Analytical investigations of plane wave propagation across a single tuned mass damper

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Abstract: The problem of wave propagation in elastic beams is presented in a general fashion. The role of tuned mass damper is also shown. In particular we will calculate reflection and transmission coefficients across a tuned mass damper and show their frequency dependence. The same analysis will be extended to the case of an impact tuned mass damper. In particular we will show the effect of impact on the equations that govern this kind of devices.

Key–Words: Second gradient, Elasticity, Variational approach, Wave, dispersion relation

1 Introduction

In [40, 55] it is proposed a general set of equations, obtained via a variational principle, accounting for surface mass density, elasticity and inertia embedded in a three-dimensional second gradient material [2, 37, 38, 42, 59]. In particular, well posed duality jump conditions to be imposed at the considered structured surfaces are introduced and discussed. In this contribution we apply the same strategy to the case of Euler beams. Euler beams are treated as second gradient materials, i.e. as materials for which the internal energy depends upon the second gradient of the placement field [3, 24, 25, 46, 52, 53, 54, 60]. In Euler beams the body is one-dimensional. Thus, the internal energy of an Euler beam is assumed to depend simply on the second derivative of the (dis)placement field. Classic variational approach has been used to derive the system of PDEs and boundary conditions. Many types of internal constraints across two semi-infinite Euler beams can be considered. In this paper we consider the case of a Tuned Mass Damper (TMD), that is a very efficient technique to reduce wave propagation [1, 56]. Tuned mass dampers are devices mounted in structures to reduce the amplitude of mechanical vibrations. They are frequently used in many fields of engineering. Various vibration control techniques may be used in order to reduce wave propagation in beams [49]. Mass–spring systems are widely used to control the response of resonant structures [48, 49, 50]. According to Ormoundroyd and Den Hartog [51], the use of TMDs was first suggested in 1909.

At its resonance frequency an undamped mass–spring system is attached to the host structure and it should be tuned to its resonance. However, to give the best effect over a frequency band under random excitation, Den Hartog [45] derived optimum values for the frequency of a damped absorber and its damping ratio in order to minimize the displacement response of the host structure. This paper gives an easier method to achieve the same goal.

2 Wave propagation in 1D second gradient elasticity (Euler beam)

2.1 Formulation of the problem

X is the coordinate of the material points of the $1D$ body in the reference configuration, L is its length and $X \in [0, L]$. t is the time and $t \in [t_i, t_f]$, where t_i and t_f are initial and final time of the dynamic process, respectively. The kinetic energy density functional $K(\dot{u})$ depends only on the first derivative $\dot{u}$, i.e. on the velocity, of the displacement field $u(X,t)$ with respect to t. The internal energy density functional $U(u'')$ depends only on the second derivative u'' of the displacement field $u(X,t)$ with respect to X. The action functional $\mathcal{A}(u(X,t))$ is given by the contributions of kinetic, internal and the external energies as follows,

$$
\begin{aligned}
\mathcal{A}(u(X,t)) = \int_{t_i}^{t_f} \{ \int_0^L [K(\dot{u}) - U(u'') \quad (1)\\
+ b^{ext} u + m^{ext} u'] dX \\
+ F_0^{ext} u(0,t) + F_L^{ext} u(L,t) \\
+ M_0^{ext} u'(0,t) + M_L^{ext} u'(L,t) \} dt
\end{aligned}
$$

where $b^{ext}(X)$ and $m^{ext}(X)$ are the external distributed force and couple, F_0^{ext}, F_L^{ext} are the external concentrated forces at $X = 0$ and at $X = L$, respectively, and M_0^{ext}, M_L^{ext} are the external concentrated couples at $X = 0$ and at $X = L$, respectively. If we assume $\delta\mathcal{A} = 0$ for any admissible variation δu then from (1) we get the final form of the system of partial differential equations, that can be explained once kinematical restrictions are defined. Kinetic energy density K is assumed to be quadratic in the velocity $\dot{u}$,

$$K(\dot{u}) = \frac{1}{2} \varrho \dot{u}^2, \tag{2}$$

where the coefficient $\varrho(X)$ is the so-called mass density of the material. Internal energy density is assumed to be quadratic in the so-called strain gradient u'',

$$U(u'') = \frac{1}{2} K_M u''^2,$$

where the coefficient $K_M(X)$ is the so-called bending stiffness of the material.

