

Green Tensor of Maxwell Transport Equations and Its Properties at Pre- and Superluminal Speeds

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Abstract. Transport solutions of Maxwell equations at pre- and superluminal speeds are considered. The Green tensor and generalized solutions are constructed in analytical form over the entire speed range, from sublight to superluminal. At superluminal velocity, electromagnetic shock waves occur in the medium. The conditions for the jumps of their fronts are obtained. Formulas for calculating the electromagnetic fields of mobile radiators of various types, useful for radio engineering applications, are given.

Key-Words: Maxwell's transport equations, Green tensor, shock electromagnetic waves

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1 Introduction

The basis of modern electrodynamics are Maxwell's equations, which make it possible to determine the electromagnetic field with knowledge of the active charges and currents (radiation sources) and vice versa. Among the existing sources, the most common are mobile ones located on platforms of various vehicles. Research in this area is not so numerous and is associated with a certain type of radiation source

In this paper, we construct fundamental and generalized solutions of Maxwell's equations for a radiation source moving at an arbitrary constant speed V . It can be less than or greater than the speed of light $c = \sqrt{1/\mu\epsilon}$, which affects the type of Maxwell's transport equations in a moving coordinate system, from elliptical to hyperbolic, depending on the ratio $V/c = M$ - the Mach number.

Therefore, the construction of solutions to these equations, as well as their properties, requires separate study for each speed range: $M < 1$ and $M \geq 1$.

2 Maxwell transport equations. Statement of the task

Mobile transport sources of electromagnetic waves are considered, which do not change their form, but move with a constant velocity V along axis X_3 . They can be described by currents of the form $\mathbf{J}(x, z)$, where $x_3 + VT = z$. In a moving coordinate (x, z) , $\mathbf{x} = (x_1, x_2)$, the system of Maxwell equations for currents has the form [1]:

$$\mathbf{M}(\partial_1, \partial_2, \partial_z)\mathbf{u}(\mathbf{x}, z) = \mathbf{J}(\mathbf{x}, z), \quad (1)$$

where the following designations are introduced:

$$\mathbf{u}(\mathbf{x}, z) = \begin{pmatrix} \mathbf{E}(\mathbf{x}, z) \\ \mathbf{H}(\mathbf{x}, z) \end{pmatrix}, \quad \mathbf{J}(\mathbf{x}, z) = \begin{pmatrix} \mathbf{j}^m(\mathbf{x}, z) \\ \mathbf{j}^e(\mathbf{x}, z) \end{pmatrix}$$

the differential matrix operator $\mathbf{M}(\partial_1, \partial_2, \partial_z)$ is equal to

$$\bar{\mathbf{M}}(\partial_1, \partial_2, \partial_z) = \begin{pmatrix} 0 & -\partial_z & \partial_2 & -\mu V \partial_z & 0 & 0 \\ \partial_z & 0 & -\partial_1 & 0 & -V\mu \partial_z & 0 \\ -\partial_2 & \partial_1 & 0 & 0 & 0 & -V\mu \partial_z \\ V\epsilon \partial_z & 0 & 0 & 0 & -\partial_z & \partial_2 \\ 0 & V\epsilon \partial_z & 0 & \partial_z & 0 & -\partial_1 \\ 0 & 0 & V\epsilon \partial_z & -\partial_2 & \partial_1 & 0 \end{pmatrix}$$

here $\mathbf{E}(\mathbf{x}, z)$, $\mathbf{H}(\mathbf{x}, z)$ are the vectors of electric and magnetic strengths of EM field; $\mathbf{j}^m(\mathbf{x}, z)$, $\mathbf{j}^e(\mathbf{x}, z)$ are the magnetic and electric current densities, $\partial_z \square \frac{\partial}{\partial z}$, $\partial_j \square \frac{\partial}{\partial x_j}$.

It is required to construct solutions of equations (1) for any Mach numbers.

