Solving problems in mechanics by mechanical and geometrical considerations

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Abstract: - One of the problems that are solved with Statistics, as division of theoretical Mechanics, is the equilibrium of the system of forces in which the conditions are determined for a system of forces to be in equilibrium. In classical mechanics, the state of equilibrium of a material body can be defined from static or dynamic point of view, both being forms of mechanic equilibrium. Mechanics depends on mathematics, in the sense that almost no problem of mechanics can be solved without mathematics. Thus, interdisciplinarity being the cooperation between various disciplines from the same curricular area, it means that the approach has as its aim forming a unitary image regarding a certain theme. This implies the combination of two or several academic disciplines in one single activity, thus simultaneously accumulating new knowledge in several domains. In this context, the paper presents aspects on solving problems of equilibrium of rigid bodies, by mechanical and geometrical considerations, namely with the condition of concurrency of three lines in a plane.

Key-Words: - mechanic, rigid body, equilibrium, geometry, lines, concurrency, plane

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1 Introduction

Statistics, as part of theoretical mechanics, studies the conditions of equilibrium of material systems under the action of applied forces, ignoring motion [1], [2].

It is recommended toapproach statistics deductively, starting from the simplest mechanical model (material point), and finishing with systems of bodies - the mechanical model with the highest degree of complexity.

From the point of view of classical mechanics models [7], statistics has two parts: statistics of material points and statistics of rigid solid bodies, which extends in the case of body systems as well.

In statistics, three categories of problems occur [4], [8], which differ especially by the modality of mathematical solving:

a) *direct problem,* in which the position of equilibrium of the studied material system is supposed to be known, and in its solving, we focus on determining the forces under the action of which the equilibrium is reached;

b) *inverse problem*, which lies on determining the position of equilibrium of the material system analyzed, when the forces acting on it are known;

c) *mixed problem*, looking for finding all the unknown elements referring both to the position of equilibrium, and to the forces that concur to it beingcarried out, in the situation in which information is known both regarding the position of equilibrium and regarding the forces.

The paper will analyze examples of the equilibrium of rigid bodies in a plane, in the solving of which cases from the category of the mixed problem are met.

In the solving of these examples, mechanical and geometrical methods will be applied, which will be succinctly described below [11].

2 Mechanical Considerations

As in the case of bound material point, in order to study the equilibrium of the rigid body submitted to bonds, the axiom of bounds is used, based on which the bond is suppressed and replaced by corresponding mechanical elements (forces or moments (reactions) [5], [10]. After all the bonds to which a solid rigid body is submitted are suppressed, directly applied exterior forces and moments, as well as bonding forces and moments, act on those [14].

Reduction torsor inpoint *O* (considered the origin of the reference system) of exterior forces is $\tau_O\left(\bar{R},\bar{M}_O\right)$, and the reduction torsor *O*of the bonding forces is $\tau'_O\left(\bar{R}',\bar{M}'_O\right)$.

In this case, vectorial conditions of equilibrium will be:

$$\bar{R}+\bar{R}'=0,\quad \bar{M}_O+\bar{M}'_O=0, \tag{1}$$

These two vectorial equations are equivalent with the scalar ones of equilibrium given by equations (3), valid, when equations (2) are considered [11]:

$$\begin{aligned}&\bar{R}=X\cdot\bar{i}+Y\cdot\bar{j}+Z\cdot\bar{k}\\&\bar{M}_O=M_{Ox}\cdot\bar{i}+M_{Oy}\cdot\bar{j}+M_{Oz}\cdot\bar{k}\\&\bar{R}'=X'\cdot\bar{i}+Y'\cdot\bar{j}+Z'\cdot\bar{k}\\&\bar{M}_O'=M'_{Ox}\cdot\bar{i}+M'_{Oy}\cdot\bar{j}+M'_{Oz}\cdot\bar{k}\end{aligned}, \tag{2}$$

$$\begin{cases}X+X'=0\\Y+Y'=0\\Z+Z'=0\end{cases};\quad \begin{cases}M_{Ox}+M'_{Ox}=0\\M_{Oy}+M'_{Oy}=0\\M_{Oz}+M'_{Oz}=0\end{cases}, \tag{3}$$

The unknown elements referring both to the equilibrium position and to the forces concurring to its being reached, are determined by solving the scalar equations, which in the case of rigid bodies in a plane are given by the equations:

$$\begin{cases}X+X'=0\\Y+Y'=0\\M_{Oz}+M'_{Oz}=0\end{cases}, \tag{4}$$

the solid rigid body being required by a system of coplanar forces.

The bonds to which a solid rigid body can be subjected are: simple support, joint, framing and fastening with wires [3].

The rigid body can be simultaneously submitted to several bonds. In such a case, the rigid solid body is released from its bonds, and mechanical elements are introduced that replace each mechanically equivalent bond. The number of scalar unknowns introduced by bonds is then estimated to see whether the problem is statically determined or not.

If the number of scalar unknowns is equal to the number of scalar equations of equilibrium, then the problem is statically determined. If the number of scalar unknowns is greater than the number of scalar equilibrium equations, then the problem is statically indeterminate [3].

The requirement for the number of scalar unknowns to be equal to the number of scalar equations of equilibrium is necessary, but not sufficient, since it is possible for the system of equations to be undetermined, although the number of scalar equations of equilibrium is equal to the number of unknowns. To study this, the conditions of equilibrium are then applied, as in the case of the free rigid solid body.

In the case of the equilibrium of the rigid solid body submitted to friction bond, the solving of the applications will be done by completing the equations of equilibrium with inequalities between the components of the bonding forces torsor, as the case may be.

