ c_5&=b_1+\mu_5+d. \end{aligned}$$

2 Equilibrium points

We calculate the points of equilibrium in the absence and presence of infection.

In the absence of infection, the system (3) has a disease-free equilibrium *E_0:*

$$E_0=(S,\hat{I},I_1,I_2,O,A,Q_1,Q_2)^T=\left(\frac{\lambda+\nu-\upsilon}{\mu+d},0,0,0,0,0,0,0\right)^T. \quad (4)$$

Define the quarantine reproduction number as

$$R_0=\frac{\beta}{c_0}\times\frac{\lambda+\nu-\upsilon}{\mu+d+a_1\ \lambda+\nu-\upsilon} \quad (5)$$

In the presence of infection, substituting in the system, Ω also contains a unique positive, endemic equilibrium

$$E_\tau^*=(S^*,I^*,I_1^*,I_2^*,O^*,A^*,Q_1^*,Q_2^*)^T$$

Where:

$$\left\{ \begin{array}{l} S^*=\frac{\lambda+\nu-\upsilon}{\mu+d}+f_2\times I^*, f_2=\left[\frac{\rho\sigma b_2 e^{-\mu_6\tau}}{c_3c_6}-c_0+b_1f_1\right]\\ I^*=\frac{f_4\times\left(\frac{\lambda+\nu-\upsilon}{\mu+d}\right)-\frac{1}{a_2}}{1-f_2f_4}, f_3=c_0+b_1f_1, f_4=\left(\frac{\beta}{f_3}\frac{}{a_2}-\frac{a_1}{a_2}\right),\\ I_1^*=\frac{\alpha_1}{c_1}\times I^*,\\ I_2^*=\left[\frac{\alpha_2}{c_2}+\frac{\theta\alpha_1}{c_1c_2}\right]\times I^*,\\ O^*=\frac{\sigma}{c_3}\times I^*,\\ A^*=\frac{1}{c_4}\times\left[\sum_{i=1}^{2}\frac{\alpha_i\gamma_i}{c_i}+\frac{\theta\alpha_1\gamma_2}{c_1c_2}\right]\times I^*,\\ Q_1^*=f_1\times I^*,\\ f_1=\frac{1}{c_5}\times\left[\delta_1+\delta_2\left(\frac{\alpha_2}{c_2}+\frac{\theta\alpha_1}{c_1c_2}\right)f+\frac{\delta_3}{c_4}\times\left[\sum_{i=1}^{2}\frac{\alpha_i\gamma_i}{c_i}+\frac{\theta\alpha_1\gamma_2}{c_1c_2}\right]\right]\\ Q_2^*=\left(\frac{\rho\sigma}{c_5c_6}\right)\times I^*. \end{array} \right. \quad (6)$$

Theorem 1

The disease-free equilibrium E_0 of the system (3) is locally asymptotically stable if $R_0<1$.

Theorem 2

If $R_0>1$, the system (3) has a unique, non-trivial equilibrium E_τ^* which is locally asymptotically stable.

3. Permanence of the epidemic

Theorem 3

Let $(S,I,I_1,I_2,O,A,Q_1,Q_2)$ be the solution of system (3). If there exists a sequence (t_n) such that

$$\begin{aligned} &t_n\to\infty, S(t_n)\to l, I(t_n)\to 0, I_1(t_n)\to 0,\\ &I_2(t_n)\to 0, O(t_n)\to 0, A(t_n)\to 0, Q_1(t_n)\to 0,\\ &Q_2(t_n)\to 0, \end{aligned}$$, then $l=\frac{\lambda+\nu-\upsilon}{\mu+d}$.

Proof

We have $0\le l\le\frac{\lambda+\nu-\upsilon}{\mu+d}$. Suppose that $0\le l<\frac{\lambda+\nu-\upsilon}{\mu+d}$.

Since

$$\begin{aligned} &S(t_n)\to l, I(t_n)\to 0, I_1(t_n)\to 0, I_2(t_n)\to 0,\\ &O(t_n)\to 0, A(t_n)\to 0, Q_1(t_n)\to 0, Q_2(t_n)\to 0. \end{aligned}$$

It follows that

$l,0,0,0,0,0,0,0 \in W\ S_0,I_0,\ I_{1\ 0},\ I_{2\ 0},O_0,A_0,\ Q_{1\ 0},\ Q_{2\ 0}$

Which is the set of W limit.

