Hydrodynamic Considerations of Micro-/Nano-Scale Flows

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Abstract:- In the present paper, hydrodynamic characteristics of the external forced convection in the slip regime over an isothermal horizontal plate at a relatively low Mach number is numerically studied. Slip flow occurs when the dimensions of the flow system are comparable to the molecular mean free path. Under this situation, the no-slip condition is replaced by slip-flow condition. Dimensionless stream function, velocity, slip velocity at the wall, wall shear stress and boundary layer thickness are presented for a range of values of the parameter characterizing the slip flow. This slip parameter is a function of the local Reynolds number, the local Knudsen number, and the tangential momentum accommodation coefficient representing the fraction of the molecules reflected diffusively at the surface. As the Knudsen number approaches zero, the slip parameter also approaches zero, and the no-slip condition is recovered. As the Knudsen number increases, the slip velocity increases. These results are in good agreement with the conclusions reached in other recent studies.

Key-Words:- Isothermal Horizontal Plate, Micro-/Nano-Scale Flows, Slip parameter, Similarity approach.

List of Symbols

a_l initial values Eq. (17)
f function defined in Eq. (5)
f_l function defined in Eq. (19)
Kn_x Knudsen number, dimensionless
K slip parameter, defined in Eq (13), dimensionless
Re_x Reynolds number at x, dimensionless
u velocity component in x, m/s
u_l free stream velocity in x, m/s
v velocity component in y, m/s
x coordinate from the leading edge, m
y coordinate normal to plate, m
z_1, z_2, z_3 variables, Eq. (14)

Greek Symbols

λ molecular mean free path, m
σ tangential momentum accommodation coefficient, dimensionless
δ_{99} boundary layer thickness, m
μ dynamic viscosity, N.s/m^2
ν kinematic viscosity, m^2/s
η similarity variables, Eq. (6)
τ_{wall} wall shear stress, N/m^2
ψ stream function, m^2/s

1 Introduction

Study of flow through micro-/nano-scales has gained interest because of potential applications of micro devices in engineering, medical, and various scientific areas [1-3].

The dimensions of micro-/nano-scale devices are comparable to the mean free path of gas molecules, and high Knudsen number fluid flows. The flow behavior through such devices greatly differs from the traditional no-slip boundary conditions at the solid-fluid interface and belongs to the slip flow regimes.

For slightly rarefied flows, the slip condition at the solid-liquid interface, with Knudsen number less than 0.1, simplifies to [4]

$$u_{wall} = \frac{2-\sigma}{\sigma}\lambda \left.\frac{\partial u}{\partial y}\right|_{wall}$$

where λ is the mean free path, and σ is the tangential momentum accommodation coefficient. It represents the portion of total wall-colliding molecules that are diffusively reflected back by the wall and have bulk velocity equal to the wall velocity after collision. Rest molecules are reflected back specularly. $\left.\frac{\partial u}{\partial y}\right|_{wall}$ is the velocity gradient normal to the wall

In the present numerical investigation, a simple accurate numerical simulation of laminar free-convection flow over an isothermal horizontal plate under slightly rarefied flow condition is developed.

The paper is organized as follows: Mathematical model of the problem, its solution procedure, development of code in Matlab, interpretation of the results.

2 Mathematical Model

We consider the flow of a fluid of velocity u_1 (of low Mach number) over an isothermal horizontal plate. The low Mach number implies that compressibility effects and heating by viscous dissipation effects are negligible. We assume the natural convection flow to be steady, laminar, two-dimensional, and the fluid to be Newtonian with constant properties, including density, with one exception: the density difference $\rho - \rho_\infty$ is to be considered since it is this density difference between the inside and the outside of the boundary layer that gives rise to buoyancy force and sustains flow, known in the literature as *Boussinesq approximation*. We take the direction along the plate to be *x*, and the direction normal to surface to be *y*, as shown in Fig. 1.