In the case of sliding friction [13], [15], the equation intervenes between the friction force in sliding $\bar{T}$ (which in a certain range hinders the movement of the body), and the normal reaction $\bar{N}$:

$$|T|\le\mu\cdot|N|, \tag{5}$$

in which the number of contact points (support), in which the sliding friction takes place, is taken in consideration.

The proportionality factor that intervenes in equation (5), μ, is called sliding friction coefficient, being a scalar value, positive, sub unitary, non-dimensional, and experimentally determined.

Rolling friction originates in the rolling of a body on a rolling surface, when in the contact area or point, the sliding friction force (rest) is higher than the tangential accelerating force (the respective body will roll with no sliding) [17-19].

The rolling friction is materialized in the form of the sliding moment in rolling $\bar{M}_r$, which, in a certain range, is opposed to the rotation tendency of a body around an axis.

Bringing back the normal reaction in the contactpoint, it results, that the effect exerted by this in a point, is expressed by force $\bar{N}$ (applied in the contact point), and by the friction moment in rolling $\bar{M}_r$:

$$\left|\bar{M}_r\right| £\ s\left|\bar{N}\right|, \tag{6}$$

where *s* is the friction coefficient in rolling, a value measured in units of length [17].

In the examples that will be shown and analyzed, except the sliding friction, no other friction occurs (rolling friction and also friction in joints and bearings, which has not been mentioned) [12].

3 Geometrical considerations

The use of concurrency of lines in a plane, namely of the condition of concurrency of three lines in a plane, has been motivated, among other things, also by the fact that the forces acting on a rigid body in a plane (directly applied and bonding), are coplanar [9], [16].

In order to apply the concurrency condition of three lines in a plane, there should be possible to write the support lines equations (as to the reference system chosen for the solving of the problem), of three vector forces, that act on the solid rigid body in a plane [4].

In the case of equilibrium with sliding friction, the components, normal $\bar{N}$ and tangential $\bar{T}$ of the bonding reaction (force), will be replaced by their resultant, considering the equilibrium at its limit. This is done to diminish the number of unknowns and to be able to know the slope (angular coefficient) of the support line that enters in the calculation of the problem solving. For the case in which more than one active force act upon the solid body (for example its weight), then the respective forces will also be replaced with their resultant. The resultant's support is the equation of the central axis, obtained as a result of the reduction of these (directly applied) active forces [6], [20].

There are also cases of exception, where the lines of the forces acting are parallel (thus they will not intersect), for the equilibrium the condition is required for the supports to coincide (if the lines coincide, then their intersection resides in all the point found on those) [12].

The lines in a Cartesian plane can be algebraically defined by linear equations and functions. In the bi-dimensional case (line in a plane), the most frequently used form is the equation of the line where the dependent variable (y, here) is expressed "function of" the independent variable (x, here); to the line in a plane, a first-degree equation corresponds, of the form:

$$y = mx + n, \tag{7}$$

Where, $m=tg\alpha$represents the slope of the line or the angular coefficient of the line (that is, the value of the tangent function of the angle between the line and the and the positive sense of the abscise (horizontal axis, Ox), n the y-axis cut (ordinate at origin (distance measured on the vertical axis, Oy, between the point of intersection of the line with Oy axis, and the origin of the system of coordinate)), and x is the independent variable. The angular coefficient of the line is the tangent of angle α, made by the line with axis Ox. Angle α takes values in the range of 0 and π, $\alpha \in [0, \pi)$.

Equation (7) is called the reduced equation of the line (or implicit equation).

If $\alpha \in (0, \pi/2)$ then the slope is positive, for $\alpha \in (\pi/2, \pi)$ the slope is negative.

If the line is parallel to x-axis, the slope is null, $m = 0$, thus the equation of the line is reduced to$y = n$.

If the line goes through the origin O, $n = 0$. The equation of the line that passes through the origin is $y = mx$, $m = tg\ \alpha$.

If the line is parallel with the y-axis, $\alpha = \pi/2$și$m = tg\ \pi/2 = \infty$. On such a line, all the points have the same abscise.

Similarly, the line can also be represented by the most general form of a first degree equation in $x$and y:

$$Ax + By + C = 0, \quad A,B,C \in \mathbb{R}, \tag{8}$$

with $A \neq 0$ or $B \neq 0$

It represents a line:

$$y = -\frac{A}{B}x - \frac{C}{B}, \quad B \neq 0, \tag{9}$$

In this case: $m = -A/B, \quad n = -C/B$.

The equation of the line that passes through a given point $M_0(x_0, y_0)$, and slope m is:

$$y - y_0 = m(x - x_0), \tag{10}$$

The equation of the line that goes through two different points $A(x_A, y_A)$, $B(x_B, y_B)$ is:

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A}, \tag{11}$$

if the denominators are not null.

The line can also be determined by a point and the direction vector, determined by a point and the normal vector to the line.

The problems regarding concurrency of lines, as well as collinearity problems of certain points, are truths that can be generally easily inferred, but their rigorous demonstration requires accurate reasoning and a wide range of specific techniques. In the first stage, their solving is based on finding the intersection point of two lines, after which, depending on the data of the problem, we shall demonstrate that a third line passes through this point. The point found will be the point of concurrency of the given lines.