Consider

$$\begin{pmatrix} S\ t\ ,I\ t\ ,I_1\ t\ ,I_2\ t \\ ,O\ t\ ,A\ t\ ,Q_1\ t\ ,Q_2\ t \end{pmatrix} \in W\begin{pmatrix} S_0,I_0,\ I_{1\ 0},\ I_{2\ 0} \\ ,O_0,A_0,\ Q_{1\ 0},\ Q_{2\ 0} \end{pmatrix}. \quad (7)$$

For every t because the set $\in W\ S_0,I_0,\ I_{1\ 0},\ I_{2\ 0},O_0,A_0,\ Q_{1\ 0},\ Q_{2\ 0}$ is invariant by (3).

We have $S\ t\ = \frac{\lambda+\nu-\upsilon}{\mu+d}+\left(S_0-\frac{\lambda+\nu-\upsilon}{\mu+d}\right)e^{-\mu+d\ t}$ and

$I\ t\ = I_1\ t\ = I_2\ t\ = O\ t\ = A\ t\ = Q_1\ t\ = Q_2\ t\ = 0,$ For every $t \in \mathbb{R}$.

Since $0 \le l \le \frac{\lambda+\nu-\upsilon}{\mu+d}$, contrary to the positivity of S in all $\mathbb{R}$. (The region Ω). □

Proposition

Let S,I,I_1,I_2,O,A,Q_1,Q_2

1- If $S_0>0,I_0>0,\ I_{1\ 0}>0,\ I_{2\ 0}>0,O_0>0,A_0>0,\ Q_{1\ 0}>0,\ Q_{2\ 0}>0.$, then for every $t \in\ 0,T$,

$S\ t\ >0,I\ t\ >0,I_1\ t\ >0,I_2\ t\ >0,$.
$O\ t\ >0,A\ t\ >0,Q_1\ t\ >0,Q_2\ t\ >0$

2- The solution S,I,I_1,I_2,O,A,Q_1,Q_2 ,

is defined in $[0,\infty$.,and $\lim_{t\to\infty}\sup N\ t\ \le \frac{\lambda+\nu-\upsilon}{\mu+d}.$.

Proof

1- We suppose there exists, $t_0 \in\ 0,T$ such that $S\ t_0\ = 0, \dot{S}\ t_0\ \le 0$ and $S\ t\ >0$ for $t \in [0,t_0$. Then $I\ t\ >0$ for $t \in [0,t_0$.

If the assumption is not correct then there exists $t_1 \in [0,t_0]$ such that $I\ t_1\ = 0, \dot{I}\ t_1\ \le 0$ and $I\ t\ >0$ for $t \in [0,t_1$.

By integrating, we have, for $t \in [0,t_1]$

$$Q_1\ t\ =\ Q_{1\ 0}e^{-c_4 t}+\delta_1\int_0^t e^{-c_4\ t-\tau}\ I\ \tau\ d\tau+$$
$$\delta_2\int_0^t e^{-c_4\ t-\tau}\ I_2\ \tau\ d\tau+\delta_3\int_0^t e^{-c_4\ t-\tau}\ A\ \tau\ d\tau>0,$$

And

$$Q_2\ t\ =\ Q_{2\ 0}e^{-c_5 t}+\delta_1\int_0^t e^{-c_5\ t-\tau}\ O\ \tau\ d\tau>0,$$

Then $\dot{I}\ t\ = b_1Q_2\ t\ >0$, contradiction.

Hence $Q_1\ t\ >0,Q_2\ t\ >0$, for $t \in [0,t_0]$.

Therefore $\dot{S}\ t_0\ = \lambda+\nu-\upsilon-b_2e^{-\mu_6 t}Q_2\ t_0\ >0$.

However, this is contradiction with supposition that $\dot{S}\ t_0\ \le 0$.

