The Quantum of Action and the Principle of Least Action

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Abstract: - In 1900, to avoid the contradictions between classical theory and experiment, in the study of the distribution of blackbody radiation energy law, Planck, with a genius intuition, stated that the classical law of interaction should be abandoned. Given the fact that this law is based on the notions of continuous energy exchange, Planck proposed that this energy exchange occurs in discrete and indivisible portions, which he called "energy quanta". He showed that the quantum of energy is proportional to the frequency ν of the radiation, which is well-known today as the Planck constant. It has the dimensions of action (energy x time) or (momentum x length). Planck realized that the quantum hypothesis was essentially not an energy hypothesis, but an action hypothesis. The quantum of energy for radiation makes sense only for periodic phenomena that have a definite frequency. But there is no doubt that Planck realized that the element of action, should have a more fundamental meaning both for non-periodic and non-stationary phenomena. He "felt" that the physical meaning of the action element could be harmonized only with the help of the principle of least action which governs all fundamental phenomena in nature. Talking about this harmony between the quantum hypothesis and the principle of least action, Planck showed that this principle should be given a more general form, which makes it applicable even to discrete phenomena. Planck's hypothesis about quanta introduces into the principle of least action a fundamental condition: complete action can always be an integer multiple of Planck's constant. Atomic systems of various natures can now be described in terms of action, and hence the quantum conditions of motion in the atomic world (Bohr-Sommerfeld conditions) can be derived, and further, according to Feynman's imagination, the principle of action on small can be interpreted in a more realistic conception as the principle of maximum probability, which means that the single trajectory, is the most probable. The principle of least action is statistical. The analysis and general methodological-scientific of the above problem are the object of this article.

Key-Words: - Classical theory, Planck’s hypothesis, quanta, principle of least action, integral of action

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1 Introduction

The principle of least action can be considered as the initial principle of mechanics. It shows unequivocally, what will be the real motion of the point or system for the initial given conditions. The principle of least action represents the most economical formulation of the laws of mechanics.

The principle of least action gives the necessary number of equations that determine the motion of the body. The tasks in which the smallest and largest values are given give each independent coordinate a separate equation.

The principle of least action, formulated by the French mathematician Maupertuis [1] in 1747 shows that the "complete action" of a particle moving from A to B will be either minimum or maximum i.e. the trajectory between points A and B will be either smaller or larger, respectively.

Later, Leibniz, Euler, Hamilton, Helmholtz, and Fermat contributed to a more general and deeper understanding of the principle of least action. Their contribution stays in the universal character of this principle regarding the movement of bodies in the absence of air resistance, as well as the application of this principle not only to particular bodies but also to the whole body systems.

Leibniz [1] emphasized that, during movement, the action usually remains maximal or minimal. A few years later, Maupertuis, passed to imagining the smallest action as a law of motion and equilibrium. According to him, the integral principle of the smallest action determines the form of the functional dependence of the position of the material point on time, ie the trajectory of the material point.

This principle in Euler's view [1] took mathematical form. According to him, it is shown that the body moving under the action of central forces from A to B with velocity v, describes the trajectory corresponding to the minimum or maximum value of the integral:

$$\int_A^B mvds$$

Euler's overall conclusion stays in the universal character of the principle of least action for the movement of bodies in the absence of environmental resistance. This principle applies not only to particular bodies but also to the system of several bodies.
Lagrange [1] generalizes this principle to any system with material n-points of mass mi: In this case, the movement of the system is determined by the demand of the smallest or largest value of the sum:

$$\sum_{i=1}^{n} m_i \int_A^B v_i ds_i$$

An important concept introduced by Lagrange was that of isoenergetic variation, the essence of which was the connection of the principle of least action with the principle of energy conservation. He compares the trajectories joining points A and B that satisfy the demand that the energy E = const and arrives at the assertion that from such points, the real trajectory will be the trajectory corresponding to the minimum size:

$$S = \int_A^B mvds$$

where S is the action of a quantity, the smallest value of which characterizes the flow of a real process. In the general case, the spatial paths between A and B for the same total energy ($E = T + U$) of the particle will be described at non-uniform time intervals. The potential energy U at different points in space will generally be the same, consequently, the kinetic energy ($E = const$) must be changed as well as the velocity of the particle. Different values of velocities mean, not uniform time intervals necessary for the displacement of the particle from A to B.
In the 1930s Hamilton [1], applying the principle of variation to the problems of dynamics, switched to another conception of the principle of least action. According to Hamilton's principle **not the path integral of the amount of motion, but the other quantity of smaller (or greater) value characterizes the true path of the particle. This is the time integral of the Lagrange function**. Compare the different paths of the particle that match the connections and join the two spatial points - the positions of the particle described at the given time at time t_0 at the first point, and at the moment t_1 at the second point. The integral of the Lagrange function for the real path will be the smallest or largest.

$$W = \int_{t_1}^{t_2} Ldt$$

In this way, here, in contrast to the principle of least action, the demand of the energy constant for the compared paths is removed, while under the sign of the integral lies another function. The size W can be not only the smallest but also the largest, in contrast to S which for the real path has the minimum value. The Lagrange function in the case when the system is conservative is equal to the difference between kinetic and potential energies:

$$L = T - U$$

In this case, Hamilton's principle corresponds to the principle of least action. If no force acts on the material point, the material point will move at a constant speed and will pass from A to B for a minimum time. Such a trajectory is a straight line.
The transition from mechanical interpretation to the principle of least action, in a more general sense, was made by Helmholtz [1], who in 1886 systematically applied this principle to the mechanics, thermodynamics, and electrodynamics problems. **He introduced the concept of kinetic potential - a quantity from which action can be taken by integration, no longer in terms of the path but in terms of time.** This site appears in all areas of the point, without any mechanical interpretation. In Helmholtz's work, the kinetic potential is interpreted not as a derived quantity - as the difference between kinetic energy and potential - but as the initial quantity. This was an important step towards the transition to the non-mechanical understanding of the principle of least action because the kinetic potential is distinguished from the mechanical concept of the $T - U$ difference. Outside of mechanics, where the difference between kinetic and potential energy loses its direct meaning, kinetic potential cannot be obtained uniformly for the given energy. The independent character of the concept of kinetic potential, therefore, allows the principle of least action to be "righted" to a universal principle of physics for reversible processes.
In 1662 Fermat [1] got the laws of refraction of light from the principle of the smallest time which later took the name Fermat principle. If the speed of light

changes continuously in the path between points A and B, the Fermat principle can express the requirement of the smallest value of the inverse of the phase velocity integral according to the way of propagation:

$$\int_A^B \frac{ds}{u} = \min$$

where u-is the phase velocity of light. This integral during light propagation will be minimal. So, the principle of least action is no longer interpreted as a mechanical principle. Planck writes that the principle of least action followed the same path as the principle of conservation of energy. "The latter was also initially regarded as a mechanical principle, and its general meaning was seen as a confirmation of the mechanical worldview. In our time the mechanical worldview was greatly curved, but no one began to seriously doubt the general character of the principle of energy conservation. If we now consider the principle of least action as a purely mechanical principle, then we would fall into bias "[1].

The concept of action is universal. Physical systems of various natures can be described in terms of action. Not only mechanics but also many other parts of physics - relativistic mechanics, electromagnetic field theory, quantum mechanics, quantum electrodynamics - are formulated more concisely, more comfortably, and more clearly with the help of variational principles.

There is the elementary quantum of action - the quantum constant. There is the speed of light - the relativistic constant. The ratio of the action of the system to the quantum of action, compared to the ratio of the velocity of the system to the relativistic constant determines the area of application of the most basic physical laws.

2 Planck's hypothesis and the principle of least action

P In constructing his theory of black radiation, Planck began by imagining matter as an assembly of electronic oscillators through which the exchange of energy between matter and radiation takes place. According to him, these oscillators are mechanical systems that are characterized by an original property: **the oscillation frequency of the oscillator which does not depend on the magnitude of its amplitude. Such an oscillator is harmonic and only in such a case the hypothesis of energy quantification is valid**. In the most general case of a system where the frequency of its oscillations is not constant but depends on the amplitude of the oscillations, the quantum of energy determined by Planck does not apply. "Planck understood the necessity of giving a more general formulation to the principle of quantification - de Broglie points out - applicable to any mechanical system and to be adapted in the special case of the harmonic oscillator mentioned above" [2].