Out of the most frequently used methods of solving of this type of problems, we mention:

a) Demonstration of concurrency by reducing to a problem of collinearity;

b) Demonstration of concurrency by reducing to known concurrencies;

c) Demonstration of concurrency by showing that the intersection point of two of the three lines meets a characteristic property of the points belonging to the third line. In other words, we show that the intersection points of two of the three lines belong to the locus made up of the points of the third line.

d) Demonstration of the concurrency of three lines d_1, d_2, d_3, showing that d_1 and d_2, and d_1 and d_3, respectively, are concurrent and their concurrency points coincide;

e) Demonstration that the three lines verify the conditions of the hypothesis of a theorem (direct or reciprocal), the conclusion of which leads to their concurrency;

f) Use of the definition of concurrent lines, namely, we show that there is a common point for the lines;

g) In order to demonstrate the concurrency of three lines, we can use the theorems referring to the concurrency of the important lines in a triangle;

h) Use of the reciprocal of Ceva's theorem;

i) For the concurrency of three lines, we demonstrate that two by two intersect and the area of the polygon obtained is null;

j) Demonstration of the concurrency of three lines, using the reciprocal of Carnot's theorem;

k) Demonstration of concurrency with the help of complex numbers;

l) Demonstrationof concurrency of three lines, with the help of geometrical transformations;

m) Demonstrationof concurrency of three lines by the vectorial method;

n) Demonstration of concurrency of three lines by analytical method (with the help of coordinates), using analytical equations of lines;

This method means showing that being given three lines by their general equations:

$$\begin{aligned} &Ax + By + C = 0 \\ &A'x + B'y + C' = 0 \\ &A''x + B''y + C'' = 0 \end{aligned} \quad , \qquad (12)$$

for these lines to be concurrent, there should be a point $M(x_0,y_0)$, which would verify these three equations. This means that the system of three equations with three unknowns from below should be compatibly determined.

$$\begin{aligned} &Ax_0 + By_0 + C = 0 \\ &A'x_0 + B'y_0 + C' = 0 \\ &A''x_0 + B''y_0 + C'' = 0 \end{aligned} \quad , \qquad (13)$$

For this, the necessary and sufficient conditions are:

a) A second-order determinant should exist, made up with the coefficients of x and y, thus one of the determinants should be other than zero.

$$\begin{vmatrix} A & B \\ A' & B' \end{vmatrix} \neq 0; \quad \begin{vmatrix} A & B \\ A'' & B'' \end{vmatrix} \neq 0; \quad \begin{vmatrix} A' & B' \\ A'' & B'' \end{vmatrix} \neq 0, \qquad (14)$$

b) the determinant of the system should be null:

$$\begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix} = 0, \qquad (15)$$

By condition *a)* we expressed analytically that two lines are concurrent.

Condition *b)* shows that the third line passes through the point of intersection of the first two.

Observation

The criterion reflects the property that the lines are concurrent if, and only if the system of linear equations is compatibly undetermined, that is, the characteristic determinant of the system is null.

In the following examples, the application of the concurrency condition of the lines in a plane, is done according to the requirements of each example in part.

Next, we shall present examples the solving of which will be done according to the requirements, both from mechanical considerations, and geometrical ones.

4 Applications solved by mechanical and geometrical considerations

The following types of examples will be solved by mechanical and geometrical considerations.

The reaction, where appropriate, will be determined only by mechanical considerations, and the other unknowns, where appropriate, will be determined by both considerations. Examples of the equilibrium of rigid bodies in a plane, submitted to bonds without friction [11] and to bonds with friction will be approached [12].

4.1 Rigid body submitted to bonds without friction

We shall consider the following example:

Homogeneous *AB* bar, *G* weight and *2l* length is leaned in *A* to a vertical wall, and in *D* to the edge of another wall, at a distance of *a* from the first (Fig. 1). The contact is without friction. Let us determine the angle θ of the bar with the horizontal surface, in its resting position, and also the reactions from supports *A* and *D* (by mechanical consideration).

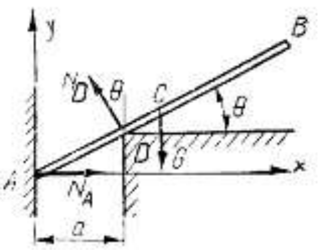

Fig.1

Solving

1) Mechanical solving

The bar is released from its bonds by introducing reactions in resting points *A* and *D*, each reaction having a normal direction at the surface which does not have singular point in the contact point.

To get as simple as possible projection equations, simple, *xAy* reference system is selected, so that, as far as possible, as many forces as possible of their true size would be projected. Scalar equations of equilibrium in respect to the reference system chosen in this case, are [17]:

$$\begin{aligned} X &\equiv N_A - N_D \sin\theta = 0 \\ Y &\equiv N_D \cos\theta - G = 0 \\ M_A &\equiv N_D \frac{a}{\cos\theta} - Gl\cos\theta = 0 \end{aligned}, \quad (16)$$

We solve the system (16):

$$\left.\begin{aligned} & N_A - N_D \sin\theta = 0 \\ & N_D \cos\theta - G = 0 \;\; \Rightarrow N_D = \frac{G}{\cos\theta} \end{aligned}\right\} \Rightarrow$$

$$\begin{aligned} & \Rightarrow N_A = G\frac{\sin\theta}{\cos\theta} = Gtg\theta \\ & \Rightarrow \frac{G}{\cos\theta}\cdot\frac{a}{\cos\theta} - Gl\cos\theta = 0 \left| \cdot \frac{\cos^2\theta}{G} \right. \Rightarrow \\ & \Rightarrow \cos\theta = \sqrt[3]{\frac{a}{l}} \Rightarrow \cos^3\theta = \frac{a}{l} \end{aligned}, \quad (17)$$

So that the equilibrium would be possible: $\sqrt[3]{a/l} \le 1$, $a \le l$. The reactions:

$$N_A = G\sqrt{\left(\frac{l}{a}\right)^{\frac{2}{3}} - 1}, \quad N_D = G\sqrt[3]{\frac{l}{a}}, \quad (18)$$