If the displacement field $u(X,t)$ is interpreted as transverse to the direction of the line defined by the reference configuration of the material body, and if such a body is composed by isotropic elastic material with Young modulus E and I is the moment of inertia of its cross section, then we have $K_M = EI$. Finally, we will consider only the case of admissible variation δu such that

$$\delta u(X, t_i) = \delta u(X, t_f) = 0 \tag{3}$$

The result is given by reporting the variation of the action functional,

$$
\begin{aligned}
\delta\mathcal{A}(u(X,t)) = -\int_{t_i}^{t_f} \{ \int_0^L [\varrho \ddot{u} + \left(K_M u^{II}\right)^{II} \quad (4)\\
- b^{ext} + \left(m^{ext}\right)^I] \delta u dX \\
+ \left[\left(K_M u^{II} - M_L^{ext}\right) \delta u'\right]_{X=L} \\
- \left[\left(K_M u^{II} + M_0^{ext}\right) \delta u'\right]_{X=0} \\
- \left[\left[\left(K_M u^{II}\right)^I + F_L^{ext} + m^{ext}\right] \delta u\right]_{X=L} \\
+ \left[\left[\left(K_M u^{II}\right)^I - F_0^{ext} + m^{ext}\right] \delta u\right]_{X=0} \} dt.
\end{aligned}
$$

2.2 Dispersion relation of the Euler beam problem

Let us assume no external actions,

$$b^{ext} = m^{ext} = 0,$$

and an indefinite length, i.e. no boundary conditions are considered. Thus, the Partial Differential Equations PDEs are,

$$\varrho \ddot{u} + \left(K_M u^{II}\right)^{II} = 0, \qquad \forall X, t. \tag{5}$$

Let us, now, look for plane wave solution, for the homogeneous ($K_M' = 0$) case, in the following form,

$$u(X,t) = \mathrm{Re}\{u_0 \exp[I(\omega t - kX)]\}, \tag{6}$$

where I is the imaginary unit, ω the frequency, k the wave number, and insert (6) into (5),

$$\mathrm{Re}\left\{\left(-\varrho\omega^2 + K_M k^4\right) u_0 \exp[I(\omega t - kX)]\right\} = 0. \tag{7}$$

Thus, the (7) is given in a more suitable way,

$$\left(-\varrho\omega^2 + K_M k^4\right) u = 0. \tag{8}$$

The (8) is satisfied for every displacement field if and only if,

$$k^4 = \frac{\varrho}{K_M}\omega^2, \tag{9}$$

that is the wanted dispersion relation.

If ω is real, then the 4 possible wave numbers (solutions of the dispersion relation (9)) are,

$$k = k_{1,2,3,4}, \qquad k_{1,2} = \pm\overline{k},$$

$$k_{3,4} = \pm I\overline{k}, \qquad \overline{k} = \sqrt[4]{\frac{\varrho}{K_M}\omega^2} \in R^+$$

that correspond to four possible waves. A general solution in terms of plane waves is therefore,

$$\begin{aligned} u(X,t) = \mathrm{Re}\{&u_{01}\exp\left[I\left(\omega t - k_1 X\right)\right] \\ &+u_{02}\exp\left[I\left(\omega t - k_2 X\right)\right] \\ &+u_{03}\exp\left[I\left(\omega t - k_3 X\right)\right] \\ &+u_{04}\exp\left[I\left(\omega t - k_4 X\right)\right]\} \end{aligned}$$

that is also

$$\begin{aligned} u(X,t) = \mathrm{Re}\{&u_{01}\exp\left[I\left(\omega t - \overline{k}X\right)\right] \\ &+u_{02}\exp\left[I\left(\omega t + \overline{k}X\right)\right] \\ &+u_{03}\exp\left[I\left(\omega t - I\overline{k}X\right)\right] \\ &+u_{04}\exp\left[I\left(\omega t + I\overline{k}X\right)\right]\} \end{aligned}$$

or

$$\begin{aligned} u(X,t) = \mathrm{Re}\{&\exp\left[I\omega t\right]\left[u_{01}\exp\left(-I\overline{k}X\right)\right. \\ &+u_{02}\exp\left(I\overline{k}X\right) + u_{03}\exp\left(\overline{k}X\right) \\ &\left.+u_{04}\exp\left(-\overline{k}X\right)\right] \end{aligned} \tag{10}$$

It is easy to show that the amplitude u_{01} corresponds to a plane wave propagating towards the positive axis X and, vice versa, the amplitude u_{02} corresponds to a plane wave propagating towards the negative axis X. Besides, the amplitudes u_{03} and u_{04} correspond to the so-called standing waves, being the standing wave associated to the amplitude u_{03} diverges to infinity at $X \longrightarrow +\infty$ and that to the amplitude u_{04} diverges at $X \longrightarrow -\infty$.