According to De Broglie, Planck reasoned as follows:

Let us consider the mechanical system in which periodic motions are performed and which is characterized by only one variable, for example, the system consisting of a particle and in which periodic motions are performed in a straight line. For such a system can be calculated the action integral according to Maupertius, corresponds to the action integral that guru in the principle of least action, taken according to the full period of motion. This quantity is the determined characteristic of periodic motion. By requiring it to be equal to the product of a whole number of Planck constants $nh = S$ where $n = 1, 2, 3, ...$ we obtain the new formulation of the principle of quantum chemistry applicable to any periodic motion of one dimension. We are easily convinced that in the special case of the harmonic oscillator, this new principle of quantum chemistry is fully equivalent to the previous principle of energy quantum chemistry. In this way, to give the principle of quantum chemistry a more general form, Planck abandoned the initial hypothesis of energy quantum chemistry and replaced it with the hypothesis of quantum action.

"What appears in the general formulation of the principle of quantum chemistry, -action- writes Louis De Broglie, - is at the same time natural but a little strange. Natural, because this quantity plays a fundamental role in all analytical mechanics, according to the principle of Hamilton and the principle of least action. This in turn, - continues De Broglie, - led to the fact that the whole apparatus of analytical mechanics seemed to be ready to define the new principle of quantification. Strange, because from a purely physical point of view, it is difficult to understand that a quantity such as an action that has an abstract character and that does not directly satisfy any conservation law can be imagined as a characteristic of the discontinuity (discrete character) of the processes of the atomic world ", he concludes. [2].

3 Quantum of action and the integral of action

Planck came up with the idea of the fundamental role of the quantum of action in all quantum phenomena. In this case, he used the imagination of the phase space introduced by Gibbs. The state of the linear oscillator is characterized by the displacement q and the pulse p, i.e. from a point in the phase plane. Then the quantum hypothesis is equivalent to the assertion that not all phase points are equivalent. While only definite states of the system are possible, in the phase plane, the points corresponding to these states with definite energy lie in a series of discrete curves (in the case of the oscillator - in concentric ellipses) which divide the plane into zones with sizes equal to $h\nu$. (Fig 1)

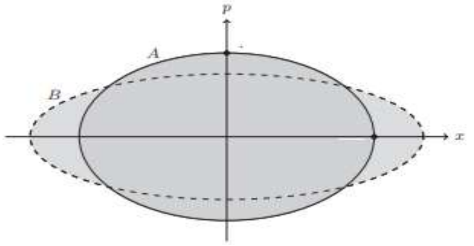

Figure 1. Oscillators in concentric ellipses

The energy of the oscillator with frequency ν in one of these selected states, will be $h\nu$. Such a phase plane structure can be mathematically characterized by the requirement that the surface of the ellipse including the n-curve ($S = nh\nu$) satisfies the condition:

$$\iint dq \cdot dp = \int p \cdot dq = nh \qquad (1)$$

This integral must be taken according to the ellipse curve in the phase plane, whereas the expression only makes sense for certain discrete values. If we think of the parameters q and p, respectively, as coordinates and as generalized pulses, **then the integral represents the general condition of quantification for any system with a degree of freedom**.

It is interesting to focus on the discussions made about this problem between Lorenz, Poincaré, Sommerfeld, and Planck.

Lorentz, interested in Planck's attitude towards the Gibbs method, asks Planck: "It is about the phase plan (p, q). Is the probability considered the same for different areas of size h "[4]?

Planck responds "The essential difference between the Gibbs method and my method stays in the fact that the element of the Gibbs plan phase is infinitesimally small, while I propose as the finite one ..." Gibbs and I assume that the same elements of the phase plan correspond to the same probabilities."

Poincaré was interested in the mathematical aspect of the issue and made some possible generalizations: “Planck presents the plan with ellipses, because this seemed to him more comfortable for the calculation of energy, but could he have presented the same result differently?" [4].