2) Geometrical solving to determine angle θ

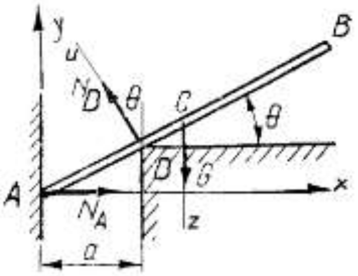

Fig.2

For the problem in Fig. 1, the equations of the lines (Fig. 2) are [11]:

$$(Ax): y = 0, \quad (19)$$

$$\begin{aligned} & (Du): y - atg\theta = tg\left(90^0 + \theta\right)(x - a) \Rightarrow \\ & \Rightarrow y = -(x - a)ctg\theta + atg\theta \end{aligned}, \quad (20)$$

$$(Cz): x = l\cos\theta, \quad (21)$$

The condition stipulated is that the three lines to be concurrent. We remove *y* from the first equations and calculations are made:

$$\begin{aligned} & -(x - a)ctg\theta + atg\theta \Rightarrow (x - a)ctg\theta = atg\theta \Rightarrow \\ & \Rightarrow x = a\frac{tg\theta + ctg\theta}{ctg\theta} \end{aligned}, (22)$$

We substitute *x* with its expression of (22) in (21), and get:

$$\begin{aligned} & l\cos\theta = a\frac{tg\theta + ctg\theta}{ctg\theta} \Rightarrow \\ & \Rightarrow l\cos\theta\frac{\cos\theta}{\sin\theta} = a\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right) \Rightarrow \\ & \Rightarrow l\frac{\cos^2\theta}{\sin\theta} = a\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}\left| \cdot \sin\theta \Rightarrow \right., \\ & \Rightarrow l\cos^2\theta = \frac{a}{\cos\theta} \Rightarrow \cos^3\theta = \frac{a}{l} \Rightarrow \\ & \Rightarrow \cos\theta = \sqrt[3]{\frac{a}{l}} \end{aligned} \quad (23)$$

In the next example the active forces should be replaced by their resultant.

A homogeneous hemisphere, of *G* weight and *R* radius is put with its convex part on a horizontal plane. In point *B* of the hemisphere, *Q* weight is appended. Angle *φ*, made in resting position by the symmetry axis of the hemisphere with the vertical (Fig. 3) will be determined [12].

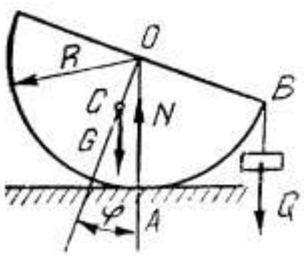

Fig.3

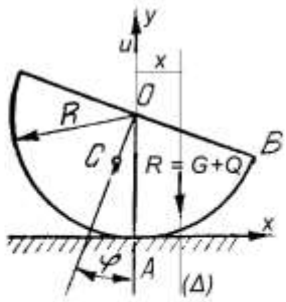

Fig.4

Solving

1) Mechanical solving

Support *A* is replaced with the normal bonding force *N*. Out of the scalar equilibrium equations only the sum of moments is used, where only the required angle *φ*appears as unknown.

$$M_A \equiv G\cdot OC\sin\varphi - Q\cdot R\cos\varphi = 0, \quad (24)$$

where $OC = 3R/8$ (see centers of gravity).

The result is:

$$tg\varphi = \frac{8Q}{3G}, \quad (25)$$

2) Geometrical solving to determine angle φ

Active forces *G* and *Q* that are parallel with *Ay* are replaced with their resultant, situated on axis *(Δ)*, at a distance to *Ay* equal with:

$$x = \frac{-\frac{3R}{8}\sin\varphi\cdot G + R\cos\varphi\cdot Q}{G + Q}, \quad (26)$$

For the problem in Fig. 3, the equations of lines (Fig. 4) in respect of the reference system *xAy* are:

$$(Au): x = 0, \quad (27)$$

$$(\Delta): x = \frac{-\frac{3R}{8}\sin\varphi \cdot G + R\cos\varphi \cdot Q}{G+Q}, \quad (28)$$

We eliminate$x$from equations (27) and (28), and calculate:

$$-\frac{3R}{8}\sin\varphi \cdot G + R\cos\varphi \cdot Q = 0 \Big| \cdot \frac{1}{R} \Rightarrow$$

$$\Rightarrow \frac{3R}{8}\sin\varphi \cdot G = R\cos\varphi \cdot Q \Big| \cdot \frac{1}{\cos\varphi} \Rightarrow, \quad (29)$$

$$\Rightarrow tg\varphi = \frac{Q}{\frac{3G}{8}} \Rightarrow tg\varphi = \frac{8Q}{3G}$$

In the following example, the support lines of the acting forces are parallel (thus they will not intersect). To ensure equilibrium, the conditionhas to be met for the supports to coincide(if the lines coincide, then their intersection consists in all the points found on those). Let us determine weight P for the equilibrium of the bar, with G weight, l length, inclined with α angle as to the horizontal, which leans without friction in point B (Fig. 5) [12].

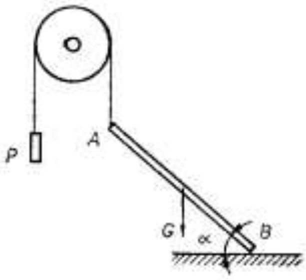

Fig.5

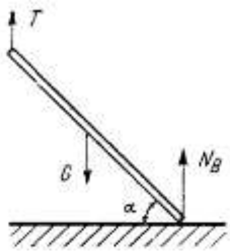

Fig.6

Solving

1) Mechanical solving

From equations (30) and (31) of static equilibrium (Fig. 5), the result is (32):

$$Y \equiv T + N_B = 0, \quad (30)$$

$$M_B \equiv -T \cdot l\cos\alpha + Gl/2\cos\alpha = 0 \quad (31)$$

$$P = T = N_B = G/2, \quad (32)$$

2) Geometrical solving for the calculation of P

To have equilibrium, the supports of the resultants of active forces and of the bonding ones (Fig. 6), should coincide. Thus, we have:

$$l\cos\alpha = \frac{(-Gl/2)\cos\alpha}{T-G} \Rightarrow P = T = \frac{G}{2}, \quad (33)$$

In the following example, reduction of forces and determination of the central axis will be done both for the active forces, and for the bonding forces.