2.3 Energy and average energy fluxes related to plane waves

The calculation of the energy fluxes is done as follows. First of all, we define the total energy density, that is the sum of kinetic and internal energy. The time derivative of the total energy density is

$$\dot{E} = \varrho\dot{u}\ddot{u} + K_M u''\dot{u}'', \tag{11}$$

and the flux H is defined in such a way that the following PDE is satisfied,

$$\dot{E} + H' = 0 \tag{12}$$

By the use of the PDE (5) of the process we have

$$\dot{E} = -\dot{u}\left(K_M u^{II}\right)^{II} + K_M u''\dot{u}''. \tag{13}$$

By the use of the chain derivative rule we have,

$$\begin{aligned} \dot{E} &= -\left[\dot{u}\left(K_M u''\right)^I\right]^I + \dot{u}'\left(K_M u''\right)^I \\ &+ \left(K_M u''\dot{u}'\right)^I - \left(K_M u''\right)^I \dot{u}' \\ &= -\left[\dot{u}\left(K_M u''\right)^I\right]^I + \left(\dot{u}' K_M u''\right)^I \\ &= \left(-\dot{u}\left(K_M u''\right)^I + \dot{u}' K_M u''\right)^I \end{aligned} \tag{14}$$

By comparison of (12) and (14), we have

$$H = \dot{u}\left(K_M u''\right)^I - \dot{u}' K_M u''. \tag{15}$$

The average flux density $\langle H\rangle$ related to a wave with frequency ω is defined by the integration of H in time over the period $T = 2\pi/\omega$ of the wave,

$$\langle H\rangle = \int_t^{t+T} H\left(X,\tilde{t}\right) d\tilde{t}. \tag{16}$$

2.3.1 Energy flux related to a propagative wave towards the positive direction of the axis X

Let us take into account the propagative wave towards the positive direction of the axis X, with $\omega, \bar{k} \in Real^+$,

$$u(X,t) = \mathrm{Re}\left(u_0 \exp\left(I\left(\omega t - \bar{k}X\right)\right)\right). \tag{17}$$

Thus, its average density flux is defined by (15) and (16),

$$\langle H\rangle = \left\langle \dot{u}\left(K_M u''\right)^I\right\rangle - \left\langle \dot{u}' K_M u''\right\rangle,$$

that for the homogenous case is,

$$\langle H\rangle = K_M\left(\left\langle \dot{u}u^{III}\right\rangle - \left\langle \dot{u}'u''\right\rangle\right). \qquad (18)$$

By insertion of (17) into (18) and by using the theorem (33) defined in the Appendix2 in Sect. 4,

$$\frac{\langle H\rangle}{K_M\omega\bar{k}^3} = \frac{1}{2}\mathrm{Re}\left(u_0u_0^*\right) + \frac{1}{2}\mathrm{Re}\left(u_0u_0^*\right) = \|u_0\|^2,$$

that is

$$\langle H\rangle = K_M\omega\bar{k}^3\,\|u_0\|^2,$$

where $\|u_0\|$ is the modulus of the amplitude u_0 and u_0^* is its complex conjugated.

Thus, we remark that the average energy flux of a propagative wave towards the positive direction of the axis X is positive.

2.3.2 Energy flux related to a propagative wave towards the negative direction of the axis X

Let us take into account the propagative wave towards the negative direction of the axis X, with $\omega, \bar{k} \in Real^+$

$$u(X,t) = \mathrm{Re}\left(u_0\exp\left(I\left(\omega t + \bar{k}X\right)\right)\right). \qquad (19)$$

Its energy flux (18) is now computed. By insertion of (19) into (18) and by using the theorem (33) defined in the Appendix2 of Sect. 4,

$$\frac{\langle H\rangle}{K_M\omega\bar{k}^3} = -\frac{1}{2}\mathrm{Re}\left(u_0u_0^*\right) - \frac{1}{2}\mathrm{Re}\left(u_0u_0^*\right) = -\|u_0\|^2,$$

that is

$$\langle H\rangle = -K_M\omega\bar{k}^3\,\|u_0\|^2,$$

where $\|u_0\|$ is the modulus of the amplitude u_0, and u_0^* is its complex conjugated.

Thus, we remark that the average energy flux of a propagative wave towards the negative direction of the axis X is negative.

2.3.3 Energy flux related to a general standing wave

Let us take into account a general standing wave, with $\omega, \bar{k} \in Real^+$

$$u(X,t) = \mathrm{Re}\left(u_0\exp\left(I\omega t \pm \bar{k}X\right)\right). \qquad (20)$$

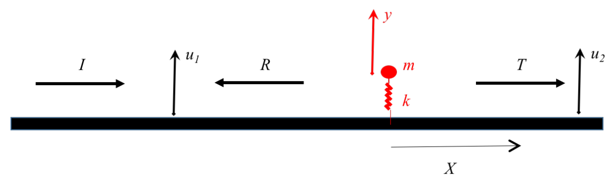

Figure 1: A mass-spring system with rigidity k and mass m inside an indefinite beam at the position $X = 0$. Incident, reflection and transmission coefficients are made explicit.

