Railway System 'Vehicle-Track': Simulation, Mathematical Modelling and Spectral Densities of Excitation and Response (Part II)

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Abstract: - The present article is a quasi-Part II of [12] and examines the relation between the Spectral Density of Excitation to the Spectral Density of response -beyond the analyses of [12]- in the case of the system "Railway Vehicle-Railway Track". It begins with the Historical Perspective, it includes some principles of Periodic, Non-Periodic Functions, Fourier Series and Probabilities Theory and examines in more detail the relationship between Excitation-Response Spectral Densities and specifically for the case of the Track vertical Defects and the Transfer Function of the Track Recording Car/Vehicle. It is based on the Mathematical Modelling of the Railway System "Vehicle-Track".

Key-Words: Excitation, Response, Spectral, Density, Railways, Track, Vehicle, Random, Functions, stationary, ergodic.

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1 Introduction

The present article is a quasi-Part II of the article "*Railway System 'Vehicle-Track': Relation Between the Spectral Density of Excitation vs Response*" [12]. It focuses on the properties of the Spectral Density of a random signal, and, especially why the Power Spectral Density of the real vertical defects along the Track can be *accurately* calculated by the Spectral Density of the recordings of the measurements of the Track defects via the Transfer Function of the Recording Vehicle.

Statistics play a significant role in spectral analysis in this case, because most signals have a noisy or random aspect. If the underlying statistical attributes of a signal were known exactly or could be determined without error from a finite interval of the signal, then spectral analysis would be an exact science. Finally, the practical reality documents that -indeed- an estimate of the spectrum can be made from even a single finite segment of an infinite signal [[16], 1, 2].

We should remember that [12]: in Railways "the Train circulation is a random dynamic phenomenon and, according to the different frequencies imposed by the loads, and the corresponding response of track superstructure. The dynamic component of the load of the vehicle on the track depends on the mechanical properties (stiffness, damping) of the system "vehicle-track", which acts as an excitation on the vehicle's motion (Figures 4, 5) and, vice-versa, the vehicle's motion acts as an excitation on the track. The most simplified approach of this motion (vehicle on Track) is simulated by a SDOF [= Single Degree Of Freedom] system (Figure 5)".

The dynamic component of the acting load is primarily caused by the motion of the vehicle's Non-Suspended (Unsprung) Masses, which are excited by the track geometry and the vertical defects, and, to a smaller degree, by the effect of the Suspended (sprung) Masses. In order to evaluate the real defects of the Track and their influence on the acting forces we use Track Recording cars whose reliability was presented recently ([10], [11]).

In order to calculate the magnitude of this dynamic component of the acting Load we use a theoretical analysis based on the Fourier Transformation, which approaches the phenomenon as the Loads -owed to forced random oscillations in systems with damping- appear. In the following, we will present this procedure. The forms of the excitations are random by nature and not deterministic.

To reply the question "*why should we use the Fourier Transform?*" we should examine the Historical perspective of the development of the Spectral Analysis.

2 Historical Perspective of the Development of the Spectral Analysis

In the sixth century BC, Pythagoras developed a relationship between the periodicity of pure sine vibrations of musical notes produced by a string of fixed tension and a number representing the length of the string (Figure 1). He believed that the essence of harmony was inherent in numbers. Pythagoras extended this empirical relationship to describe the harmonic motion of heavenly bodies, describing it as the "*music of the spheres*".

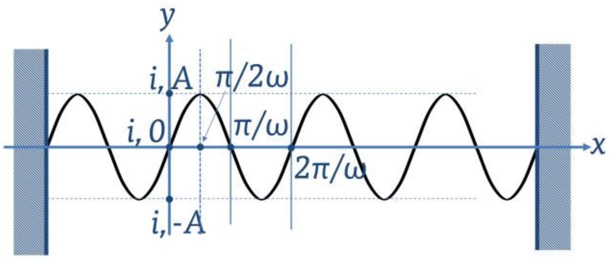

Figure 1. A music string vibrating in a sinusoidal form. This is an example of a "periodic function" (see § 3). The point *i* is a random point on the axis of *x*; it could represent the beginning of the observations (*i,0 = 0,0*).

The mathematical basis for modern spectral estimation has its origins in the seventeenth-century work of the scientist Sir Isaac Newton (1671 AD). He observed that sunlight passing through a glass prism was expanded into a band of many colors and gave the name *spectrum*. Spectrum is a variant of the Latin word "specter," meaning image or ghostly apparition. The adjective associated with spectrum is spectral. Thus, spectral estimation, rather than spectrum estimation, is the preferred terminology. Newton presented in his major work Principia (1687 AD) the first mathematical treatment of the periodicity of wave motion that Pythagoras (6th century BC) had empirically observed.

The solution to the wave equation for the vibrating musical string was developed by Daniel Bernoulli (1738 AD), a mathematician who discovered the general solution for the displacement $u(x, t)$ of the string at time t_i and position x_i (the endpoints of the string are at $x = 0$ and $x = \pi$) in the wave equation to be (**Eqn. 2.1**):

$$u(x,t) = A_0 + \sum_{k=1}^{\infty} \sin kx \left(A_k \cos kct + B_k \sin kct \right)$$

where c is a physical quantity characteristic of the material of the string which represents the velocity of the traveling waves on the string. The term A_0 is normally zero (since it is dependent on the choice of the position of the axes) and we will assume this here. The mathematician L. Euler (1755 AD) demonstrated that the coefficients A_k and B_k in the series given by Eqn. (2.1), which -approximately half a century later- would be called the Fourier series and, were found as solutions to

$$A_k = \frac{2}{\pi} \int_0^{\pi} u(x,0) \sin kx dx, \qquad B_k = \frac{2}{\pi} \int_0^{\pi} u(x,0) \cos kx dx \tag{2.2}$$

for which $t0 = \pi/2kc$. The French engineer Jean Baptiste Joseph Fourier in his thesis Analytical Theory of Heat (1822 AD) extended the wave equation results by asserting that any arbitrary function $u(x)$, even one with a finite number of discontinuities, could be represented as an infinite summation of *sine* and *cosine* terms:

$$u(x) = \sum_{k=1}^{\infty} \left(A_k \cos kax + B_k \sin kax \right) \tag{2.3}$$

The mathematics of taking a function $u(x)$, or its samples, and determining its A_k and B_k coefficients has become known as *harmonic analysis*, due to the harmonic indexing of the frequencies in the *sine* and *cosine* terms [[16], 3-4]. However, it is possible to represent non-periodic (random) functions using any class of periodic ones. In Fourier analysis, the periodic functions used are *sine* and *cosine* functions. They have the important properties that an approximation consisting of a given number of terms achieves ***-the minimum mean square error between the signal and the approximation-***, and also that they are orthogonal, so the coefficients may be determined independently of one another [[15], 17].

Furthermore, the analytic techniques developed by Fourier are particularly important in three applications:(**a**) for studying periodic solutions to physical problems described by differential equations, especially partial differential equations -for example, the study of wave motion of plucked strings or the transmission of electromagnetic waves in waveguides or cables; (**b**) as an operational device for solving differential equations -for example, ordinary differential equations with constant coefficients may be converted into algebraic equations by Fourier transformation; (**c**) for approximating non-periodic functions, which is our scope in this article [[15], 16].