2- We have $\dot{N}=\lambda+\nu-\upsilon-\mu S-\mu_0 I-\mu_1 I_1-\mu_2 I_2$
$-\mu_3 O-\mu_4 A-\mu_5 Q_1-\mu_6 Q_2-dN.$

$\dot{N}\le\lambda+\nu-\upsilon-\ \mu+d\ N.$

$t \in [0,T$

$N\le\frac{\lambda+\nu-\upsilon}{\mu+d}-\ 1-e^{-\ \mu+d\ t}\ ,$

$N\le 2\times\frac{\lambda+\nu-\upsilon}{\mu+d}.$

The solution S,I,I_1,I_2,O,A,Q_1,Q_2 bounded in $[0,T$ .For $t \in [0,\infty\ ,N\le\frac{\lambda+\nu-\upsilon}{\mu+d}-\ 1-e^{-\ \mu+d\ t}$

Finally

$\lim_{t\to\infty}\sup N\ t\ \le\frac{\lambda+\nu-\upsilon}{\mu+d}.$

□

4 Stochastic stability of the free disease equilibrium

We limit ourselves here to perturbing only the contact rate so we replace β by $\beta + a$ b(t), where b(t) is white noise (Brownian motion). The system (3) transformed to the following Itô stochastic differential equations:

$$\begin{cases} \dot{S}(t)=\lambda+\nu-\upsilon-\ \mu+d\ \mathrm{S(t)}-\beta\frac{\mathrm{S(t)I(t)}}{1+\mathrm{a}_1S(\mathrm{t})+\mathrm{a}_2I(\mathrm{t})} \\ +b_2e^{-\mu_6\tau}\mathrm{Q}_2(\mathrm{t}-\tau)-a\frac{\mathrm{S(t)I(t)}}{1+\mathrm{a}_1S(\mathrm{t})+\mathrm{a}_2I(\mathrm{t})}\mathrm{db}, \\ \dot{I}(t)=\beta\frac{\mathrm{S(t)I(t)}}{1+\mathrm{a}_1S(\mathrm{t})+\mathrm{a}_2I(\mathrm{t})}+b_1Q_1(t)-c_0I(\mathrm{t})+a\frac{\mathrm{S(t)I(t)}}{1+\mathrm{a}_1S(\mathrm{t})+\mathrm{a}_2I(\mathrm{t})}\mathrm{db}, \\ \dot{I}_1(t)=\alpha_1\mathrm{I(t)}-c_1\mathrm{I}_1(\mathrm{t}), \\ \dot{I}_2(t)=\alpha_2\mathrm{I(t)}+\theta\mathrm{I}_1(\mathrm{t})-\mathrm{c}_2\,\mathrm{I}_2(\mathrm{t}), \\ \dot{O}(t)=\sigma\mathrm{I(t)}-\mathrm{c}_3\,O(\mathrm{t}), \\ \dot{A}(t)=\sum_{i=1}^{2}\gamma_i\mathrm{I}_i(\mathrm{t})-\mathrm{c}_4\,A(\mathrm{t}), \\ \dot{Q}_1(t)=\delta_1\mathrm{I(t)}+\delta_2\mathrm{I}_2(\mathrm{t})+\delta_3\,\mathrm{A(t)}-\mathrm{c}_5\,Q_1(\mathrm{t}), \\ \dot{Q}_2(t)=\rho O(t)-\left(c_6+d\right)Q_2(t)-b_2e^{-\mu_6\tau}\mathrm{Q}_2(\mathrm{t}-\tau). \end{cases} \quad (7)$$

Theorem 4:

If $R_0<1$, I(t),Q_1(t) and Q_2(t) are exponentially almost surely stable.

Proof:

Let w such that

$ac_0+\frac{c_0}{\beta}\times\frac{\mu+d}{\lambda+\nu-\upsilon}>0$

With Itô's formula, we obtain

$$L=d\log\left(I+w_1Q_1+w_2Q_2\right)=\frac{\dot{I}+w_1\dot{Q}_1+w_2\dot{Q}_2}{I+w_1Q_1+w_2Q_2}$$

$$L = \frac{1}{I + w_1Q_1 + w_2Q_2}\begin{bmatrix} \frac{\beta \mathrm{SI}}{1+\mathrm{a}_1S + \mathrm{a}_2I} + w_1\delta_2 \mathrm{I}_2 \\ + w_1\delta_3 \mathrm{A} + w_2\rho O \\ -[c_0 - w_1\delta_1]I \\ -[w_1c_5 - b_1]Q_1 \\ -\left[w_2(\mu_6 + d) + w_2b_2\mathrm{e}^{-\mu_6\tau}\right]Q_2 \\ -\frac{1}{2}\times\frac{a^2}{I + w_1Q_1 + w_2Q_2}\times\left(\frac{\mathrm{SI}}{1+\mathrm{a}_1S + \mathrm{a}_2I}\right)^2 \end{bmatrix} dt$$