Planck answers: "The ellipse representation is not arbitrary, this is required to calculate the probability for the given oscillator energy. If it were a question of the probability of another quantity having a given value, then it is understood that the representation would be different "[4]. Meanwhile, Poincaré asked: "What if we had some degree of freedom? Imagine a resonator that can emit oscillations in all directions, so it has three degrees of freedom but is isotropic, and consequently, the period is the same for all three axes. If we project according to the three axes, then for the parallel movement with the x-axis we get the energy $\alpha h\nu$ where α - integer, according to the y and z axes respectively the energies $\beta h\nu$ $\gamma h\nu$ and where β and γ -are integers. Now we change the axes. Then for the new axes the energy αJhν, βJhν, γJhν must be taken, where αJ βJ γJ , - integers and this regardless of what the new axes are; but this is impossible "[4].

Poincaré posed two basic problems: the problem of the equivalence of presentation in phase elements and the problem of the extension of the results obtained for the case with many degrees of freedom. To the latter Planck responds: "The quantum hypothesis of some degree of freedom has not yet been formulated, but I do not find it impossible to arrive at this." [4]. And, indeed this was achieved by Sommerfeld.

In this regard, Einstein would declare at the meeting of the Association of Physicists on May 11, 1917: "In addition, Sommerfeld's work on spectrum theory unequivocally proves that, for systems with some degree of freedom, instead of a quantum condition, as many new quantum conditions as the degree of freedom contained in the system must emerge."

Planck started from general statistical imaginations, and Sommerfeld focused on application in Bohr's theory. According to Bohr's initial theory, the frequency of each spectral line is determined by the transition of the electron from one circular orbit to another. The existence of a large number of approximate frequencies led to the conclusion of the existence of a larger number of orbits than predicted by Bohr's theory. Since it was known that, under the action of central forces, elliptical orbitals are possible, **Sommerfeld set himself the task of finding the conditions for the quantification of such orbits.** Since the ellipse is determined by two parameters, Sommerfeld introduced two quantification conditions. Compared to Bohr's previous one:

$$\int_0^{2\pi} p_\varphi d\phi = nh$$

he wrote about the vector radius:

$$\int_0^{a} p_r = dr = n' h$$

In the most general case for k - degrees of freedom:

$$\int p_k dq_k = n_k h \qquad (2)$$

During the application of integral (2) for each particular coordinate, the impulse p_k can be imagined as the function of the corresponding coordinate q_k. In this case, the mechanical system is called degenerate. In this expression by general coordinates q_k, are meant such coordinates, in which the system can be imagined degenerate. "If there are several quantities - Sommerfeld points out - that allow the system to split, then, in this case, the respective phase orbits defined by the phase integral will be different, while the energies which correspond to these orbits will be the same"[6]. Such was Sommerfeld's response to Poincaré's remark. Sommerfeld tried to show that quanta of action at least do not contradict the classical picture. This effort was made in 1911 [6].

Given the fact that the quantum of action has the dimensions [Energy x time], Sommerfeld wrote: "The most general property of all molecules, which radiate, is not in the appearance of a definite amount of energy, but in the time of energy exchange, ... greater amounts of energy are absorbed and emitted in a more time short, smaller amounts of energy, in a longer time, so the definition of energy with time or integral according to energy time is determined with the size $h\nu$ [6]. And further, he went on; we will get the exact expression for magnitude [Energy $\times$time] if we start from Planck's very well-known "quantum of action" label. This leads us to what the integral over time $\int (T-U)dt$ which we encounter in Hamilton's principle to call integral of action ". [6]. In the general case, if L is the kinetic potential (according to Helmholtz terminology), then $\delta \int Ldt = 0$ expresses the principle of Hamilton. And, "if we consider together with Helmholtz-Planck, the principle of action as the most fundamental law of mechanics and physics, then we must establish the relationship between the fundamental constant of radiation and the integral $\int Ldt$ having the same dimensions of action. We then arrive at the fundamental hypothesis on the importance universal of h. For every pure molecular process, it will be absorbed and emitted from the molecule a universally defined magnitude of the action and precisely the magnitude $\int_0^{\tau} Ldt \frac{h}{2\pi}$ where τ the length of the process of action is ". [6]

Sommerfeld tried to derive this foundational hypothesis also by relying on the theory of relativity, [6] and concluded that: Energy or also the time integral of energy has no other absolute physical meaning than the meaning of the magnitude of action. It gives us the only possibility to connect the mechanics of the material point with the universal constant; the formula written for our basic hypothesis is the analytical expression of this magnitude [6].