The critical result from the Fourier theory is that: ***any random function even one with a finite number of discontinuities, could be represented as an infinite summation of sine and cosine terms***.

This infers that any random (*excitation and/or response*) *function can be analyzed in an infinite summation of harmonic oscillations*. Many real-world signals can be characterized as being random (from the observer's viewpoint). Briefly speaking, this means that the variation of a random signal outside the observed interval cannot be determined exactly but can only be specified in statistical terms of averages [[26], 2]; the measurement(s) of the rail running table geometry gives/give continuous measured values and belong to a continuous in time-space "signal", namely the excitation applied to the running wheel. "The essence of the spectral analysis problem is captured by the following informal fonnulation: '*From a finite record of a stationary data sequence, estimate how the total power is distributed over frequency*'" [[26], 1].

3 Periodic, Non-Periodic Functions and Fourier Series and Integral

When we speak of a wave-like structure we usually have in mind -by intuition- something like the pattern of ocean waves, that is a pattern which, more or less, repeats itself after certain intervals. This idea may be expressed more precisely in terms of what is called a "periodic function" (Figure 1): Suppose that a pattern of faults of a Railway Track happened to repeat itself perfectly at intervals of, say, p meters or more precisely, that the section of the surface of the *runway* (*= rail running table or surface*) with some vertical plane repeated itself perfectly at intervals of p meters. Then if we measure distance along a horizontal line in the vertical plane, and let $f(x)$ denote the height of the surface (measured from some fixed level) at a point whose distance is x meters (we speak in millimeters in reality) -from some fixed origin-, we express the repetitive nature of the pattern by means of the equation:

$$f(x) = f(x + kp) \qquad (3.1.)$$

valid for all x, and k may take any integral value 0, ±1, ±2,…… Generally, if a function $f(x)$ satisfies an equation of the above form it is said to be periodic, and if p is the smallest number such that equation (3.1) holds for all x, p is called the period of the function. If there is no value of p (other than zero) such that (3.1) holds for all x, the function is called non-periodic. The most familiar periodic functions which we encounter are the *sine* and *cosine* functions, since, of course, $A \bullet sin\omega x$ and $A \bullet cos\omega x$ are both periodic, each with period $p = (2\pi/\omega)$ [Figure 1]. The quantity $\omega = 2\pi/p$ is called the angular frequency of $sin\omega x$ (or $cos\omega x$) and the constant A is called the *amplitude*. The theory of periodic functions states that any "well behaved" periodic function can be expressed as ***a (possibly infinite) sum of sine and cosine functions***, thus according to Fourier's theorem any function $f(x)$, with period p may be written as a Fourier series:

$$f(x) = \sum_{r=0}^{\infty} \left[a_r \cdot \cos\left(\frac{2\pi rx}{p}\right) + b_r \cdot \sin\left(\frac{2\pi rx}{p}\right) \right] \qquad (3.2a)$$

where a_o, a_1, a_2,.., b_o, b_1, b_2,..., are constants which may be determined from the form of $f(x)$. The various terms which appear in the summation on the right-hand side of Eqn. (3.2) may be described in the following way. The first term (corresponding to r = 0) is simply a constant. The second term (with r = 1) represents *cosine* and *sine* waves with the basic period p, the next term (r = 2) represents *cosine* and *sine* waves with period $p/2$, the next term (r = 3) represents *cosine* and *sine* waves with period $p/3$, and so on. Noting that any *cosine* and *sine* wave whose period is an integral fraction of p will also repeat itself after intervals of p units, we see immediately that each of the terms in the summation repeat their values after intervals of p units, and therefore the sum is periodic with period p. This argument does not, of course, prove Fourier's theorem, but merely indicates its plausibility [[21], 3-5].

Eqn. (3.2a) can be written also under the form:

$$f(x) = \sum_{r=-\infty}^{+\infty} A_r \cdot e^{i\omega_r x}, \qquad where \qquad (3.2b)$$

$$A_r = \frac{1}{2}(a_r - ib_r), \qquad when \qquad r > 0$$
$$A_r = a_0, \qquad when \qquad r = 0 \qquad (3.2c)$$
$$A_r = \frac{1}{2}\left(a_{|r|} - ib_{|r|}\right), \qquad when \qquad r < 0$$

and $\omega_r = 2\pi r/p$, r=0, ±1, ±2,……

One way of looking at a non-periodic function is to regard it as a periodic function with an infinite period. Reasoning in this way, we might suppose that non-periodic functions could also be represented as a sum of the form of Eqn. (3.2) if we let $p \rightarrow \infty$. In other words, *we might attempt to approximate to a non-periodic function by a sequence of periodic functions with longer and, even more, longer periods*. It turns out that as the values of the previous coefficients a_i and b_i are reduced the distance among the frequencies $2\pi r/p$, $2\pi(r+1)/p$, of neighboring terms in Eqn. (3.2)

→0; in the limit the summation becomes an integral. A non-periodic function could be written (decomposed) as:

$$f(x)=\int_0^{\infty}\{g(\omega)\cdot\cos\omega x+k(\omega)\sin\omega x\}d\omega \quad (3.3a)$$

where *g(ω)* and *k(ω)* are functions whose forms may be determined from the form of *f(x)*, under the precondition that *f(x)* is “absolutely integrable”, that is if:

$$\int_{-\infty}^{+\infty}|f(x)|\cdot dx<\infty$$

Eqn. (3.3a) can be written also under the form:

$$f(x)=\int_{-\infty}^{+\infty}p(\omega)\cdot e^{i\omega x}d\omega, \qquad where \qquad (3.3b)$$

$$p(\omega)=\frac{1}{2}\left(g(\omega)-i\cdot k(\omega)\right), \qquad when \qquad \omega>0$$

$$p(\omega)=g(0), \qquad when \qquad \omega=0 \qquad (3.3c)$$

$$p(\omega)=\frac{1}{2}\left(g(|\omega|)+i\cdot k(|\omega|)\right), \qquad when \qquad \omega<0$$

The function *p(ω)* is called the Fourier transform of *f(x)* and is of fundamental importance in spectral analysis.

The essential difference between the decomposition of periodic and of non-periodic functions is that, while a periodic function can be expressed as a sum of *cosines* and *sines* terms, over a discrete set of frequencies $\omega_0, \omega_1, \omega_2, \omega_3, \ldots$, a non-periodic function can be expressed only in terms of *cosines* and *sines* ***which cover the whole continuous range of frequencies, that is from 0 to*** ∞ [[21], 5-6].

4 Probability Theory: Basic Elements

4.1 Definitions

First of all, in the following when we will use the term “experiment(s)” we mean an operation of establishing certain conditions which may produce one of several possible outcomes or results. This use of the term “‘experiment” is more general than its customary interpretation, and it certainly includes what would normally be regarded as “experiments” (*such as measuring pressures or stresses, the defects of the Railway Track, but also temperatures, currents, voltages*, *etc.*).