Let bar AB, of weight G, length l, supported in $C(AC = l/8)$ by a vertical cord, and in B by a cord passing through a ring (Fig. 7). Determine $d=DB$ where a weight $2G$ should be put, so that the tensions in the vertical cord and in the one passing through the ring be equal [15].

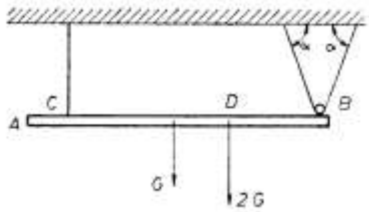

Fig.7

Solving

1) Mechanical solving

Static equilibrium equations are written (Fig. 8):

$$X \equiv -T\cos\alpha + T\cos\alpha = 0 \Rightarrow 0 \equiv 0, \quad (34)$$

$$Y \equiv 2T\sin\alpha + T - G - 2G = 0, \quad (35)$$

$$M_A \equiv T\frac{l}{8} + 2Tl\sin\alpha - G\frac{l}{2} + 2G(l-d) = 0 \quad (36)$$

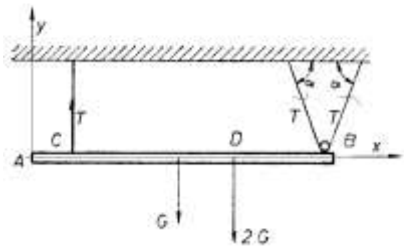

Fig.8

(35) gives:

$$T(1+2\sin\alpha) = 3G \Rightarrow T = \frac{3G}{1+2\sin\alpha}, \quad (37)$$

and from (36), and considering the value of T from equation (37), gives:

$$T\left(\frac{l}{8} + 2l\sin\alpha\right) = G\frac{l}{2} + 2Gl - 2Gd \Rightarrow$$

$$\Rightarrow \frac{3G}{1+2\sin\alpha}\left(\frac{l}{8} + 2l\sin\alpha\right) =$$

$$= \frac{5Gl}{2} - 2Gd \Big| \cdot \frac{1}{G} \Rightarrow$$

$$\Rightarrow \frac{5l}{2} - \frac{3}{1+2\sin\alpha}\left(\frac{l}{8} + 2l\sin\alpha\right) = 2d \Rightarrow \quad , \quad (38)$$

$$\Rightarrow \frac{5l}{2} - \frac{3}{8\cdot(1+2\sin\alpha)} - \frac{6l\sin\alpha}{1+2\sin\alpha} = 2d \Rightarrow$$

$$\Rightarrow \frac{20l + 40l\sin\alpha - 3l - 48l\sin\alpha}{8\cdot(1+2\sin\alpha)} = 2d \Rightarrow$$

$$\Rightarrow d = \frac{17 - 8\sin\alpha}{16\cdot(1+2\sin\alpha)} l$$

2) Geometrical solving for the calculation of d

In order to have equilibrium, the supports of the resultants of active forces, and of the bonding ones, should coincide (if the lines coincide, then their intersection consists in all the points found on them). Thus, we have:

The equation of the central axis for the resultant of active forces G and $2G$ is [12]:

$$(\Delta_C): x = \frac{\frac{5l}{2} - 2d}{3}, \tag{39}$$

and the equation of the central axis for the resultant of the forces related to the vertical cord and from the one that passes through the ring is:

$$(\Delta'_C): x = \frac{\frac{l}{8} + 2l\sin\alpha}{1 + 2\sin\alpha}, \tag{40}$$

The equality of the equations (39) and (40) gives:

$$\begin{aligned}
&\frac{\frac{5l}{2} - 2d}{3} = \frac{\frac{l}{8} + 2l\sin\alpha}{1 + 2\sin\alpha} \Rightarrow \\
&\Rightarrow \frac{3l}{8} + 6l\sin\alpha = \\
&= \frac{5l}{2}\left(1 + 2\sin\alpha\right) - 2d\left(1 + 2\sin\alpha\right) \Rightarrow \\
&\Rightarrow 2d\left(1 + 2\sin\alpha\right) = \\
&= \frac{5l}{2} + \frac{10l\sin\alpha}{2} - \frac{3l}{8} - 6l\sin\alpha \Rightarrow \\
&\Rightarrow 2d\left(1 + 2\sin\alpha\right) = \frac{17l}{8} - \frac{8l\sin\alpha}{8} \Rightarrow \\
&\Rightarrow d = \frac{17 - 8\sin\alpha}{16\cdot\left(1 + 2\sin\alpha\right)} l
\end{aligned} \tag{41}$$

Next, we shall consider a system of bodies in a plane:

Bar AB, of weight G, length l is propped without friction to a hemisphere of radius R and to a vertical wall (Fig. 9). Determine angle θ, angle α being known [15].

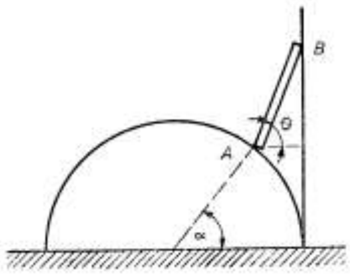

Fig.9

Solving

1) Mechanical solving

The bodies are separated. On the bar the following forces act: G - weight of the bar; N_B - normal reaction of the wall; N_A - reaction of the hemisphere (Fig. 10).