Its energy flux (18) is now computed. By insertion of (20) into (18), and by using the theorem (33) defined in the Appendix, we have

$$\langle H\rangle = 0,$$

where $\|u_0\|$ is the modulus of the amplitude u_0, and u_0^* is its complex conjugated.

Thus, we remark that the average energy flux of a general standing wave is null.

2.4 Reflection and transmission coefficients

Reflection R and transmission T coefficients are related to the average energy fluxes of the reflected and transmitted waves, respectively. Thus they do not consider the standing waves solution, because their average density flux is null. If we assume that H_i is the energy flux of an incident wave and H_r andH_t are the energy fluxes of the reflected and transmitted waves, the following definition hold,

$$R = \left\|\frac{\langle H_r\rangle}{\langle H_i\rangle}\right\|, \qquad T = \left\|\frac{\langle H_t\rangle}{\langle H_i\rangle}\right\|. \qquad (21)$$

3 The case of an indefinite beam with a tuned mass damper

3.1 Formulation of the problem

Let us consider a mass-spring system with rigidity k and mass m inside an indefinite beam at the position $X = 0$, see the Fig. 1. The displacement field must now be treated with two independent functions, $u_1(X,t)$ and $u_2(X,t)$. The first related to the displacement field on the left-hand side and the second to that of the right-hand side. The action $\mathcal{A}\left(u_1\left(X,t\right),u_2\left(X,t\right)\right)$ is the superposition of $(\mathcal{A}_1\left(u_1\left(X,t\right)\right)$,

$$\mathcal{A}_1\left(u_1\left(X,t\right)\right)=\int_{t_i}^{t_f}\left\{\int_{-L}^{0}\left[\frac{1}{2}\varrho\dot{u}_1^2-\frac{1}{2}K_M u_1''^2\right]dX\right\}$$

and $\mathcal{A}_2\left(u_2\left(X,t\right)\right))$,

$$\mathcal{A}_2\left(u_2\left(X,t\right)\right)=\int_{t_i}^{t_f}\left\{\int_{0}^{L}\left[\frac{1}{2}\varrho\dot{u}_2^2-\frac{1}{2}K_M u_2''^2\right]\right\}$$

related to both sides of the indefinite beam and to that of the internal constraint spring mass system $\mathcal{A}_{IC}$,

$$\mathcal{A}=\mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_{IC} \tag{22}$$

The vertical displacement of the mass m is not constrained to be attached to the beam but it is given by its vertical coordinate $y\left(t\right)$, that is a function of time. The internal clamp constrained is considered

$$u_1\left(0,t\right)=u_2\left(0,t\right), \qquad u_1'\left(0,t\right)=u_2'\left(0,t\right), \tag{23}$$

between the two sides of the indefinite beam. and the action of the mass spring system is therefore,

$$\mathcal{A}_{IC}\left(\cdot\right)=\frac{1}{2}m\left[\dot{y}\left(t\right)\right]^2-\frac{1}{2}k\left(y\left(t\right)-u_1\left(0,t\right)\right)^2 \tag{24}$$

where no distributed nor concentrated external actions are considered, the length L will be assumed to be arbitrary large, i.e. $L\rightarrow\infty$ and the mass-spring system is concentrated at $X=0$ where the displacement $u_1\left(0,t\right)=u_2\left(0,t\right)$ and the velocity $\dot{u}_1\left(0,t\right)=\dot{u}_2\left(0,t\right)$ are the same in the two independent branches of the beam. The variation of (24),

$$\begin{aligned}\delta\mathcal{A}_{IC}\left(\cdot\right)=\delta y\left[-m\ddot{y}-ky+ku_1\right]\\+\delta u_1\left(0,t\right)k\left[y\left(t\right)-u_1\left(0,t\right)\right]\end{aligned}$$

is used to derive the variation of (22),

$$\begin{aligned}&\delta\mathcal{A}\left(u_1\left(X,t\right),u_2\left(X,t\right),y\left(t\right)\right)=\\&-\int_{t_i}^{t_f}\{\int_{0}^{L}\{\delta u_1\left[\varrho\ddot{u}_1+\left(K_M u_1^{II}\right)^{II}\right]\\&\delta u_2\left[\varrho\ddot{u}_2+\left(K_M u_2^{II}\right)^{II}\right]\}dX\\&+\left[\left(K_M u_1^{II}\right)\delta u_1'-\left(K_M u_2^{II}\right)\delta u_2'\right]_{X=0}\\&-\left[\left[\left(K_M u_1^{II}\right)^{I}\right]\delta u_1-\left[\left(K_M u_2^{II}\right)^{I}\right]\delta u_2\right]_{X=0}\\&\delta y\left[-m\ddot{y}-ky+ku_1\right]\\&+\delta u_1\left(0,t\right)k\left[y\left(t\right)-u_1\left(0,t\right)\right]\}dt.\end{aligned}$$