The subject of probability theory is concerned with those experiments which involve “random phenomena”, i.e. experiments whose outcomes cannot be predicted with certainty. Although we cannot say definitely then, whether or not a particular event will occur, we may have reason to believe that some events are “more likely” to occur than others. The question now arises: how likely is the occurrence of a particular event? We attempt to answer this question by associating “probabilities” with each event [cf. [[21], 28-30]]. For comprehensive axiomatic approach to probability and random variables and *distribution functions*, the interested reader can read [[21], 31-47].

4.2 Distribution Functions, Means, Variances, Moments

Let us consider the case of a discrete valued function *X* with *n* values resulted from an experiment (measurement) and that *X* took the value x_1 n_1 times, the value x_2 n_2 times, …, and the value x_k n_k times; where $n+n_1+n_2+\ldots+n_k = n$. The *arithmetic mean* of these *n* values of *X* is:

$$\overline{x}=\frac{1}{n}\cdot\sum_{i=1}^{k}n_i\cdot x_i \qquad (4.1)$$

and is called *a random sample of n measurements on X*. The *sample variance* -which is the usual quantity to calculate the “spread” of the values of the sample is:

$$s^2=\frac{1}{n}\cdot\sum_{i=1}^{k}n_i\cdot\left(x_i-\overline{x}\right)^2 \qquad (4.2)$$

which is *the mean square deviation* of the *n* measurements around their *arithmetic mean value* (Eqn. 4.4) and has the dimension of x^2; to obtain a measure of the “spread” we consider the square root of s^2:

$$s=\sqrt{s^2}=\left(\frac{1}{n}\cdot\sum_{i=1}^{k}n_i\cdot\left(x_i-\overline{x}\right)^2\right)^{1/2} \qquad (4.3)$$

and it is called *the sample standard deviation*.

If we suppose that we let *n* as the number of repetitions of the measurements and that *n*→∞, we should expect that as *n* increases, then:

$$\left(\frac{n_i}{n}\right)\sim p\left[X=x_i\right]=p_i \qquad p=probability$$

If we replace n_i/n by p_i in Eqns (4.1) and (4.2) then:

$$\overline{x}=\mu=\sum_{i=1}^{k}p_i\cdot x_i \qquad =mean \qquad (4.4)$$

$$s^2=\sigma^2=\sum_{i=1}^{k}p_i\cdot\left(x_i-\mu\right)^2 \qquad (4.5)$$

and the square root of the variance σ^2 is also named the standard deviation of the distribution. At this point we should define that given *a random variable X* the *distribution function X* (or sometimes named *cumulative distribution function*), *F(x)* is given by:

$$F(x)=p[X\le x] \qquad p=probabiblity \quad (4.6)$$

In the case of a continuous random variable X with probability density function $f(x)$ the relevant Eqns (4.4) and (4.5) become:

$$\mu = \int_{-\infty}^{+\infty} x \cdot f(x) \cdot dx \tag{4.7}$$

$$\sigma^2 = \int_{-\infty}^{+\infty} (x-\mu)^2 \cdot f(x) \cdot dx \tag{4.8}$$

The quantities μ and σ^2 are extremely useful in providing a rough summary and estimation of the form of a probability distribution (discrete or continuous), representing, loosely speaking, the "central value" μ, that is the *mean value*, and σ^2 the variance of the distribution about this mean value. However, it should be clear that we could never hope to describe fully the function $f(x)$ or the sequence p_i merely in terms of the two number μ and σ^2. Indeed, we could construct many distributions all of which have the same values of μ and σ^2, but with their shapes differing in other issues. To distinguish between such distributions, we now introduce a further sequence of constants, called the moments:

The *rth moment* around $m_r^{'}$ **:**

$$\mu_r^{'} = \sum_i x_i^{'} \cdot p_i \text{ , if X is discrete, and} \tag{4.9a}$$

$$\mu_r^{'} = \int_{-\hbar}^{+\infty} x^r \cdot f(x) \cdot dx, \text{ if } X \text{ is continuous} \tag{4.9b}$$

The first two moments (corresponding to $r=1, 2$) add nothing to the information provided by μ and σ^2, since substituting $r=1, 2$, in Eqns (4.7)-(4.9a and b), we obtain the μ and σ^2.

We may similarly define moments for a random sample of n measurements:

$$m_r^{'} = \frac{1}{n}\sum_i n_i \cdot x_i^{'} \qquad r = 1,2,3,.... \tag{4.10}$$

And the rth moment around the mean $m_r^{'}$**:**

$$m_r^{'} = \frac{1}{n}\sum_i n_i \cdot \left(x_i^{'} - \overline{x}\right)^{'} \qquad r = 1,2,3,.... \tag{4.11}$$

and,

$$s^2 = m_2^{'} - \overline{x}^2 \tag{4.12}$$

Which is very useful in numerical calculations of sample variances (e.g. in measurements).

For continuous variables the mean $\overline{\mu}$ is defined by:

$$\int_{-\infty}^{\overline{\mu}} f(x) \cdot dx = \int_{\overline{\mu}}^{+\infty} f(x) \cdot dx = \frac{1}{2} \tag{4.13}$$

and also:

$$p\left[X \le \overline{\mu}\right] = p\left[X \ge \overline{\mu}\right] = \frac{1}{2} \tag{4.14}$$

In place of the standard deviation, σ, we sometimes use as an alternative measure of spread *the mean deviation*, defined by (in the continuous case),

$$\overline{\sigma} = \int_{-\infty}^{+\infty} |x-\mu| \cdot f(x) \cdot dx \tag{4.15}$$

Chebyshev's inequality [see [13]] can be derived easily.

4.3 Bivariate Distributions - Covariance

The probability distributions discussed till now are of one variable, but there are cases where the measurements require several random variables for their description and analysis. Most of the new requirements involved may be analyzed by considering the case of bivariate distributions, i.e. distributions relating to two random variables, X, Y.

In this case, the random variables (X, Y) are restricted to a *discrete set of possible values*, x_i, y_j, i=1,2,...; j=1,2,.... The set of numbers,

$$p_{ij}=p[X=x_i, Y=y_j] \tag{4.16}$$

is called the *discrete bivariate probability distribution* of (X, Y), and in all cases satisfies,

$$0 \le p_{ij} \le 1, \qquad \sum_i \sum_j p_{ij} = 1 \text{ for all i, j} \tag{4.17}$$

and in continuous case, when (X, Y) are continuous variables we define the bivariate probability density $f(x, y)$ by (**Eqn. 4.18**):

$$p\left\lfloor x < X \le x + \delta x, \quad y < Y \le y + \delta y \right\rfloor = f(x,y)\delta x \delta y$$

in all cases $f(x,y)$ satisfies the equations:

$$f(x,y) \ge 0, \quad \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y) \cdot dx \cdot dy = 1, \text{ for all } x,y.$$

and *Moments*:

$$\mu_{r,s}^{'} = E\left\lfloor X^r Y^s \right\rfloor = \sum_i \sum_j x_i^r \cdot y_j^s \cdot p_{ij} \qquad \textit{discrete} \tag{4.18a}$$

$$\mu_{r,s}^{'} = E\left[X^r Y^s\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^r \cdot y^s \cdot f(x,y) \cdot dx \cdot dy$$

for continuous case (**Eqns 4.19**), and in particular, the *covariance* (when $r = s = 1$) [[21], [[3], 12]]:

$$\mu_{11} = \text{cov}(X,Y) = E\left\lfloor (X-\mu_X)\cdot(Y-\mu_Y)\right\rfloor \Rightarrow$$

$$\Rightarrow \mu_{11} = \text{cov}(X,Y) = \sum_i \sum_j (x_i - \mu_X)\cdot(y_i - \mu_Y)\cdot p_{ij}$$

for discrete case and for continuous (**Eqns. 4.18b**):

$$\mu_{11} = \operatorname{cov}(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{+\infty}(x-\mu_X)\cdot(y-\mu_Y)\cdot f(x,y)\cdot dx\cdot dy$$