2 The Green tensor of Maxwell transport equations

Using the Fourier transform of generalized functions, the Green tensor is constructed $\mathbf{U}(\mathbf{x}, z)$, which satisfies (1) and the radiation conditions at $\mathbf{J}(\mathbf{x}, z) = \delta(\mathbf{x})\delta(z)\delta_{ij}$, $i, j = 1, \dots, 6$. It has the following form [1]:

$$\mathbf{U}(\mathbf{x}, z) = \begin{pmatrix} 0 & -\partial_3 f_0 & \partial_2 f_0 & b & \frac{\partial_1 \partial_2 f_0}{\mu V} & \frac{\partial_1 f_0}{\mu V} \\ \partial_3 f_0 & 0 & -\partial_1 f_0 & \frac{\partial_1 \partial_2 f_1}{\mu \cdot V} & b & \frac{\partial_2 f_0}{\mu V} \\ -\partial_2 f_0 & \partial_1 f_0 & 0 & \frac{-\partial_1 f_0}{\mu \cdot V} & \frac{\partial_2 f_0}{\mu V} & \frac{m^2 \partial_3 f_0}{\mu V} \\ a & \frac{-\partial_1 \partial_2 f_1}{\varepsilon V} & \frac{\partial_1 f_0}{\varepsilon V} & 0 & -\partial_3 f_0 & \partial_2 f_0 \\ \frac{\partial_1 \partial_2 f_1}{\varepsilon \cdot V} & -a & \frac{\partial_2 f_0}{\varepsilon V} & \partial_3 f_0 & 0 & -\partial_1 f_0 \\ \frac{-\partial_1 f_0}{\varepsilon V} & \frac{-\partial_2 f_0}{\varepsilon V} & \frac{-m^2 \partial_3 f_0}{\varepsilon V} & -\partial_2 f_0 & \partial_1 f_0 & 0 \end{pmatrix}$$

where

$$\frac{(\partial_1^2 - M^2 \partial_3^2) f_1}{\varepsilon V} = a, \quad \frac{(\partial_2^2 - M^2 \partial_3^2) f_1}{\mu V} = b.$$

Here the basic functions $f_0(\mathbf{x}, z)$, $f_1(\mathbf{x}, z)$ have different values depending on the Mach number [2]. At $M < 1$ (sublight speed):

$$f_0(\mathbf{x}, z) = \frac{1}{4\pi\sqrt{m^2 r^2 + z^2}},$$

$$f_1(\mathbf{x}, z) = \frac{\text{sgn}(z)}{4\pi} \ln \left(\frac{|z| + \sqrt{m^2 r^2 + z^2}}{mr} \right) \quad (3)$$

where $m = \sqrt{1 - M^2}$, $r = \sqrt{x_1^2 + x_2^2}$.

At $M > 1$ (superlight speed):

$$f_0(\mathbf{x}, z) = \frac{-H(z - mr)}{2\pi\sqrt{z^2 - m^2 r^2}},$$

$$f_1(\mathbf{x}, z) = \frac{-H(z - mr)}{2\pi} \ln \left(\frac{z + \sqrt{z^2 - m^2 r^2}}{mr} \right) \quad (4)$$

where $m = \sqrt{M^2 - 1}$.

At $M = 1$ (light speed):

$$f_0(\mathbf{x}, z) = -\frac{1}{2\pi} \delta(z) \ln r,$$

$$f_1(\mathbf{x}, z) = -\frac{1}{2\pi} H(z) \ln r. \quad (5)$$

Here $H(z)$ is Heaviside step function

3 Electromagnetic shock waves. Conditions on the fronts

At light and superluminal speeds, the system of Maxwell's transport equations (1) is strictly hyperbolic. The radiation source generates shock electromagnetic waves, at the fronts of which the following conditions are met for jumps in the intensity of the electromagnetic field:

$$\mu V [\mathbf{H}(\mathbf{x}, z)]_F = [\mathbf{E}]_F \times \mathbf{n}(\mathbf{x}, z), \quad (6)$$

$$\varepsilon V [\mathbf{E}(\mathbf{x}, z)]_F = [\mathbf{H}]_F \times \mathbf{n}(\mathbf{x}, z), \quad (7)$$

where on the right in the equations are the vector products of the jumps of the intensity vectors at the front F of the EM wave to the $\mathbf{n}(\mathbf{x}, z)$ - normal to the front (wave vector).