Thus, the PDEs are

$$\varrho\ddot{u}_1+\left(K_M u_1^{II}\right)^{II}=\varrho\ddot{u}_2+\left(K_M u_2^{II}\right)^{II}=0$$

and the boundary conditions at $X=0$, because of the kinematical restrictions in (23) are

$$\begin{aligned}&u_1\left(0,t\right)=u_2\left(0,t\right), \qquad u_1'\left(0,t\right)=u_2'\left(0,t\right),\\&u_1^{II}\left(0,t\right)=u_2^{II}\left(0,t\right),\\&-m\ddot{y}\left(t\right)-ky\left(t\right)+ku_1\left(0,t\right)=0 &(25)\\&-K_M u_1^{III}\left(0,t\right)+K_M u_2^{III}\left(0,t\right)\\&+k\left[y\left(t\right)-u_1\left(0,t\right)\right]=0 &(26)\end{aligned}$$

3.2 A plane wave solution

Let us assume a propagative wave towards the right-hand side (i.e. towards the positive direction of the X axis) into the left-hand side of the beam. In this way, such a propagative wave complies the mass-spring system and interact with it. Such an incident wave has, therefore, the form of (17), i.e., (with $\omega,\bar{k}\in Real^{+}$)

$$u_i\left(X,t\right)=\mathrm{Re}\left(u_i^p\exp\left(I\left(\omega t-\bar{k}X\right)\right)\right). \tag{27}$$

Once the incident wave complies the concentrated spring-mass system, a transmitted and a reflected wave is generated. However, not all the 4 kind of waves in (10) can be generated on both sides of the concentrated mass. First of all, the standing wave that diverges at $X\longrightarrow-\infty$ is not possible (Sommerfield condition) for the reflected wave and the standing wave that diverges at $X\longrightarrow+\infty$ is not possible (Sommerfield condition) for the transmitted wave. Thus the reflected wave can only be of the following form,

$$\begin{aligned}u_r\left(X,t\right)=\mathrm{Re}(u_r^p\exp\left(I\left(\omega t+\bar{k}X\right)\right)\\+u_r^s\exp\left(I\omega t+\bar{k}X\right)), \qquad \omega,\bar{k}\in Real^{+}.\end{aligned}$$

and the transmitted wave is given by,

$$\begin{aligned}u_t\left(X,t\right)=\mathrm{Re}(u_t^p\exp\left(I\left(\omega t-\bar{k}X\right)\right)+\\+u_t^s\exp\left(I\omega t-\bar{k}X\right)), \qquad \omega,\bar{k}\in Real^{+}.\end{aligned}$$

Thus, in the vicinity of the mass resonator, the solution is the superposition $u_i\left(X,t\right)+u_r\left(X,t\right)$ of incident $u_i\left(X,t\right)$ and reflected $u_r\left(X,t\right)$ waves for the left-hand side of the concentrated mass,

$$u_1\left(X,t\right)=u_i\left(X,t\right)+u_r\left(X,t\right)=$$
$$\mathrm{Re}(u_i^p\exp\left(I\left(\omega t-\bar{k}X\right)\right)+u_r^p\exp\left(I\left(\omega t+\bar{k}X\right)\right)$$
$$+u_r^s\exp\left(I\omega t+\bar{k}X\right))$$

and simply the transmitted wave for the right-hand side of the concentrated mass, (with $\omega,\bar{k}\in Real^+$)

$$u_2\left(X,t\right)=u_t\left(X,t\right)=$$
$$\mathrm{Re}\left(u_t^p\exp\left(I\left(\omega t-\bar{k}X\right)\right)+u_t^s\exp\left(I\omega t-\bar{k}X\right)\right).$$