4.4 The Variance and Covariance Functions of a Stochastic Process

At the beginning of the present paragraph, we should quote [12] for General Random Excitation (§3) and Stationary Ergodic Processes (§4), in stochastic processes. Stochastic (random) process loosely means that the measured signal (or sample) looks different every time, when an experiment is repeated. However, *the process that generates the signal is still the same* [[3], 1].

For each t, $X(t)$ is a random variable and has a range of possible values, some of which may be more likely to take place than others. Accordingly, for each t, $X(t)$ will have some probability distribution. If $X(t)$ is a discrete random variable i.e. if its possible values form a discrete set $\{x_i\}$, its properties will be described by a discrete probability distribution, $P_t(x_i)$. However, in the more usual situation $X(t)$ will be a *continuous random variable* (with a continuous range of possible values), and its properties will then be described by its probability density function, $f_t(x)$. Consequently, unless the contrary is explicitly stated, we will assume that, for each t, $X(t)$ is a continuous random variable with probability density function $f_t(x)$ defined for all x, so that, for example, the mean and variance of $X(t)$ will be given by (**Eqn. 4.19**):

$$mean\left[X(t)\right] = E\left[X(t)\right] = \int_{-\infty}^{+\infty} x\cdot f_t(x)\cdot dx = \mu(t)$$

and,

$$\operatorname{var}\lfloor X(t)\rfloor = E\lfloor \{X(t)-\mu(t)\}^2 \rfloor =$$

$$= \int_{-\infty}^{+\infty}\left(x-\mu(t)\right)^2\cdot f_t(x)\cdot dx = \sigma^2(t) \qquad (4.20)$$

where, both $\mu(t)$ and $\sigma^2(t)$ are functions of t.

Definition: *The process $X(t)$ is said to be stationary up to order m if, for any admissible $t_1, t_2, \ldots, t_n$, and any k, all the joint moments up to order m of $E[\{X(t_1), X(t_2), \ldots, X(t_n)\}$ exist and equal the corresponding joint moments up to order m of $\{X(t_1+k), X(t_2+k), \ldots, X(t_n+k)\}$* [[21], 105].

Complete stationarity is, however, a severe requirement, and we therefore relax this by introducing the notion of "*stationarity up to order m*", which is a weaker condition but nevertheless describes roughly the same type of physical behaviour.

Consequently,

$$E\lfloor \{X(t_1)\}^{m_1}\{X(t_2)\}^{m_2}\ldots\ldots\{X(t_n)\}^{m_n}\rfloor = \qquad (4.21)$$

$$= E\lfloor \{X(t_1+k)\}^{m_1}\{X(t_2+k)\}^{m_2}\ldots\ldots\{X(t_n+k)\}^{m_n}\rfloor$$

for any k, and all positive integers $m_1, m_2, \ldots, m_n$, satisfying $m_1+m_2+\ldots+m_n<m$. In particular, setting $m_2=m_3=\ldots=m_n=0$, we have that, for any t and all $m_1<m$, and putting $k=-t$ we derive (**Eqn. 4.22**):

$$E\lfloor \{X(t)\}^{m_1}\rfloor = E\lfloor \{X(0)\}^{m_1}\rfloor = c \quad indepedent \quad of \quad t$$

Also, for any t, s and all m_1, m_2 satisfying $m_1+m_2 \le m$, we have:

$$E\lfloor \{X(t)\}^{m_1}\{X(s)\}^{m_2}\rfloor = E\lfloor \{X(0)\}^{m_1}\{X(s-t)\}^{m_2}\rfloor =$$

$$= f(s-t) \qquad (4.23)$$

that is a *function of (s-t)* only.

The following special cases are valid:

(i)-Stationarity up to order 1 ($m=1$): this implies that only $E[X(t)]=\mu$ is a constant independent of t.

(ii)- Stationarity up to order 2 ($m=2$), then the following are valid:

-$E[X(t)]=\mu$ is a constant independent of t.

-$E[X^2(t)]=\mu'_2$ is constant independent of t.

-$var[X(t)]=\mu'_2-\mu^2=\sigma^2$, also constant independent of t.

And for any t, s:

-E[X(t), X(s)]= *function of (t-s) only*, consequently, *$cov[X(t), X(s)]=E[X(t)\,X(s)]-\mu^2$ =function of (t-s) only*.

To summarize, if a process is stationary up to order 2, then:

(**a**) it has the same mean value, μ, at all time-instants;

(**b**) it has the same variance, σ^2, at all time-instants; and,

(**c**) the covariance between the values at any two time-instants, s, t, depends only on (s—t), the *time-interval* between the time-instants, and not on the location of the points (instants) along the time-axis.

At this point we should clarify that "Stationary" indicates that the statistical property/ies of a signal are constant in time. The properties of a stochastic signal are fully described by the joint probability density function of the observations (measurements). This density would give all information about the signal, if it could be estimated from the observations (measurements). Unfortunately, that is generally not possible without very much additional knowledge about the process that generated the observations (measurements). General characteristics that can always be estimated are the power spectral density that describes the frequency content of a signal and the autocovariance function that indicates how fast a signal can change in time. Estimation of spectrum or autocovariance is the main purpose of time series identification. This knowledge is sufficient for an exact description of the joint probability density function of normally distributed observations (measurements). For observations with other densities, it is also useful information. A time series is a stochastic signal with chronologically ordered observations at regular intervals. Time series appear in physical data, in economic or financial data, and in environmental, meteorologic and hydrologic data. Observations are made every second, every hour, day, week, month, year or every, e.g., 0,25 m [[3], 1].