Some of Sommerfeld's conclusions are:

- Sommerfeld considers the principle of action as the most fundamental law of physics.
- He introduces a definite physical hypothesis on the character of the molecular process which is mathematically expressed, (as opposed to classical physics which requires the Hamiltonian extreme integral) by Sommerfeld with the value of this integral equal to $\frac{h}{2\pi}$.
- Energy has no "absolute" physical meaning, while the basic concept is the concept of action. Action emerges as the key to the laws of physical phenomena.
- This site allows connecting classical physics with new (microscopic) physics.
- The fact that the dimensions of the quantum of action and the size of the sub-integral correspond to Hamilton's principle leads to the thought of finding in this fact the manifestation of their internal connection. This connection can be expressed by the application of the apparatus of variation mechanics to atomic problems in the

discovery of any particular understanding of the magnitude of action in the physical picture of the world. After the creation of quantum mechanics, many scientists exposed solutions to the problem of the possible connection of Planck's principle of action and constant to find in this connection the sources of the further development of physics.

4 De Broglie's hypothesis and the principle of least action conclusion

De Broglie carried portraits of Fermat, Maupertuis, and Hamilton in his cabinet. This is shown by the ideas about the wave nature of the subject related to the principle of least action, expressed in his dissertation which he presented in November 1923. It was published in 1924 [7]. In his dissertation, De Broglie starts from the analogy between two broad generalizations of classical physics: **the principle of the shortest time of light propagation and the principle of the smallest integral of the mechanical velocity for the motion of the material point**. Both principles are variable. The fact that De Broglie started from these principles shows not only the historical connection between classical physics and modern physics but also the special role of variational principles with the transition of old physics to new physics. When we talked about the principle of least action, we showed that the laws of geometric optics can be derived from the requirement of the shortest time, ie, from the Fermat principle. If we have the media where the speed of light is constantly changing, we will take the integral of the inverse magnitudes of the phase velocity u according to the path of light propagation:

$$\int_A^B \frac{ds}{u} = \min$$

This integral for real propagation will be the smallest.

In the Maupertuis principle, the mechanical velocity v plays the same role as the inverse magnitude of the phase velocity in the Fermat principle for optics. The Maupertuis principle can be derived from the Fermat principle, if we assume that v is proportional $1/u$. This implies that the mechanical velocity v is the velocity of any wave process with phase velocity u and these magnitudes v and u are related by a coefficient of proportionality. **We can assume the analogy between light scattering and particle motion**. So far we are dealing with the radius from one side, and the trajectory of the particle on the other. The radius is nothing but normal to the wavefront (spherical wave surface with the same phase). Respectively in mechanics, we can consider the vector of the amount of motion as normal to the velocity of the same action S. **We can express the principle of least action in the waveform**. Imagine the "wave" propagation of the action or otherwise: imagine the action as the phase of a wave from the source on all sides. Yes phase, size without dimensions. Therefore in phase mechanics, the ratio of the action S to the constant h corresponding to the action dimensions corresponds to: $\phi = \frac{S}{h}$

The propagation velocity of the "action phase" ie the velocity of propagation of the same action surface (analogous to the wavefront in optics) is equal to:

$$u = \frac{E}{p} = \frac{E}{mv}$$

where E is the energy of the moving particle, and p is the absolute magnitude of the pulse. This speed is understood to be inversely proportional to the velocity of the particle. Recall the phase velocity $u = \frac{v}{k}$. In mechanics, for this speed, we have phase velocity for the "action wave".

$$u = \frac{E}{p} = \frac{c^2}{v}$$

So, according to De Broglie, the particle possesses wave properties; its motion with velocity v corresponds to the propagation of the velocity with velocity $u = \frac{c^2}{v}$. Meanwhile, we know that the mechanical velocity $v \ll c$. Therefore, $u \succ c$ which do not contradict the principle of relativity. This principle limits the speed of energy and mass transport, which can only serve as signals.