$$+\frac{aSI}{I + w_1Q_1 + w_2Q_2}db$$

$$L \le \frac{1}{I + w_1Q_1 + w_2Q_2}\begin{bmatrix} -\left[c_0 - \frac{\beta(\lambda + \nu - \upsilon)}{\mu + d + \mathrm{a}_1(\lambda + \nu - \upsilon) + \mathrm{a}_2I} - w_1\delta_1\right]I \\ -[w_1c_5 - b_1]Q_1 \\ -\left[w_2(\mu_6 + d) + w_2b_2\mathrm{e}^{-\mu_6\tau}\right]Q_2 \\ -\frac{1}{2}\times\frac{a^2}{I + w_1Q_1 + w_2Q_2}\times\left(\frac{\mathrm{SI}}{1+\mathrm{a}_1S + \mathrm{a}_2I}\right)^2 \end{bmatrix} dt$$

$$+\frac{aSI}{I + w_1Q_1 + w_2Q_2}db$$

$$L \le \frac{-1}{I + w_1Q_1 + w_2Q_2}\begin{bmatrix} \left[c_0 - \frac{\beta(\lambda + \nu - \upsilon)}{\mu + d + \mathrm{a}_1(\lambda + \nu - \upsilon) + \mathrm{a}_2I} - w_1\delta_1\right]I \\ +[w_1c_5 - b_1]Q_1 \\ +\left[w_2(\mu_6 + d) + w_2b_2\mathrm{e}^{-\mu_6\tau}\right]Q_2 \end{bmatrix} dt \cdot$$

$$+\frac{aSI}{I + w_1Q_1 + w_2Q_2}db$$

We suppose that

$$L_1 = \min\left\{\begin{matrix}\left(c_0 - \frac{\beta(\lambda + \nu - \upsilon)}{\mu + d + \mathrm{a}_1(\lambda + \nu - \upsilon) + \mathrm{a}_2I} - w_1\delta_1\right), \\ \left(w_1c_5 - b_1\right), \left(w_2(\mu_6 + d) + w_2b_2\mathrm{e}^{-\mu_6\tau}\right)\end{matrix}\right\}.$$

Then

$$L \le -L_1dt + \frac{aSI}{I + w_1Q_1 + w_2Q_2}db$$

With integration, we obtain

$$\log(I + w_1Q_1 + w_2Q_2) \le -L_1dt + a\int_0^t \frac{S(\nu)I(\nu)}{(I(\nu) + w_1Q_1(\nu) + w_2Q_2(\nu))}db(\nu) \cdot$$

We have

$$\left(\frac{S(\nu)I(\nu)}{\left(I(\nu) + w_1Q_1(\nu) + w_2Q_2(\nu)\right)}\right)^2, \text{ is bounded.}$$

So,

$$\underset{t\to\infty}{Lim}\int_0^t \frac{S(\nu)I(\nu)}{\left(I(\nu) + w_1Q_1(\nu) + w_2Q_2(\nu)\right)}db(\nu) = 0,$$

almost surely.

The following form from Doob's martingale inequality combined with Itô isometry see [18,].

$$\underset{t\to\infty}{Lim}\sup\frac{1}{t}\log\left(I + w_1Q_1 + w_2Q_2\right) \le -L_1 \text{ Alm ost surely.}$$

Finally

$$\underset{t\to\infty}{Lim}\sup\frac{1}{t}\log I \le -L_1, \text{ so I is almost surely.}$$

$$\underset{t\to\infty}{Lim}\sup\frac{1}{t}\log Q_1 \le -L_1, \textit{ so } \mathrm{Q}_1 \text{ is almost surely.}$$

And

$$\underset{t\to\infty}{Lim}\sup\frac{1}{t}\log Q_2 \le -L_1, \textit{ so } \mathrm{Q}_2 \text{ is almost surely.}$$

□

5 Conclusion

This paper addresses the equilibrium and stability of the epidemic model with temporary immunity and saturated incidence rate. Both trivial and endemic equilibrium are founds. The disease-free equilibrium E_0 is globally asymptotically stable if $R_0<1$, and the system has a unique non-trivial equilibrium E_τ^* which is globally asymptotically stable if $R_0>1$. We study Permanence of the epidemic and stochastic stability under some conditions.

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