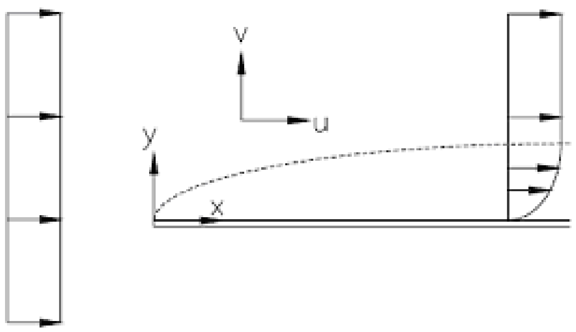

Fig. 1. Physical Model and its coordinate system

The equations governing the flow are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \qquad (2)$$

The boundary conditions on the solution are:

$$at \quad y = o \quad , \qquad u_{wall} = \frac{2-\sigma}{\sigma}\lambda\left.\frac{\partial u}{\partial y}\right|_{wall} \quad , \quad v = 0$$

For large y: $u \to u_1$ (3)

The continuity Eq. (1) is automatically satisfied through introduction of the stream function:

$$u \equiv \frac{\partial \psi}{\partial y} \qquad v \equiv -\frac{\partial \psi}{\partial x} \qquad (4)$$

A similarity solution is possible if

$$\psi = u_1\sqrt{\frac{\nu x}{u_1}}\, f(\eta) \qquad (5)$$

where, η is the similarity variable

$$\eta = \frac{y}{x}\sqrt{\mathrm{Re}_x} = y\sqrt{\frac{u_1}{\nu x}} \qquad (6)$$

From Eqs. (4) through (6), we get

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} = u_1\sqrt{\frac{\nu x}{u_1}}\frac{df}{d\eta}\sqrt{\frac{u_1}{\nu x}} = u_1\frac{df}{d\eta} \qquad (7)$$

and

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial x} = -(u_1\sqrt{\frac{\nu x}{u_1}}\frac{df}{dx} + \frac{u_1}{2}\sqrt{\frac{\nu}{u_1 x}}f)$$

$$= \frac{1}{2}\sqrt{\frac{\nu u_1}{x}}(\eta\frac{df}{d\eta} - f) \qquad (8)$$

By differentiating the velocity components, it may also be shown that

$$\frac{\partial u}{\partial x} = -\frac{u_1}{2x}\eta\frac{d^2 f}{d\eta^2} \qquad (9)$$

$$\frac{\partial u}{\partial y} = u_1\sqrt{\frac{u_1}{\nu x}}\frac{d^2 f}{d\eta^2} \qquad (10)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_1^2}{\nu x}\frac{d^3 f}{d\eta^3} \qquad (11)$$

Substituting these expressions into Eq. (2), we then obtain (with a prime denoting differentiation with respect to η)

$$2f''' + ff'' = 0 \qquad (12)$$

Hence the velocity boundary layer problem is reduced to an ordinary differential equation. The appropriate boundary conditions are:

$$at \quad y = 0, \quad u = 0, \quad i.e., \quad at \quad \eta = 0, \quad f'(0) = \frac{2-\sigma}{\sigma}Kn_x\,\mathrm{Re}_x^{1/2}\, f''(0) = Kf''(0)$$

at y = 0: v = 0 i.e., at η = 0: f = 0

for large y: $u \to u_1$ i.e., for large η: $f' \to 1$ (13)

where Kn_x and Re_x are the Knudsen and Reynolds numbers based on x, and $K = \frac{2-\sigma}{\sigma}Kn_x\,\mathrm{Re}_x^{1/2}$ is a non-dimensional parameter that describes the behaviour at the surface. The slip coefficient, k is a dimensionless parameter of the amount of slip, ranging from zero (no-slip) to infinity (full slip).

3 Solution Procedure

Eqs. (12) is nonlinear ordinary differential equations for the velocity function f'. No analytic solution is known, so numerical integration is necessary [5]. There is one unknown initial value at the wall. One must find the proper value of $f''(0)$ which cause the velocity to its free stream values for large η.

3.1 Reduction of Equations to First-order System

This is done easily by defining new variables:

$$z_1 = f$$
$$z_2 = z_1' = f'$$
$$z_3 = z_2' = z_1'' = f''$$
$$z_3' = z_2'' = z_1''' = f''' = -\frac{1}{2} ff'' = -\frac{1}{2} z_1 z_3 \tag{14}$$

Therefore from Eqs (12), we get the following set of differential Eqs. (15)

$$z_1' = f'$$
$$z_2' = z_1'' = f''$$
$$z_3' = z_2'' = z_1''' = f''' = -\frac{1}{2} ff'' = -\frac{1}{2} z_1 z_2 \tag{15}$$

with the following boundary conditions:

$$z_1(0) = f(0) = 0$$
$$z_2(0) = z_1'(0) = f'(0) = Kf''(0) = Kz_3(0)$$
$$z_2(\infty) = z_1'(\infty) = f'(\infty) = 1 \tag{16}$$

Eq (12) is third-order and is replaced by three first-order Eqs. (15).