The oscillation of the internal resonator is also achieved,

$$y\left(t\right)=y_0\exp\left(I\omega t\right) \tag{28}$$

3.3 Reflection and transmission condition across a tuned mass damper

The amplitude of the propagative incident wave u_i^p is again considered a datum of the problem. In the following we will find both the amplitudes of reflected u_r^p and u_r^s and of the transmitted u_t^p and u_t^s waves, as well as the amplitude y_0 of the internal resonator. The boundary conditions are the 5 included in eqns. (25), (25) and (26). By insertion of (28), (28) and (28) into eqns. (25), (25) and (26) we derive the five equations that are needed to derive the unknowns of reflected u_r^p and u_r^s, of the transmitted u_t^p and u_t^s and of the internal resonator $y_0$waves. Keeping in mind that

$$u_1'=\mathrm{Re}(u_i^p\left(-I\bar{k}\right)\exp\left(I\left(\omega t-\bar{k}X\right)\right)$$
$$+u_r^p\left(I\bar{k}\right)\exp\left(I\left(\omega t+\bar{k}X\right)\right)$$
$$+u_r^s\left(\bar{k}\right)\exp\left(I\omega t+\bar{k}X\right))$$
$$u_1''=\mathrm{Re}(u_i^p\left(-I\bar{k}\right)^2\exp\left(I\left(\omega t-\bar{k}X\right)\right)$$
$$+u_r^p\left(I\bar{k}\right)^2\exp\left(I\left(\omega t+\bar{k}X\right)\right)$$
$$+u_r^s\bar{k}^2\exp\left(I\omega t+\bar{k}X\right))$$
$$u_1^{III}=\mathrm{Re}(u_i^p\left(-I\bar{k}\right)^3\exp\left(I\left(\omega t-\bar{k}X\right)\right)$$
$$+u_r^p\left(I\bar{k}\right)^3\exp\left(I\left(\omega t+\bar{k}X\right)\right)$$
$$+u_r^s\bar{k}^3\exp\left(I\omega t+\bar{k}X\right))$$
$$\dot{u}_1=\mathrm{Re}(u_i^p\left(I\omega\right)\exp\left(I\left(\omega t-\bar{k}X\right)\right)$$
$$+u_r^p\left(I\omega\right)\exp\left(I\left(\omega t+\bar{k}X\right)\right)$$
$$+u_r^s\left(I\omega\right)\exp\left(I\omega t+\bar{k}X\right))$$
$$\ddot{u}_1=\mathrm{Re}(u_i^p\left(I\omega\right)^2\exp\left(I\left(\omega t-\bar{k}X\right)\right)$$
$$+u_r^p\left(I\omega\right)^2\exp\left(I\left(\omega t+\bar{k}X\right)\right)$$
$$+u_r^s\left(I\omega\right)^2\exp\left(I\omega t+\bar{k}X\right))$$

and

$$u_2'=\mathrm{Re}(u_t^p\left(-I\bar{k}\right)\exp\left(I\left(\omega t-\bar{k}X\right)\right)$$
$$+u_t^s\left(-\bar{k}\right)\exp\left(I\omega t-\bar{k}X\right))$$
$$u_2''=\mathrm{Re}(u_t^p\left(-I\bar{k}\right)^2\exp\left(I\left(\omega t-\bar{k}X\right)\right)$$
$$+u_t^s\left(-\bar{k}\right)^2\exp\left(I\omega t-\bar{k}X\right))$$
$$u_2^{III}=\mathrm{Re}(u_t^p\left(-I\bar{k}\right)^3\exp\left(I\left(\omega t-\bar{k}X\right)\right)$$
$$+u_t^s\left(-\bar{k}\right)^3\exp\left(I\omega t-\bar{k}X\right))$$
$$\dot{u}_2=\mathrm{Re}(u_t^p\left(I\omega\right)\exp\left(I\left(\omega t-\bar{k}X\right)\right)$$
$$+u_t^s\left(I\omega\right)\exp\left(I\omega t-\bar{k}X\right))$$
$$\ddot{u}_2=\mathrm{Re}(u_t^p\left(I\omega\right)^2\exp\left(I\left(\omega t-\bar{k}X\right)\right)$$
$$+u_t^s\left(I\omega\right)^2\exp\left(I\omega t-\bar{k}X\right))$$

and, without loss of generality, assuming the time $t=0$ and position $X=0$, the five conditions (25), (25) and (26) are as follows,

$$
\begin{aligned}
& u_i^p + u_r^p + u_r^s = u_t^p + u_t^s, \\
& u_i^p \left(-I\bar{k}\right) + u_r^p \left(I\bar{k}\right) + u_r^s \left(\bar{k}\right) \\
& = u_t^p \left(-I\bar{k}\right) + u_t^s \left(-\bar{k}\right), \\
& u_i^p \left(-I\bar{k}\right)^2 + u_r^p \left(I\bar{k}\right)^2 + u_r^s \bar{k}^2 \\
& = u_t^p \left(-I\bar{k}\right)^2 + u_t^s \left(-\bar{k}\right)^2, \\
& u_i^p \left(-I\bar{k}\right)^3 + u_r^p \left(I\bar{k}\right)^3 + u_r^s \bar{k}^3 \\
& = u_t^p \left(-I\bar{k}\right)^3 + u_t^s \left(-\bar{k}\right)^3 \\
& + \frac{k}{K_M} \left[y_0 - u_i^p - u_r^p - u_r^s\right] \\
& - m y_0 \left(I\omega\right)^2 - k y_0 + k \left(u_i^p + u_r^p + u_r^s\right)
\end{aligned}
$$