4.5 The Autocovariance and Autocorrelation Functions

In a X(t) stationary function up to order 2, then:

(i)-for continuous processes:

$$\operatorname{cov}\left\lfloor X(t), X(t+\tau)\right\rfloor = E\left\lfloor \{X(t)-\mu\}\{X(t+\tau)-\mu\}\right\rfloor$$

From previously presented analysis we know that this quantity depends only on the value of τ and is independent of t, and we arrive at:

$$R(\tau) = E\left\lfloor \{X(t)-\mu\}\{X(t+\tau)-\mu\right\rfloor \quad (4.24)$$

For a continuous parameter process $R(\tau)$ is defined for all values of r and is called *the autocovariance function of X(t)*. For each τ, the function $R(\tau)$ measures the covariance between pairs of values of the same process separated by a time-interval of length τ, the quantity τ usually being termed the "lag". Now we can write for each τ [[21], [[3], 13]:

$$\rho(\tau) = \Phi_x(\Delta t) = \frac{R(\tau)}{R(0)} \quad (4.25a)$$

where:

$$R(0) = E\left\lfloor \{X(t)-\mu\}^2\right\rfloor = \operatorname{var}\{X(t)\} = \sigma^2 \quad (4.26)$$

The function $\rho(\tau)$ is called autocorrelation function of $X(t)$ and can be written:

$$\rho(\tau) = \Phi_x(\Delta t) = \frac{\operatorname{cov}\left\lfloor X(t), X(t+\tau)\right\rfloor}{\left[\operatorname{var}\{X(t)\}\operatorname{var} X(t+\tau)\right]^{1/2}} \quad (4.25b)$$

Hence, each pair τ, $\rho(\tau)$ represents the *correlation coefficient* between pairs of values of $X(t)$ separated by an interval of length τ. By engineer's intuition, we may interpret $\rho(\tau)$ as a measure of the "similarity" between a realization of $X(t)$ and the same realization shifted to the left by τ units.

As τ increases we would expect the correlation between $X(t)$ and $X(t+\tau)$ to decrease. If τ is large then in general the process will, loosely speaking, have "forgotten" at time $(t+\tau)$ the value it assumed at time t. Consequently, we would expect both $R(\tau)$ and $\rho(\tau)$ to decay to zero as $|\tau| \to\infty$. The typical form of an autocorrelation function is shown in Figure 2.

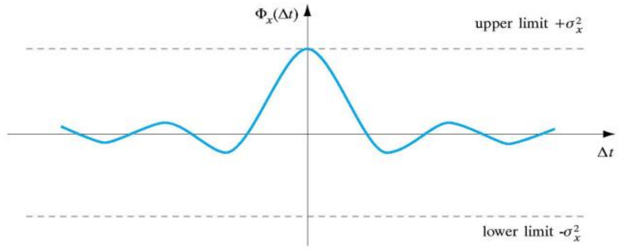

Figure 2. Typical auto-correlation function for stationary process with zero mean value (cf. [[13], 136], [12]).

(ii)-for discrete processes:
The autocovariance function R(τ) may be written (**Eqn. 4.26**):

$$R(\tau) = E\left[\{X_t-\mu\}\{X_{t+\tau}-\mu\right], \ \tau=0, \pm1, \pm2,..$$

and the autocorrelation function will be (Figure 3):
ρ(τ)=R(τ)/R(0), τ=0, ±1, ±2,...

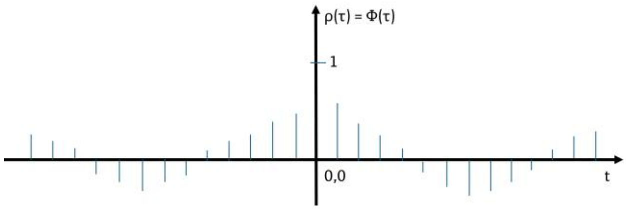

Figure 3. Auto-correlation function of a discrete process [cf. [[21], 109].

(iii)-for stationary processes:
If we use the definition of the *covariance of a stationary process x(t)*, namely the variance of the process in two time-instants t and $t+\tau$ (Eqn. (3.5.) where the mean value is represented either as μ or as $\bar{x}$) (**Eqn. 4.27**):

$$\operatorname{cov}\{x(t), x(t+\tau)\} = E[\{x(t)-\mu\}\{x(t+\tau)-\mu\}$$

5 Mathematical Modelling of the Railway System "Vehicle-Track"

A more-or-less complete model of the system "Railway-Vehicle" and "Railway Track" is depicted in Figure 4; in the present article we will approach the issue described in the title of this paragraph with the simplified model of Figure 5.

We try to evaluate the influence of the longitudinal vertical defects along the Railway Track on the vertical oscillations of a wheel based on this simplified model (Figure 5, a simplified model), but the development of the calculations are -in principle- as in the real model [cf. [7]]. The simulation of this model has: m_{NSM} the Non-Suspended (Unsprung)

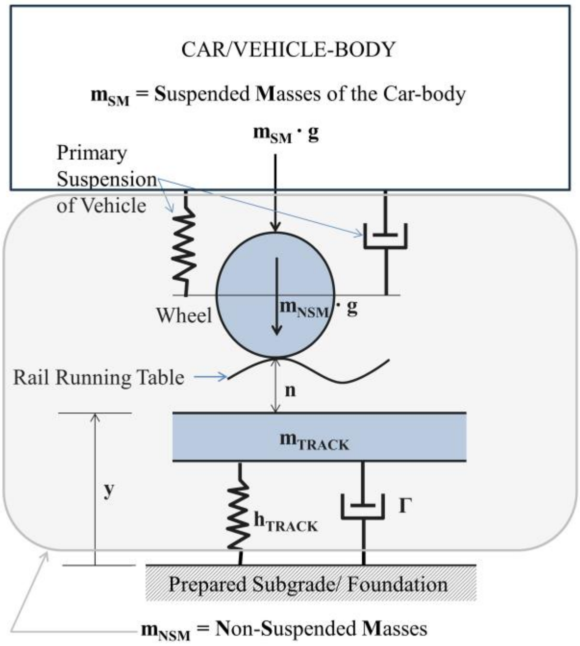

Figure 4. Schematic mapping of a vehicle/car on a Raiway Track: m_{NSM} the Non-Suspended Masses (under the primary suspension) of the vehicle (the not depicted secondary suspension is between the bogie-frame and the car-body); m_{TRACK} the mass of the track that participates in the motion of the Non-Suspended Masses (m_{NSM}); m_{SM} the Suspended Masses of the vehicle/car-body (above the primary suspension); Γ damping constant of the track; h_{TRACK} the total dynamic stiffness coefficient of the track; n the fault ordinate of the rail running table, and y the deflection of the track. The dynamic component is owed to the NSM and the SM; [cf. [12]].

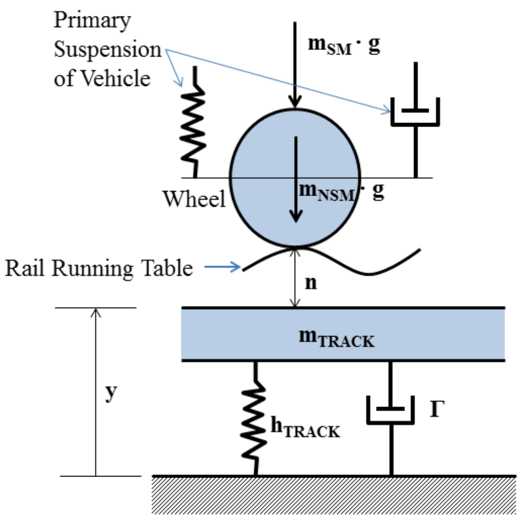

Figure 5. Simplified model of a rolling wheel (without the carbody) on the rail running table; the symbols of the coefficients are explained in the Legend of Figure 4 [cf. [8]].