De Broglie's wave propagation velocity is extremely large, hundreds and billions of times faster than the mechanical velocity of classical objects. How can these giant velocities of "matter waves" (De Broglie waves) be related to such small mechanical velocities? De Broglie used the concept of group velocity to explain this. De Broglie identified the velocity of motion of the particle v with the velocity of motion of the wave "group". The particle, in this theory, is considered a "wave packet" - a wave is limited in space and time, where at its center there is a maximum value of energy as a result of wave interference, while at the edges it has a value of zero. Such a wave packet moves with speed v_g.

Now compare the group velocity in optics and the particle velocity in mechanics: The velocity of the group of light waves is $\frac{\delta v}{\delta k}$, while the velocity of the particle is obtained by the differentiation of the phase of the velocity of the "action wave": $\frac{\delta E}{\delta p}$

In this way, we can switch from optics to mechanics by replacing the frequency ν with energy E and the wave vector $\vec{k}$ with impulse $\vec{p}$. Energy and momentum belong to the particle, while frequency and wave vector belong to the wave. They are related to each other by the coefficient h which has the dimensions of the action.

$$\nu = \frac{E}{h}, \vec{k} = \frac{\vec{p}}{h}$$

The culmination of De Broglie's general thought was **the assumption that this coefficient corresponds to Planck's constant - the quantum of action**. Then the energy of the particle is expressed through the frequency, and the impulse through the wave vector:

$$E = h\nu,\ \vec{p} = h\vec{k}$$

Hence, the expression for the wavelength λ of De Broglie.

The absolute magnitude of the wave vector is inverse of the wavelength; therefore substituting $\frac{1}{\lambda}$ in the expression for the impulse instead of k we get:

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

Such are the fundamental assertions of De Broglie's theory, derived from the principle of least action.

5 Feynman and the principle of least action

The action can be not only minimal but also maximal. But the fact that the more general law of nature affirms the extreme character of action for the real trajectories of the movement of bodies, remained unclear. Edington [8] was the first to show interest in this before quantum mechanics was born. Only the fact of the existence of the quantum of action - Planck's constant h. If we divide the action S into the quantum of the action h, then we get a dimensionless number which as well as the action itself will play in theory an important role thanks to the existence of the principle of least action. What other sizeless dimension plays an equally important role in physics? Of course, it is probability. Then according to Edington, we can give a new interpretation to **the principle of least action as the principle of maximum probability**. Feynman proved this [9].

At the heart of Feynman's new conception of quantum mechanics lies the postulate that allows us to determine the probability of a particle passing from one point in space to another. Assume that the particle at the moment t_a is a point a. It is asked what is the probability $P(b,a)$ of what a moment t_b it is at point b? To answer this question we must consider all possible trajectories from a b to, while each trajectory contributes to the required probability. According to Feynman's basic postulate, the contributions of particular trajectories are the same in size but differ only in stages. The contribution phase of any given trajectory is equal to the classical action S for the given trajectory expressed in units of action h. Quantitatively this is formulated as follows:

The probability amplitude of the given trajectory is:

$$\varphi[r(t)] = const.\exp.\left\{\frac{iS[\vec{r}(t)]}{\hbar}\right\}$$

where S - the classical action for the given trajectory $\vec{r}(t)$.

The full probability amplitude is equal to the sum of all the trajectories, thus:

$$A = (b,a) = \sum \varphi_i[\vec{r}(t)]$$

while the probability:

$$P(b,a) = \left|A(b,a)\right|^2$$

hence the square of the amplitude modulus. The norming condition defines the constant in the expression for $\varphi[\vec{r}(t)]$.