3.2 Solution to Initial Value Problems

To solve Eqs (15), we denote the unknown initial value $f''(0)$ by a_1, the set of initial conditions is then:

$$z_1(0) = f(0) = 0$$
$$z_3(0) = z_2'(0) = z_1''(0) = f''(0) = a_1$$
$$z_2(0) = z_1'(0) = f'(0) = Ka_1 \tag{17}$$

If Eqs (15) are solved with adaptive Runge-Kutta method using the initial conditions in (17), the computed boundary values at $\eta = \infty$ depend on the choice of a_1. We express this dependence as

$$z_2(\infty) = z_1'(\infty) = f'(\infty) = f_1(a_1) \tag{18}$$

The correct choice of a_1 yields the given boundary conditions at $\eta = \infty$; that is, it satisfies the equations

$$f_1(a_1) = 1 \tag{19}$$

This nonlinear equation can be solved by the Newton-Raphson method. A value of 10 is fine for infinity, even if we integrate further nothing will change.

3.3 Program Details

This section describes a set of Matlab routines for the solution of Eqs. (15) along with the boundary conditions (17). They are listed in Table 1.

Table 1. A set of Matlab routines used sequentially to solve Equations (15).

Matlab code	Brief Description
deqs.m	Defines the differential Eqs. (15).
incond.m	Describes initial values for integration, a_1 is guessed value. Eq (17)
runKut5.m	Integrates as initial value problem using adaptive Runge-Kutta method.
residual.m	Provides boundary residuals and approximate solutions.
newtonraphson.m	Provides correct values a_1 and a_2 using approximate solutions from residual.m
runKut5.m	Again integrates Eqs. (15) using correct values of a_1 and a_2.

The final output of the code runKut5.m gives the tabulated values of f, f', f'' as function of η for velocity profile as function of η. K is a parameter.

4 Computational Results for Fluid Flow

Physical quantities are related to the dimensionless stream function f through Eqs. (5), (6) (7) and (8). f is now known. Some accurate initial values of $f'(0)$ and $f''(0)$ from this computation are listed in Table 2.

Table 2 Computed parameters from Eqs. (15)

K	$f'(0)$	$f''(0)$
0	0	0.3321
0.1	0.033	0.3315
0.2	0.066	0.3298
0.3	0.098	0.3272
0.4	0.130	0.3238
0.5	0.160	0.3198
0.6	0.189	0.3152
0.7	0.217	0.3102
0.8	0.244	0.3050
0.9	0.270	0.2995
1.0	0.294	0.2939
2.0	0.479	0.2397
3.0	0.593	0.1976
4.0	0.667	0.1667
5.0	0.718	0.1437
10	0.843	0.0843
20	0.916	0.0458
30	0.942	0.0314
40	0.956	0.0239
50	0.965	0.0193

Variations of dimensionless stream function f with η for K ranging from 0 (no slip) to 50 obtained from the code are shown in Fig. 2. It is evident from this figure that for any value of η in the boundary layer, f increases as the flow becomes more rarefied, i.e., as K increases.

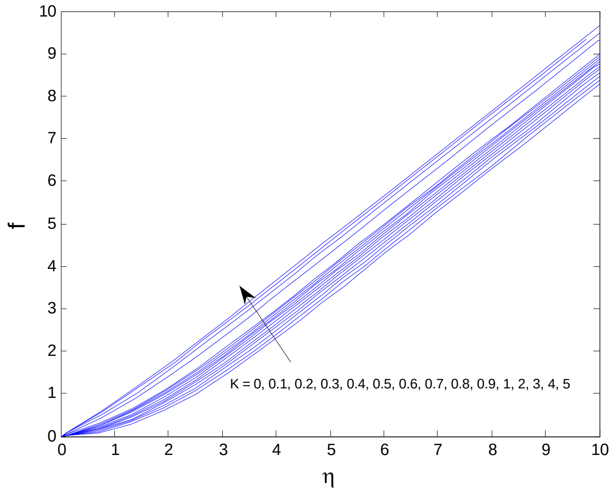

Fig. 2. Dimensionless stream function f as a function of η

The variation of dimensionless x component of the velocity, f' as a function of η with K as parameter is shown in Fig. 3. Figure 3 illustrates the fact that as the flow becomes more rarefied (K increases), the slip velocity f'(0) increases and so does the x component of the velocity for any value of η.