Masses of the vehicle, m_{TRACK} the mass of the track that participates in the motion, m_{SM} the Suspended (Sprung) Masses of the vehicle that are cited above the primary suspension of the vehicle, Γ damping constant of the track, h_{TRACK} the total dynamic stiffness coefficient of the track (for its calculation see [13]), n the fault ordinate of the rail running table and y the total deflection of the track [8].

The equation for the interaction between the vehicle's axle and the track-panel becomes [[24], [9]]:

$$\left(m_{NSM}+m_{TRACK}\right)\cdot\frac{d^2y}{dt^2}+\Gamma\cdot\frac{dy}{dt}+h_{TRACK}\cdot y= \\ =-m_{NSM}\cdot\frac{d^2n}{dt^2}+\left(m_{NSM}+m_{SM}\right)\cdot g \quad (5.1)$$

In Figure 5 the rail running table depicts a longitudinal vertical fault/defect of the rail surface. In the above equation, the oscillation of the axle is damped after its passage over the defect. Viscous damping, due to the ballast, enters the above equation under the condition that it is proportional to the variation of the deflection dy/dt. To simplify the investigation, if we ignore the track mass (for its calculation see [13]) in relation to the much larger Vehicle's Non-Suspended Mass and bearing in mind that $y + n$ is the total subsidence of the wheel during its motion (since the y and n are added algebraically), we can approach the problem of the random excitation, from a cosine defect (provided that $V << V_{critical} = 500\ km/h$):

$$\eta = a\cdot\cos\omega t = a\cdot\cos\left(2\pi\cdot\frac{V\cdot t}{\lambda}\right) \quad (5.2)$$

Where V the speed of the vehicle, $T = 2\pi/\omega \rightarrow \omega t = 2\pi/(Tt) = 2\pi\bullet V\bullet t/\lambda$, where λ the length of the defect, run by the wheel in:

$$T=\frac{\lambda}{V}\Rightarrow\lambda=T\cdot V \quad (5.3)$$

If we set:

$$y=z+\frac{m_{SM}+m_{NSM}}{h_{TRACK}}\cdot g\Rightarrow\frac{dy}{dt}=\frac{dz}{dt} \quad and \quad \frac{d^2y}{dt^2}=\frac{d^2z}{dt^2}$$

where the quantity $\frac{m_{SM}+m_{NSM}}{h_{TRACK}}\cdot g$ represents the subsidence due to the static loads only, and z random (see [7]) due to the dynamic loads. Eqn. (5.1) becomes:

$$m_{NSM}\frac{d^2z}{dt^2}+\Gamma\cdot\frac{dz}{dt}+h_{TRACK}\cdot z=-m_{NSM}\cdot\frac{d^2n}{dt^2}\Rightarrow \quad (5.4a)$$

$$\Rightarrow m_{NSM}\left(\frac{d^2z}{dt^2}+\frac{d^2n}{dt^2}\right)+\Gamma\cdot\frac{dz}{dt}+h_{TRACK}\cdot z=0 \quad (5.4b)$$

Since, in this case, we are examining the dynamic loads only, in order to approach their effect, we could narrow the study of equation (5.4b), by changing the variable:

$$u=n+z\Rightarrow\frac{d^2u}{dt^2}=\frac{d^2n}{dt^2}+\frac{d^2z}{dt^2}$$

Equation (5.4) becomes:

$$m_{NSM}\frac{d^2u}{dt^2}+\Gamma\cdot\frac{dz}{dt}+h_{TRACK}\cdot z=0\Rightarrow \quad (5.5a)$$

$$\Rightarrow m_{NSM}\frac{d^2u}{dt^2}+\Gamma\cdot\frac{d(u-n)}{dt}+h_{TRACK}\cdot(u-n)=0 \quad (5.5b)$$

Where, u is the trajectory of the wheel over the vertical fault in the longitudinal profile of the rail.

If we apply the Fourier transform to Eqn. (5.4a) (see relevantly [23] for solving the second order differential equation with the Fourier transform):

$$(i\omega)^2\cdot Z(\omega)+\frac{\Gamma\cdot(i\omega)}{m_{NSM}}\cdot Z(\omega)+\frac{h_{TRACK}}{m_{NSM}}\cdot Z(\omega)=-(i\omega)^2\cdot N(\omega)\Rightarrow$$

$$H(\omega)=\frac{Z(\omega)}{N(\omega)},\ |H(\omega)|^2=\frac{m_{NSM}^2\cdot\omega^4}{\left(m_{NSM}\cdot\omega^2-h_{TRACK}\right)^2+\Gamma^2\cdot\omega^2} \quad (5.6)$$

H(ω) is a complex transfer function, *called frequency response function* [23], that makes it possible to pass from the *defect/fault n* to the subsidence *Z*. If we apply the Fourier transform to equation (5.5a):

$$(i\omega)^2\cdot U(\omega)+\Gamma\cdot(i\omega)\cdot Z(\omega)+h_{TRACK}\cdot(i\omega)^0\cdot Z(\omega)=0\Rightarrow$$

$$G(\omega)=\frac{U(\omega)}{Z(\omega)},|G(\omega)|^2=\frac{h_{TRACK}^2+\Gamma^2\cdot\omega^2}{m_{NSM}^2\cdot\omega^4} \quad (5.7)$$

G(ω) is a complex transfer function, *the frequency response function,* that makes it possible to pass from Z to $Z + n$.

If we name U the Fourier transform of u, N the Fourier transform of n, $p = 2\pi\bullet i\bullet v = i\omega$ the variable of frequency and $\hat{\Delta}Q$ the Fourier transform of ΔQ and apply the Fourier transform at equation (5.5b):

$$Eq.(7)\Rightarrow m_{NSM}\frac{d^2u}{dt^2}+\Gamma\cdot\frac{du}{dt}+h_{TRACK}\cdot u=\Gamma\cdot\frac{dn}{dt}+h_{TRACK}\cdot n\Rightarrow$$

$$\left(m_{NSM}\cdot p^2+\Gamma\cdot p+h_{TRACK}\right)\cdot U=\left(\Gamma\cdot p+h_{TRACK}\right)\cdot N\Rightarrow$$

$$U(\omega)=\underbrace{\frac{\Gamma\cdot p+h_{TRACK}}{m_{NSM}\cdot p^2+\Gamma\cdot p+h_{TRACK}}}_{B(\omega)}\cdot N(\omega) \quad (5.8a)$$

$$|B(\omega)|^2=\frac{\Gamma^2\cdot\omega^2+h_{TRACK}^2}{\left(m_{NSM}\cdot\omega^2-h\right)^2+\Gamma^2\cdot\omega^2} \quad (5.8b)$$