The equilibrium conditions are:

$$X \equiv -N_B + N_A\cos\alpha = 0, \tag{42}$$

$$Y \equiv N_A\sin\alpha - G = 0, \tag{43}$$

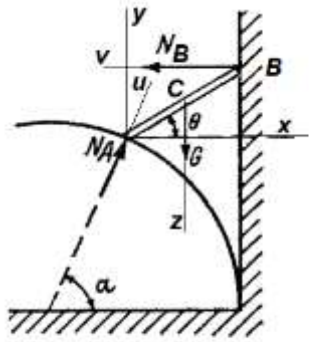

Fig.10

$$M_A \equiv N_B l\sin\theta - G\frac{l}{2}\cos\theta = 0, \tag{44}$$

From equations (43) and (42) it results:

$$N_A\sin\alpha = \frac{G}{\sin\alpha}, \quad N_B = N_A\cos\alpha = Gctg\alpha, \tag{45}$$

(44) gives:

$$\cancel{G}ctg\alpha l\sin\theta = \cancel{G}\frac{l}{2}\cos\theta \Rightarrow tg\theta = \frac{1}{2}tg\alpha, \tag{46}$$

2) Geometrical solving to determine angle θ

For the problem in Fig. 9, the equations of the lines as to system xAy (Fig. 10) are:

$$(Au): y = xtg\alpha, \tag{47}$$

$$(Bv): y = l\sin\theta, \tag{48}$$

$$(Cz): x = \frac{l}{2}\cos\theta, \tag{49}$$

Equations (47), (48) and (49) give:

$$\frac{l}{2}\cos\theta tg\alpha = l\sin\theta \left| \cdot \frac{1}{l\cos\theta} \right. \Rightarrow tg\theta = \frac{1}{2}tg\alpha, \tag{50}$$

4.2 Rigid body submitted to bonds with friction

In the following examples the rigid body is also submitted to friction bonds as well, or only to friction bonds. In this example, we shall consider only one of the bonds with friction. We mention that friction is sliding friction and it is considered at the limit of equilibrium. We remind you that in the case of the equilibrium at the limit, normal components $\overline{T}$, and tangential component $\overline{N}$ of the bonding force (reaction) will be replaced with their resultant [12].

A homogeneous bar AB, weight G and length l, is propped against the inside surface of a cylinder (μ friction coefficient), and in point O it leans without friction to a simple support. Knowing that $R<l<2R$, where R is the radius of the cylinder, let us determine the reactions and angle α, made by the bar with the vertical, in position of rest (Fig. 11) [10].

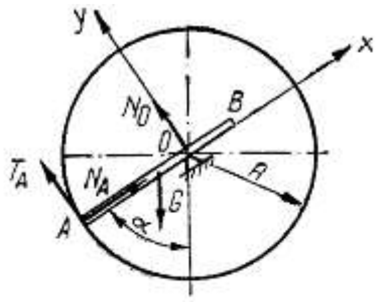

Fig.11

Solving

1) Mechanical solving

The bonds are removed from the bar, the corresponding reactions being entered, and we write the scalar equations of equilibrium in the *xOy* system of reference chosen in the figure.

$$X \equiv N_A - G\cos\alpha = 0$$
$$Y \equiv T_A - G\sin\alpha + N_O = 0 \ , \qquad (51)$$
$$M_A \equiv N_O R - G\frac{l}{2}\sin\alpha = 0$$

The friction force at the limit is $T_A = \mu N_A$.

Solving thesystem of equations (51), where the expression of the friction force is also taken in consideration, we get:

$$N_A = G\frac{2R-l}{\sqrt{(2R-l)^2 + 4\mu^2 R^2}}$$
$$T_A = G\frac{\mu(2R-l)}{\sqrt{(2R-l)^2 + 4\mu^2 R^2}} \ , \qquad (52)$$
$$N_O = G\frac{\mu l}{\sqrt{(2R-l)^2 + 4\mu^2 R^2}}$$
$$tg\alpha = \mu\frac{2R}{2R-l}$$

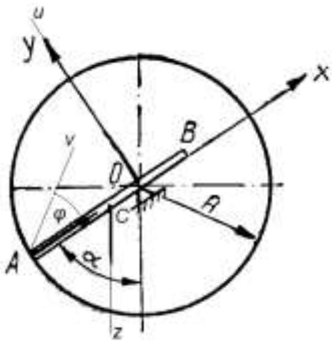

Fig.12

2) Geometrical solving to determine α angle.

For the problem in Fig. 11, the equations of the lines (Fig. 12) in relation with *xOy* reference system are:

$$(Av): y = tg\varphi \cdot (x - R), \qquad (53)$$
$$(Cz): y = -tg\left(180^0 + \theta\right)\left[x - (R - l/2)\right], \qquad (54)$$
$$(Ou): x = 0\ , \qquad (55)$$

The normal reaction N_A of the cylindrical support surface applied in *A* and the friction force in a *A*, T_A as tangential force with the sense opposed to the sliding tendency, have been replaced by their resultant, so that by replacing the two forces with their resultant, we got to apply the concurrency condition of three lines.

We eliminate *y* in equations (53), (54) and replace *x* with its expression in (55), and calculate:

$$tg\varphi \cdot (x - R) = -tg(\pi + \theta)\left[x - \left(R - \frac{l}{2}\right)\right] \Rightarrow$$
$$\Rightarrow tg\varphi \cdot R = tg\theta \cdot \left(R - \frac{l}{2}\right) \Rightarrow$$
$$\Rightarrow tg\theta = tg\varphi\frac{R}{R - \frac{l}{2}} \Rightarrow \ , \qquad (56)$$
$$\Rightarrow tg\theta = \mu\frac{2R}{2R-l}$$

In the following example, all bonds are with sliding friction [12].