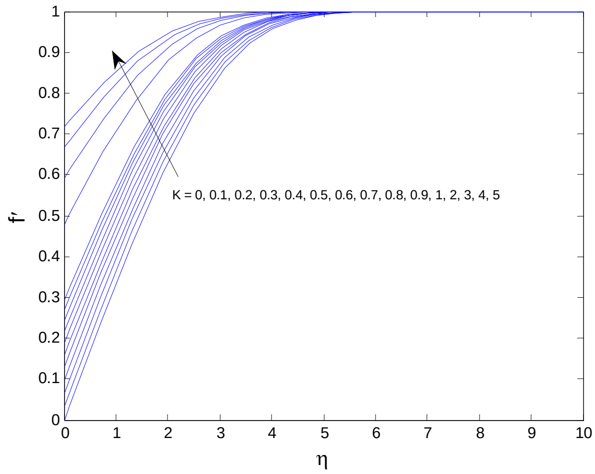

Fig. 3. The velocity profile f' within the boundary layer, K parameter

One result that can be seen in Fig. 3 is that even as the wall velocity changes drastically, the overall boundary layer thickness does not change as rapidly. The boundary layer thickness, δ_{99}, is defined as the value of y at which $u=0.99u_1$. From Eq. (6), the physical thickness of the boundary layer can be written as

$$\delta_{99} = \eta_{99} x \,\mathrm{Re}_x^{-1/2} \tag{20}$$

For the no-slip boundary layer, the boundary layer thickness η_{99} is a constant with a value of 5.0 [5]. For a boundary layer with slip, η_{99} varies along the plate. Figure 4 shows the value of η_{99}, as a function of K.

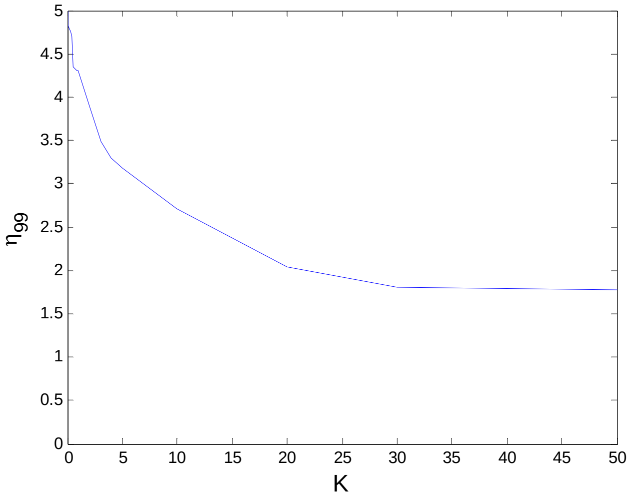

Fig. 4. Boundary layer thickness, η_{99} as a function of K

Fig. 5 shows the effect of increasing rarefaction on the slip velocity. As mentioned earlier, the slip velocity increases as the flow becomes more rarefied. As the Knudsen number approaches zero, K also approaches zero, where the no-slip condition and the classical boundary layer solution are recovered. As the Knudsen number becomes large, K approaches infinity, and the nondimensional slip velocity approaches 1, indicating 100% slip at the wall.