B(ω) is a complex transfer function, *the frequency response function,* that makes it possible to pass from the fault n to the $u = n + z$. Practically it is verified also by the equation:

$$|B(\omega)|^2=|H(\omega)|^2\cdot|G(\omega)|^2=\frac{h_{TRACK}^2+\Gamma^2\cdot\omega^2}{\left(m_{NSM}\cdot\omega^2-h_{TRACK}\right)^2+\Gamma^2\cdot\omega^2} \quad (5.8c)$$

passing from n to Z through *H(ω)* and afterwards from Z to $n + Z$ through *G(ω)*. This is a formula that characterizes the *transfer function between the wheel trajectory and the fault in the longitudinal level* and enables, thereafter, the calculation of the *transfer function* between *the dynamic load* and the *track vertical defect (fault)*. The transfer function of the second derivative of ($Z + n$) in relation to time:

$\frac{d^2(Z+n)}{dt^2}$, that is the acceleration γ, will be calculated below (and is equal to $\omega\bullet B(\omega)$). The increase of the vertical load on the track due to the Non-Suspended Masses, according to the principle *force = mass x acceleration*, is given by:

$$\Delta Q=m_{NSM}\cdot\frac{d^2u}{dt^2}=m_{NSM}\cdot\frac{d^2(n+Z)}{dt^2} \quad (5.9)$$

If we apply the Fourier transform to Eqn (5.9):

$$\hat{\Delta}Q=m_{NSM}\cdot p^2\cdot U(\omega)=m_{NSM}\cdot p^2\cdot\hat{f}_{Z+n}(\omega)\Rightarrow \quad (5.10a)$$

$$\left|\hat{\Delta}Q\right|=m_{NSM}\cdot|p|^2\cdot|B(\omega)|=m_{NSM}\cdot\beta^2\cdot\omega_n^2\cdot|B(\omega)|\cdot|N(\omega)| \quad (5.10b)$$

The transfer function *B(ω)* allows us to calculate the effect of a *spectrum of sinusoidal faults*, like the undulatory wear. If we replace $\omega/\omega_n = \rho$, where ω_n = *the circular eigenfrequency (or natural cyclic frequency) of the oscillation*, and:

$$\omega_n^2=\frac{h_{TRACK}}{m_{NSM}},\ \ \omega=\frac{2\pi V}{\lambda},\ \ 2\zeta\omega_n=\frac{\Gamma}{m_{NSM}},\ \ \beta=\frac{\omega}{\omega_n}$$

where, ζ is the damping coefficient. Equation (10b) is transformed:

$$|B(\omega)|^2=|B_n(\beta)|^2=\frac{1+4\zeta^2\cdot\beta^2}{\left(1-\beta^2\right)^2+4\zeta^2\cdot\beta^2} \quad (5.11)$$

6 Spectral Density in Random Processes: Mathematical Calculations

We saw [[12], §3] that periodic and certain types of non-periodic functions could be expressed as sums or integrals of *sine* and *cosine* terms with different amplitudes and frequencies. The importance of this so-called "spectral representation"' of a function lies in the fact that if the function represents some physical process, the total energy dissipated by the process in any time interval is equal to the sum of the amounts of energy dissipated by each of the *sine* and *cosine* terms. The energy carried by a *sine* or *cosine* term is proportional to the square of the amplitude. Consequently, in the case of periodic processes, the contribution of a term of the form ($a_r \bullet cos\omega_r x$ + $b_r \bullet sin\omega_r x$) to the total energy of the process is proportional to $(a_r^2 + b_r^2) = |A_r|^2$. If, therefore, we plot the squared amplitudes $|A_r|^2$ against the frequencies ω_r, the graph we obtain shows the relative contribution of the various *sine* and *cosine* terms to the total energy. For the moment, we will call this type of graph simply an ‘‘energy spectrum”, while, in fact, it would be more precise to describe it as a “power spectrum” [[21], 8-9], as we will analyze below.

The Power spectral density function S_x is of great importance to the analysis of stochastic processes. Figure 6 illustrates characteristic cases of spectral density of random functions.

In the theory of stochastic systems, the spectra are linked to Fourier transforms. For deterministic systems the spectra and the Fourier transformation are used to represent a function as superposition of exponential functions. For random systems (or signals) the concept of spectrum has two interpretations.

a.-The first one includes transforms of averages, and is essentially deterministic.

b.-*The second one* ***includes the representation of the (random) process as a superposition of exponential functions (namely of a sum of infinite sine and cosine functions) with random coefficients***.

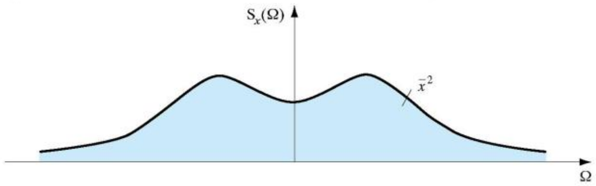

Figure 6. Spectral Density or Power Spectrum $S_x(\Omega)$ and mean square value (average value $\overline{x^2}$) (cf. [[13], 138]).

The *power spectrum* or *spectral density* of a stochastic system that is described by a function *x(t)* is *the Fourier transform of the system’s normalized* (see Eqn. 4.25a) *autocorrelation* $\Phi_x(t) = \rho(\tau)$ [[[26], 4], [[21], 216], [[13], 175-176]]. The knowledge of the power spectrum of {*X*} is important if we wish to compute optimal predictors for random processes [[21], 26]. S(Ω) represents the distribution of sequence energy as a function of frequency [[26], 4].

If *x(t)* represents the excitation and since a stochastic process, at least theoretically, can last indefinitely, it is not a prerequisite that the following equation will apply:

$$\int_{-\infty}^{+\infty} \left|x(t)\right| \cdot dt < +\infty \qquad (6.1)$$

even if the mean value $\bar{x} = 0$.

There is greater possibility that $\Phi_x(\Delta t)$ will be finite, that is the absolute value of $\Phi_x(\Delta t)$, which is the area below the curve (Figure 6). From the continuity of curve *x(t)*, and for small Δt, it can be concluded that both *x(t)* and $x(t+\Delta t)$ have the same sign and therefore $\Phi_x(\Delta t)$ must increase with time. There is no predictable behaviour or relationship in a random process between *x(t)* and $x(t+\Delta t)$ for great values of Δt. $\Phi_x(\Delta t) \rightarrow 0$ for great values of Δt, because contradictive values may arise in this case.

Let us consider a non-periodic auto-correlation function $\Phi_x(\Delta t)$ that satisfies the equation (5.1). When this equation is valid it allows unlimited use of the Fourier integral (with no restrictions) for the mapping of any function *x(t)*. To calculate this function we can apply the integral for the calculation of the coefficients of the Fourier series [[13], 121, Eqn. 393],

$$f_i(\omega) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} P_i(t) \cdot e^{-i \cdot \omega \cdot t} \cdot dt \qquad (6.2)$$

which is the frequency spectrum of the excitation for non-random processes ([[4], 121-122], [[4], pp. 319]) and we get:

$$\sigma^2(x) = \Phi_x(0) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} S_x(\omega) \cdot d\omega = \overline{x}^2 \qquad (6.3.a)$$

where:

$$S_x(\omega) = \int_{-\infty}^{+\infty} \Phi_x(\Delta t) \cdot e^{-i\omega\Delta t} \cdot d(\Delta t) \qquad (6.3.b)$$

$S_x(\omega)$ is the spectral density function. Furthermore, $S_x(\omega)$ is the Fourier transform of the function $\Phi_x(\Delta t)$ since $\Phi_x(\Delta t)$ is the inverse Fourier transform of $S_x(\omega)$. It should be noted that, if $\Phi_x(\Delta t)$ is defined as $\Phi_x(\Delta t) = E[x(t_1) \cdot x(t_1+\Delta t)]$, then the factor $1/2\pi$ before the integral does not exist in the Eqn. (5.3.a) but in (5.3.b) (see [[2], 269]).