A homogeneous *AB* ladder, *G* weight, and *l* length is leaned, with friction in *A* on horizontal plane (μ_1 coefficient), against a vertical *B* wall (μ_2 coefficient). Let us determine *θ* angle made by the ladder with the vertical surface in position of rest, as well as the reaction (Fig. 13).

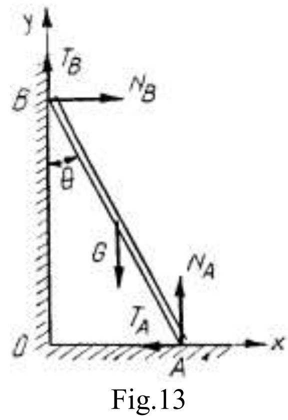

Fig.13

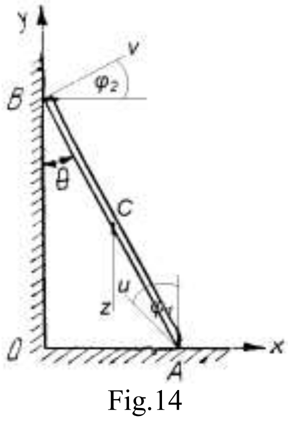

Fig.14

Solving

1) Mechanical solving

The ladder links are removed, the adequate reactions being inserted. The scalar equilibrium equations are written:

$$X \equiv N_B - T_A = 0$$
$$Y \equiv T_B - G + N_A = 0 \ , \qquad (57)$$
$$M_O \equiv N_A l\sin\theta - G\frac{l}{2}\sin\theta - N_B l\cos\theta = 0$$

Equations $T_A = \mu_1 N_A,\ T_A = \mu_2 N_B$ are added.

Solving system (57) and considering the two equations, we get:

$$N_A = \frac{G}{1+\mu_1\mu_2},\quad N_B = \frac{\mu_1 G}{1+\mu_1\mu_2},\quad tg\theta = \frac{2\mu_1}{1-\mu_1\mu_2} \tag{58}$$

2) Geometrical solving to determine angle θ

For the problem in Fig. 13, the equation of lines (Fig. 14) in respect to the *xOy* reference system, are:

$$(Au): y = -tg\left(90^0 + \varphi_1\right)\cdot\left(x - l\sin\theta\right), \tag{59}$$

$$(Cz): x = \frac{l}{2}\sin\theta, \tag{60}$$

$$(Bv): y - l\cos\theta = xtg\varphi_2, \tag{61}$$

The normal reaction N_A of the horizontal support surface applied in *A*, and the friction force in *A*, T_A as tangential force in the opposite sense to the sliding tendency, and the normal reaction N_B of the vertical support surface in *B*, and the friction force in *B*, T_B as tangential force in the opposite sense of the sliding tendency, have been replaced by their resultants, and thus, by the replacement of the four forces with their resultants, the concurrency condition of three lines in a plane has been reached.

We remove *y* from equations (59), (61), and replacing *x* with its expression from (60), the following calculations are made:

$$\begin{aligned} & -ctg\varphi_1\cdot\left(x - l\sin\theta\right) = tg\varphi_2 x + l\cos\theta \Rightarrow \\ & \Rightarrow -ctg\varphi_1\cdot\frac{l}{2}\sin\theta + ctg\varphi_1\cdot l\sin\theta = \\ & = tg\varphi_2\cdot\frac{l}{2}\sin\theta + l\cos\theta \left| \frac{2}{l\cos\theta} \right. \Rightarrow \\ & \Rightarrow ctg\varphi_1\cdot tg\theta = tg\varphi_2\cdot tg\theta + 2 \Rightarrow \\ & \Rightarrow tg\theta\left(\frac{1}{tg\varphi_1} - tg\varphi_2\right) = 2 \Rightarrow \\ & \Rightarrow tg\theta\left(\frac{1}{\mu_1} - \mu_2\right) = 2 \Rightarrow tg\theta\frac{1-\mu_1\mu_2}{\mu_1} = 2 \Rightarrow \\ & \Rightarrow tg\theta = \frac{2\mu_1}{1-\mu_1\mu_2} \end{aligned} \tag{62}$$

In the next example there are two friction bonds and two directly applied forces that have to be replaced with their resultant [12].

A homogeneous bar rests by two symmetrical supports as to the weight center *C* (Fig. 15) on a horizontal plane. The weight of the bar is *P*, and the friction coefficients μ_1 and μ_2. Let us determine the maximum value of the horizontal force *Q*, for which the bar stays in equilibrium.

Solving

1) Mechanical solving

The scalar equations of the equilibrium of the bar are:

$$\begin{aligned} & X \equiv Q - F_1 - F_2 = 0 \\ & Y \equiv N_1 + N_2 - P = 0 \\ & M_B \equiv -Qa + \frac{Pb}{2} - N_1 b = 0 \end{aligned} \tag{63}$$

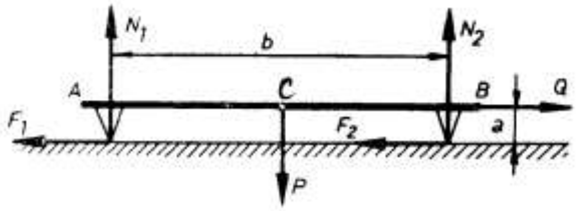

Fig.15

Since the maximum value of force *Q* is required, for which the bar stays in equilibrium, the friction forces are maximum, that is $F_1 = \mu_1 N_1$, $F_2 = \mu_2 N_2$.