that is possible to solve analytically,

$$
\begin{aligned}
& u_r^p = -\frac{(1+I)\, k m \omega^2}{D_1} u_i^p, \\
& D_1 = 4(-1+I)\, k K_M \bar{k}^3 \\
& +2\left(k + 2(1-I) K_M \bar{k}^3\right) m\omega^2
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
& u_r^s = -\frac{(1-I)\, k m \omega^2}{D_2} u_i^p, \\
& D_2 = 4(-1+I)\, k K_M \bar{k}^3 \\
& +2\left(k + 2(1-I) K_M \bar{k}^3\right) m\omega^2 \\
& u_t^p = \frac{(1+I)\left(4 k K_M \bar{k}^3 - \left(k + 4 K_M \bar{k}^3\right) m\omega^2\right)}{D_3} u_i^p, \\
& D_3 = 4(1+I)\, k K_M \bar{k}^3 \\
& -2I\left(k + 2(1-I) K_M \bar{k}^3\right) m\omega^2 \\
& u_t^s = -\frac{(1-I)\, k m \omega^2}{D_4} u_i^p, \\
& D_4 = (-1+I)\, k K_M \bar{k}^3 \\
& +2\left(k + 2(1-I) K_M \bar{k}^3\right) m\omega^2 \\
& y_0 = \frac{2(1+I)\, k K_M \bar{k}^3}{D_5} u_i^p \\
& D_5 = 2(1+I)\, k K_M \bar{k}^3 \\
& -I\left(k + 2(1-I) K_M \bar{k}^3\right) m\omega^2,
\end{aligned} \tag{30}
$$

3.4 Numerical representation of reflection and transmission coefficients

By the use of the definitions (21) of the reflection ant transmission coefficients, it is possible to verify the

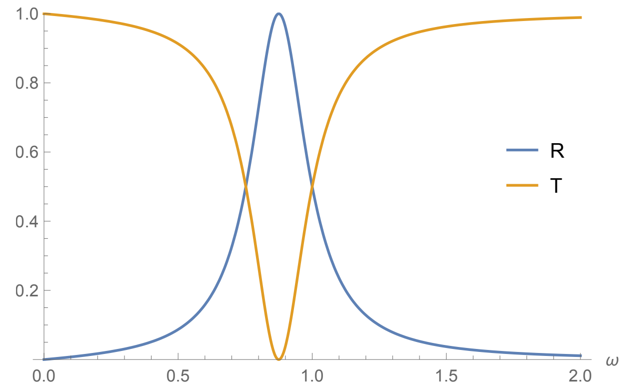

Figure 2: Reflection and Transmission coefficients with the following numerical characterization,$K_M = 1$,$\rho = 1$, $m = 1$, $k = 1$.

energy conservation,

$$R + T = 1.$$

Besides, reflection and transmission coefficients can be plotted in the following figures 2, 3, 4, 5 and 6.

4 Towards the design of piece-wise smooth tuned mass damper and outlook

Finally, we consider the case of the piece-wise smooth tuned mass dampers [5, 6, 7, 8, 9, 11, 12]. In particular we assume that the mass damper can only vibrate at the top of the beam. This will be modeled by a different assumption of the internal constraint action (24),

$$
\begin{aligned}
& \mathcal{A}_{IC} = \frac{1}{2} m \left[\dot{y}(t)\right]^2 \\
& -\frac{1}{2} k H\left(y(t) - u_1(0,t)\right)\left(y(t) - u_1(0,t)\right)^2 \\
& -\frac{1}{2} k_{imp} H\left(u_1(0,t) - y(t)\right)\left(y(t) - u_1(0,t)\right)^2
\end{aligned} \tag{31}
$$

where the Heaviside function H has been used and the rigidity $k_{imp} >> k$ is the so called impact rigidity[4, 10, 13, 14, 15, 57].

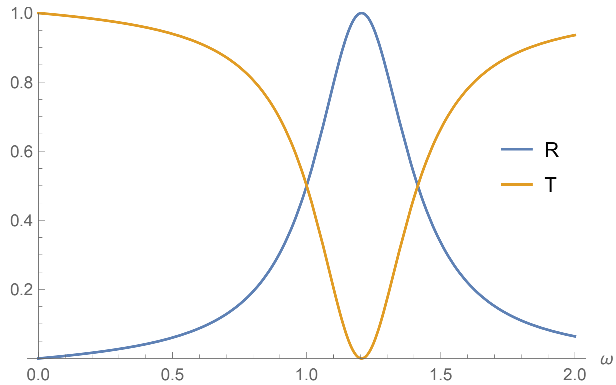

Figure 3: Reflection and Transmission coefficients with the following numerical characterization,$K_M = 1$,$\rho = 1$, $m = 1$, $k = 2$.