It follows that the electric field and magnetic field strength jumps are orthogonal to each other and orthogonal to the normal to the wave front. If in front of the wave front $\mathbf{E}(\mathbf{x}, z) = 0$, $\mathbf{H}(\mathbf{x}, z) = 0$ then (6), (7) follows:

$$\mathbf{H}(\mathbf{x}, z)|_F = \frac{1}{\mu V} [\mathbf{E} \times \mathbf{n}(\mathbf{x}, z)]|_F,$$

$$\mathbf{E}(\mathbf{x}, z)|_F = \frac{1}{\varepsilon V} [\mathbf{H} \times \mathbf{n}(\mathbf{x}, z)]|_F,$$

That is, the shock electromagnetic waves are transverse and the vectors of electric and magnetic intensity at the front of the shock electromagnetic wave are orthogonal to each other and lie in a tangent bundle to it.

4 Construction of solutions to Maxwell's transport equations

The solution of equations (1) satisfying the radiation conditions has the form of a tensor-functional convolution of the Green tensor with the right side of equation (1):

$$\mathbf{u}(\mathbf{x}, z) = \mathbf{U}(\mathbf{x}, z) * \mathbf{J}(\mathbf{x}, z), \quad (7)$$

From here follow

$$E_j = \sum_{k=1}^3 U_{jk} * j_k^m(\mathbf{x}, z) + \sum_{k=4}^6 U_{jk} * j_{k-3}^e(\mathbf{x}, z),$$

$$H_j = \sum_{k=1}^3 U_{(j+3)k} * j_k^m(\mathbf{x}, z) + \sum_{k=4}^6 U_{(j+3)k} * j_{k-3}^e(\mathbf{x}, z).$$

$$j = 1, 2, 3.$$

The integral representation of this solution depends on the type $\mathbf{J}(\mathbf{x}, z)$. Integral representations for radiation sources are obtained, which are described by regular functions, as well as by singular simple layers on the surfaces of S : $\mathbf{J} = \mathbf{J}(\mathbf{x}, z)\delta_S(\mathbf{x}, z)$ and curves L : $\mathbf{J} = \mathbf{J}(\mathbf{x}, z)\delta_L(\mathbf{x}, z)$ (see [1]).

This makes it possible to study the EM fields of moving emitters focused on surfaces and curves.

5 Conclusion

The results obtained can be used to study the electromagnetic fields of various light emitters and radio wave emitters located on mobile objects (trains, cars, ships, etc.), as well as to solve transport boundary value problems of electrodynamics in limited areas based on the method of boundary integral equations, which the authors plan to do in the near future using the method of generalized functions, similar to [3,4].

References:

[1] Alexeyeva L.A., Kanymgaziyeva I.A. Maxwell transport equations, their fundamental and generalized solutions at a constant speed of the radiation source, Technical Physics, No, 2024, pp.539-546.

[2] Алексеева Л.А. Generalized solutions of boundary value problems for one class of traveling solutions of the wave equation, Mathematical Journal, Vol 8, No 2(30), 2008, pp. 1-19.

[3] Alexeyeva L.A., Sautbekov S.S. Boundary integral equations of stationary boundary value problems for Maxwell equations, Computational mathematics and mathematical physics, No 4(40), 2000, pp. 619-630.

[4] Alexeyeva L.A. Generalized solutions of nonstationary boundary value problems for Maxwell equations, Computational mathematics and mathematical physics, No 1 (42), 2002, pp. 75-87.

Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

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Alexeyeva L.A. was involved in problem formulation and mathematical interpretation.

Kanymgaziyeva I.A. built and obtained the inverse matrix.

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