The given postulate, as Feynman showed, derives all ordinary quantum mechanics and especially the Schrödinger equation. The formulation of quantum mechanics with the help of integrals according to trajectories allows very simple to pass to the approximation of classical mechanics when two points in the given field of forces can be connected only to a single trajectory, during which the action is extreme. In the classical case, the action is very large, so $S \gg h$. Therefore the phase $\frac{S}{h}$ for each trajectory is very large. From Euler's formula:

$$\exp\left(\frac{iS}{\hbar}\right) = \cos\left(\frac{S}{\hbar}\right) + i\sin\left(\frac{S}{\hbar}\right)$$

whereas both the real and virtual parts of the amplitude $\varphi[\vec{r}(t)]$ can be both positive and negative. For a very small displacement of the trajectory $S_{\vec{r}(t)}$, the change of the action turns out large compared to h. Consequently, small trajectory excitations lead to large phase changes, and the function $\cos\left(\frac{S}{\hbar}\right)$, $\sin\left(\frac{S}{\hbar}\right)$ will perform rapid oscillations between positive and negative values. If one trajectory makes a positive contribution to the probability amplitude, the other approximately approximates, a negative contribution, then in general the contribution is zero. Therefore we can not calculate the given trajectory, if the neighbors with it have other values of action.

Their mutual contributions will be eliminated. However, the small displacement $\delta_{\vec{r}(t)}$ of a $\vec{r}_{kl}(t)$ trajectory for which the action is extreme, in the first approximation, does not change the action S (this is also shown by the existence of the extreme). The stages of all the trajectory contributions that occur in this area differ very little from each other; they are equal to $\frac{S_{kl}}{\hbar}$ and are not mutually eliminated. In the classical approximation $S \gg \hbar$, we must consider this trajectory as the only possible one. In this way, the principle of least action in classical form: $\delta S = 0$ is derived from Feynman's quantum principle.

If S is comparable to h, then we must calculate all trajectories. None of them is more privileged than the other. The only trajectory is the most probable. It is realized as the only one with the condition $S \gg \hbar$. The principle of least action is statistical.

6 Conclusions

1. Planck's hypothesis gave a new meaning to the full mechanical action S. It was imagined as a multiple of the elementary action of the Planck constant h $S = nh$. It moved on to a new generalization: **he replaced the hypothesis of energy quantification with the quantification of action**. This new hypothesis had a more general character than the "energy quantum" hypothesis because it applies to all mechanical systems and not just oscillators, while in the case of harmonic oscillators, it is reduced to the "energy quantum" hypothesis. The new hypothesis was also more abstract than the previous one because the action is a less concrete physical quantity, which is not subject to any conservation law and yet it has atomic properties.
2. Sommerfeld in his attempts to connect the quantum of action with classical mechanics proposed a new principle, at first glance very strange whose essence can be expressed in this way: **the necessary time it takes for matter to absorb or emit an amount of energy is the shorter, the greater this energy, so that the product (Energy $\times$ time) is determined by the constant h**. To give this assertion a more concrete formulation, Sommerfeld applied Hamilton's principle. Matching the size of the quantities that lie below the integral on the principle of Hamilton, with the dimensions of Planck's quantum of action, served Sommerfeld as the starting point for obtaining the conditions for the quantification of the motion of the electron in the atom. (Bohr-Sommerfeld conditions).
3. Such an idea led De Broglie to the idea of **connecting wave processes in a continuous environment on the one hand and the movement of discrete particles on the other hand**. From this idea of De Broglie later theoretically the conditions of the quantum motion of the electron for the atom were derived (Bohr- Sommerfeld conditions).
4. Eddington shifted to another conception of the principle of least action, the statistical one. According to Eddington the principle of least action can be imagined as the principle of maximum probability. Feynman tried to prove this.

5. Another important consequence of the quantum of action, perhaps more revolutionary than the quantum hypothesis itself, is the specific relationship between space, time, and dynamic quantities (energy-impulse) which we try to locate in space and time.
6. According to classical physics, since the physical quantities that characterize the interaction vary continuously, then sizes such as the size of the localization particle area, must always become infinitesimally small and, in the limit, tend to zero.
7. It was found that some physical sizes which characterize the state of the object, for some interactions, are changed with "hop" and for each real condition they do not become zero, but gain a finite indivisible and minimal value of "h". The action of physical systems as well as the exchange of action between physical systems is a discrete process, with "hop" associated with the existence of the indivisible quantum of action "h".
8. This problem constitutes a completely new connection, absolutely foreign to the concepts of classical physics, which necessarily leads to a new way of describing physical phenomena in the microworld.

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