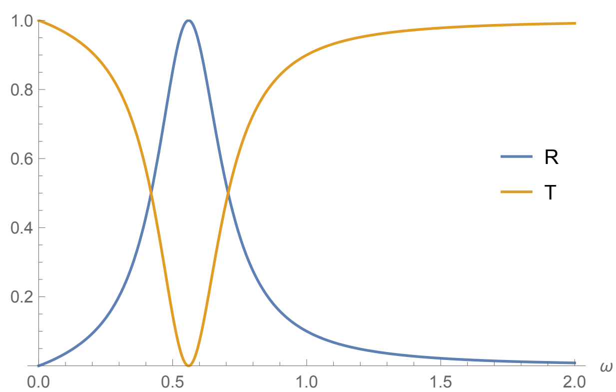

Figure 4: Reflection and Transmission coefficients with the following numerical characterization,$K_M = 1$,$\rho = 1$, $m = 2$, $k = 1$.

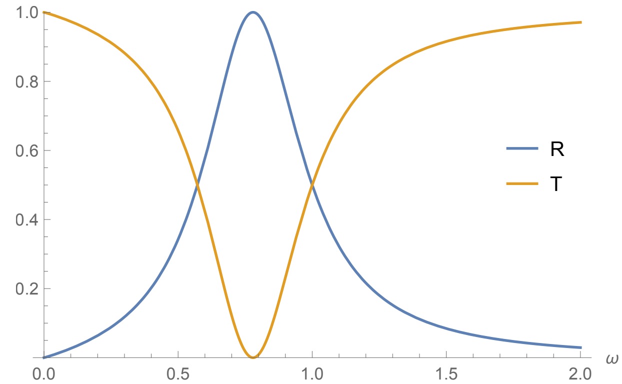

Figure 5: Reflection and Transmission coefficients with the following numerical characterization,$K_M = 0.1$,$\rho = 1$, $m = 1$, $k = 1$.

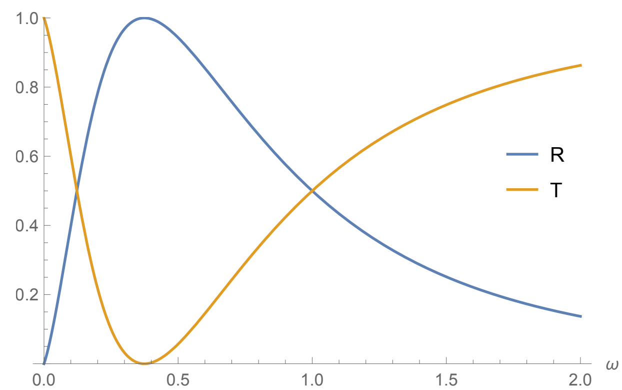

Figure 6: Reflection and Transmission coefficients with the following numerical characterization,$K_M = 1$,$\rho = 0.1$, $m = 1$, $k = 1$.

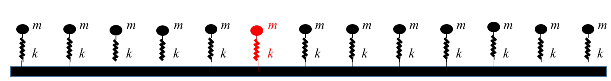

Figure 7: Distributed mass-spring system with rigidity k and mass m attached to an indefinite beam.

Distributed systems, see e.g., Fig. 7, are not new in the literature on the applications of Riccati equation [19, 22, 23] and are discussed at length in the literature on transmission lines [16, 17, 18, 20, 21]. Thus, further developments will be achieved considering a system of distributed tuned mass dampers (e.g., the oscillation of fluid into the pore space[39, 41] of different interesting metamaterials [36, 43, 44]), that could be mathematically treated with the help of special functions [26, 27, 28, 29, 30, 31, 32, 33, 34, 35].

Appendix: A theorem for the evaluation of the average flux density

Let F and f be two general complex function of this form

$$F = F_0 \exp\left(I\omega t - \gamma_1\right), \qquad f = f_0 \exp\left(I\omega t - \gamma_2\right), \qquad F_0, f_0 \in Real \tag{32}$$

then it is possible to prove that

$$\int_t^{t+T} \mathrm{Re}\left(F\right) \mathrm{Re}\left(f\right) d\tilde{t} = \frac{1}{2} \mathrm{Re}\left(F f^*\right), \tag{33}$$

where f^* is the complex conjugated of f.

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