N_1 results from the third equation of the system (63):

$$\begin{aligned} & -Qa + \frac{Pb}{2} - N_1 b = 0 \Rightarrow N_1 = \frac{\frac{Pb}{2} - Qa}{b} \Rightarrow \\ & \Rightarrow N_1 = \frac{P}{2} - Q\frac{a}{b} \end{aligned} \tag{64}$$

From the second equation of the system (63), N_2 is determined:

$$\begin{aligned} & N_1 + N_2 - P = 0 \Rightarrow N_2 = P - N_1 \Rightarrow \\ & \Rightarrow N_2 = P - \frac{P}{2} + Q\frac{a}{b} \Rightarrow N_2 = \frac{P}{2} + Q\frac{a}{b} \end{aligned} \tag{65}$$

Replacing the values of N_1 and N_2 in the first equation of the system (63), *Q* is determined:

$$\begin{aligned} & Q - F_1 - F_2 = 0 \Rightarrow Q = \mu_1 N_1 + \mu_2 N_2 \Rightarrow \\ & \Rightarrow Q = \frac{\mu_1 P}{2} - Q\frac{\mu_1 a}{b} + \frac{\mu_2 P}{2} + Q\frac{\mu_2 a}{b} \left| \cdot 2b \right. \Rightarrow \\ & \Rightarrow Q\cdot 2\left[b + \left(\mu_1 - \mu_2\right)a\right] = Pb\left(\mu_1 + \mu_2\right) \Rightarrow \\ & \Rightarrow Q = \frac{b\left(\mu_1 + \mu_2\right)}{2\left[b + \left(\mu_1 - \mu_2\right)a\right]}P \end{aligned} \tag{66}$$

2) Geometrical solving to determine weight Q

Normal reactions N_1, N_2 of the horizontal support surface in *A*and*B*and the friction forces *A* and *B*, F_1, F_2 as tangential forces, the sense being inverse to the sliding tendency, will be replaced by their resultant.

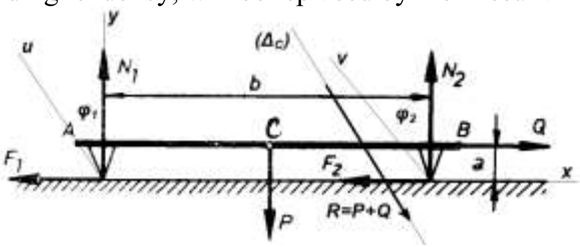

Fig.16

Similarly, the system made up of the weight forces P and Q is reduced to their resultant, and we shall determine the central axis equation on which their resultant is found.

Thus, as to system xAy (Fig.16), we have:

$$\bar{Q} = Q\bar{i}, \bar{P} = -P\bar{j}, \quad \bar{R} = \bar{Q} + \bar{P} = Q\bar{i} - P\bar{j}, \quad \bar{M}_A = -\left(Pb/2 + Qa\right)\bar{k} \tag{67}$$

The equation of the central axis is:

$$\left(\Delta_C\right): -xP - yQ = -P\frac{b}{2} - Qa, \Rightarrow \Rightarrow \left(\Delta_C\right): y = \frac{P\frac{b}{2} + Qa - xP}{Q}, \tag{68}$$

For the problem in Fig. 15, the equations of lines (Fig. 16) are:

$$\left(Au\right): y = xtg\left(90^0 + \varphi_1\right) = -xctg\varphi_1, \tag{69}$$

$$\left(Bv\right): y = tg\left(90^0 + \varphi_2\right)\left(x - b\right) \Rightarrow \Rightarrow y = -ctg\varphi_2\left(x - b\right), \tag{70}$$

$$\left(\Delta_C\right): y = \frac{P\frac{b}{2} + Qa - xP}{Q}, \tag{71}$$

Solving the system made up of equations (69) and (70) gives:

$$x = \frac{\mu_1 b}{\mu_1 - \mu_2}, \quad y = -\frac{b}{\mu_1 - \mu_2}, \tag{72}$$

Substituting the values obtained for x and y, equations (72), in (71), gives:

$$-\frac{b}{\mu_1 - \mu_2} = \frac{P\frac{b}{2} + Qa - \frac{\mu_1 b}{\mu_1 - \mu_2}P}{Q} \Rightarrow \Rightarrow -Qb = P\frac{b}{2}\left(\mu_1 - \mu_2\right) + +Qa\left(\mu_1 - \mu_2\right) - \mu_1 bP \Rightarrow \Rightarrow Q\left[b + a\left(\mu_1 - \mu_2\right)\right] = P\frac{b}{2}\left(\mu_1 + \mu_2\right) \Rightarrow \Rightarrow Q = \frac{b\left(\mu_1 + \mu_2\right)}{2\left[b + a\left(\mu_1 - \mu_2\right)\right]}P, \tag{73}$$

5 Conclusion

Correlation of knowledge from various disciplines substantially contribute to education, training and development of flexibility in the thinking process, of abilities of implementing knowledge in practice, it significantly helps in fixing and systematizing knowledge, one discipline helping the other to be better acquired.

Examples of problems in the equilibrium of rigid bodies have been considered, submitted to frictionless bonds, solved by mechanical and geometrical considerations, namely with the condition of concurrency of three lines in a plane.

The concept of interdisciplinarity is a component of the training process, by which active and formative aspects can be provided, of directing learning, and it has gained field more and more in the approach of modern teaching.

By the examples approached, it is mentioned that determined unknowns by geometrical considerations can be distances (lengths), angles, friction forces or coefficients. No reactions (bonding forces) are determined.

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It is an optional section where the authors may write a short text on what should be acknowledged regarding their manuscript.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Itu Razvan Bogdan took care of the mechanical aspects of the article choosing the right examples, compatible with the geometric aspects.
Toderas Mihaela realized the geometrical aspects related to the theme of the article and the geometrical solution of the problems chosen as examples.