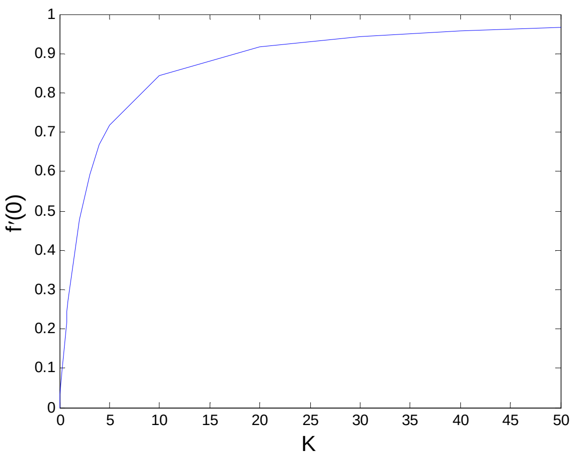

Fig. 5. Effect of k on f'(0)

From Eq. (10), wall shear stress may be expressed as

$$\tau_{wall} = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \mu u_1 \sqrt{\frac{u_1}{\nu x}} \frac{d^2 f}{d\eta^2}\bigg|_{\eta=0} = \mu u_1 \sqrt{\frac{u_1}{\nu x}} f''(0) \tag{21}$$

The nondimensional wall shear stress $f''(0)$ is shown in Fig. 6. $f''(0)$ returns to the no-slip value as K approaches 0 and asymptotically approaches zero as K approaches infinity.

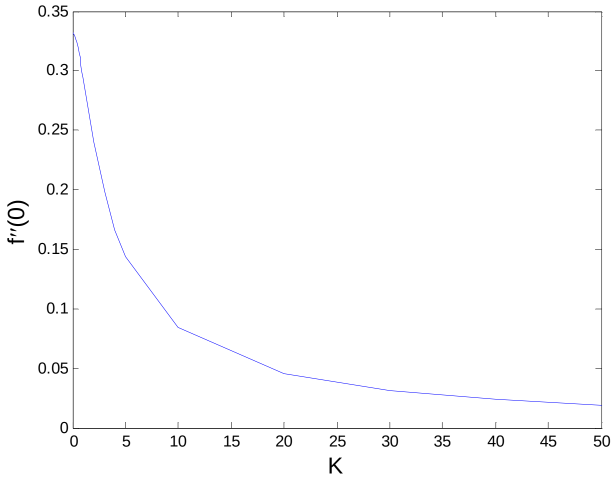

Fig. 6. Wall stress, $f''(0)$ as a function of K

All the above computational results are in good agreement with recent studies [6, 7]

5 Conclusion

Hydrodynamic characteristics of the external forced convection in the slip flow regime over an isothermal horizontal plate at a relatively low Mach number is numerically studied using local similarity approach.

This approach consists to fix the slip parameter K at any x-location along the plate, i.e., K = constant and ignoring the variation of the velocity field with K. This is equivalent to ignore the upstream history of the flow. Consequently, the original partial differential boundary layer equations become ordinary differential equations.

Dimensionless stream function, velocity, slip velocity at the wall, wall shear stress distributions and boundary layer thickness are presented for a range of values of the parameter characterizing the slip flow. As the Knudsen number approaches zero, the slip parameter also approaches zero, and the no-slip condition is recovered. As the Knudsen number increases, the slip velocity increases but the wall shear decreases. These results are in good agreement with the conclusions reached in other recent studies.

References:
[1] Ho, C. M., and Tai, Y. C., "Micro-Electro-Mechanical Systems (MEMS) and Fluid Flows," *Annual Review of Fluid Mechanics*, Vol. 30, 1998, pp. 579–612.
[2] Gad-el-Hak, M., "The Fluid Mechanics of Microdevices—The Freeman Scholar Lecture," *Journal of Fluids Engineering*, Vol. 121, 1999, pp. 5–33.
[3] N.T. Nguyen, S.T. Wereley, Artech House, "Fundamentals and Applications of Microfluidics" 2002, pp. 1–10.
[4] Gad-el-Hak, Mohamed, *Journal of Fluids Engineering,* Vol. 121, 1999, pp. 5-33.
[5] Mitra A., "Numerical Simulation on Laminar Convectionion Flow and Heat Transfer Over an Isothermal Horizontal Plate," *J. Mech. Cont. & Math. Sci*., Vol. 10, 2016, pp 1521-1534
[6] Martin MJ, Boyd ID. "Momentum and heat transfer in laminar boundary layer with slip flow". *J Thermophys Heat Transfer*, Vol. 20, 2006, pp. 710–719.
[7] Aziz A., "Hydrodynamic and thermal slip flow boundary layers over a flat plate with constant heat flux boundary condition," *Commun Nonlinear Sci Numer Simulat*, Vol. 15, 2010, pp. 573–580.