The *Wiener-Khinchin theorem* [[13], 237] is derived from (3.9) and (4.3.a) of [12]:

$$\Phi_x(\Delta t) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} S_x(\omega) \cdot e^{i\omega\Delta t} d\omega = E\left[x(t+r) \cdot x(t)\right] \qquad (6.4)$$

The shaded area below the spectral density function (Figure 6) *represents the mean square value of the process*.

Using complex numbers in (5.3.b) and the Euler equation for complex numbers, since the imaginary part is eliminated, because $\Phi_x(\Delta t)$ is symmetrical and $sin(\omega\Delta t)$ is anti-symmetrical with respect to $\Delta t = 0$ and as a result the areas below the anti-symmetrical integral cancel one another out [[2], 271], we get:

$$S_x(\omega) = \int_{-\infty}^{+\infty} \Phi_x(\Delta t)\cdot\cos(\omega\Delta t)\cdot d(\Delta t) -$$

$$-i\cdot\int_{-\infty}^{+\infty} \Phi_x(\Delta t)\cdot\sin(\omega\Delta t)\cdot d(\Delta t) =$$

$$= \frac{1}{2\pi}\cdot\int_{-\infty}^{+\infty} \Phi_x(\Delta t)\cdot\cos(\omega\Delta t)\cdot d(\Delta t) \quad (6.5)$$

It can be easily proved that:

$$S_x(\omega) = S_x(-\omega) \quad (6.6)$$

The spectral density function is real, it does not contain an imaginary part and is symmetrical around position ω = 0, as the autocorrelation function also is. The Spectral Density is the Fourier Transform of the autocovariance function [[[21], 210-215, 216, 219]; [[15], 7]].

The excitation (*rail irregularities*), in reality, is random and neither periodic nor analytically defined, like the Eqn. (4). It can be defined by its autocorrelation function in space and its spectral density [[[1], 58], [[7], 700], [19]]. If $f(x)$ is a signal with determined total energy and $F(v)$ its Fourier transform, from Parseval's modulus theorem [27], the total energy is [23]:

$$\int_{-\infty}^{+\infty} \left|f(x)\right|^2\cdot dx = \int_{-\infty}^{+\infty} \left|F(v)\right|^2\cdot dv \quad (6.7a)$$

where, $F(v) = A(v)\bullet e^{i\varphi(v)}$ and the *power spectral density*:

$$S(\omega) = \left|F(v)\right|^2 = A^2(v) \quad (6.8)$$

Reference [23] solves equation (14a) as:

$$\int_{-\infty}^{+\infty} f(t)^2\cdot dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)^2\cdot d\omega \quad (6.7b)$$

The square of the modulus $F(\omega)$ is called the energy spectrum of the signal because $F^2(\omega)\bullet\Delta(\omega)$ represents the amount of energy in any ΔΩ segment of the frequency spectrum, and the integral of $F^2(\omega)$ over (-∞, +∞) gives the total energy of the signal. An input signal -*like the running rail table*- creates through the vehicle an output signal: the wheel trajectory. The output spectral density and the input spectral density of the excitation are related through equation [[5], [13]]:

$$S_{out}(\bar{\omega}) = \left|H(i\bar{\omega})\right|^2\cdot S_{in}(\bar{\omega}) \quad (6.8a)$$

In order to relate the temporal spectrum with the spectrum in space we use the following equation:

$$\omega\cdot t = \frac{2\pi Vt}{\lambda} \Rightarrow \omega = \frac{2\pi}{\lambda}\cdot V \Rightarrow \omega = \Omega\cdot V \quad (6.8b)$$

where λ is the wavelength of the defect. This means that *circular frequency in space Ω* is the wave number k of the equation of oscillation, and [23]:

$$\int_0^{\infty} S(\Omega)\cdot d\Omega = \int_0^{\infty} s(\omega)\cdot d\omega, \quad (6.9)$$

$$\mathbb{F}\left[f(ax)\right] = \frac{1}{|a|}\cdot\hat{f}\left(\frac{v}{a}\right) \Rightarrow S(\omega) = S\left(\frac{\Omega}{V}\right) = \frac{1}{V}\cdot S(\Omega)$$

where $\mathbb{F}$ is the symbol for the application of the Fourier transform of f and $\hat{f}$ the function after the transform. *This is a property of the Fourier transform*.

7 Variance and the Spectrum of the (Real) Vertical Defects of the Track: Mathematical Analysis

The Variance or mean square value $\sigma^2(x)$ of the function is given by (6.3a) (cf. [20], [25])

$$\sigma^2(x) = \frac{1}{2\pi}\cdot\int_{-\infty}^{+\infty} S(\omega)\cdot d\omega = \bar{x}^2 \quad (7.1)$$

and $\sigma(x)$ is the *standard deviation* of the function.

The Power Spectral density and the variance of a function are depicted in Figure 6.

From equation (7.1 17) we derive:

$$\sigma^2(n) = \frac{1}{\pi}\int_0^{+\infty} S_n(\omega)\cdot d\omega,\ \sigma^2(z) = \frac{1}{\pi}\int_0^{+\infty} S_z(\omega)\cdot d\omega \quad (7.2a)$$

$$\sigma^2(\Delta Q) = \frac{1}{\pi}\int_0^{+\infty} S_{\Delta Q}(\omega)\cdot d\omega \quad (7.2b)$$

where n is the random variable of the defect (input), z the subsidence of the wheel (output) and ΔQ the dynamic component of the Load that is added to the Static Load of the wheel due to the Non-Suspended Masses (output also).

From these equations and the analytic form of the spectrum of the vertical defects/faults, we can calculate the mean square value of the dynamic component of the Load due to the Non-Suspended Masses that is added to the Static Load of the wheel

From the power spectral density and the variance functions (the Spectral Density is the Fourier

Transform of the autocovariance function) and their definitions [13]:

$$S_{\Delta Q}(\omega) = S_n(\omega)\cdot\left|B(\omega)\right|^2 \quad (7.3)$$

$$\Delta Q = m_{NSM}\cdot\Delta\gamma \Rightarrow \sigma^2(\Delta Q) = m_{NSM}\cdot\sigma^2(\gamma) \Rightarrow$$
$$\Rightarrow \sigma^2(\gamma) = \frac{\sigma^2(\Delta Q)}{m_{NSM}}$$

and using the Eqns. (7.3 19), (5.9) and (5.10b) 11-12b):

$$\sigma^2(\gamma) = \frac{1}{m_{NSM}}\cdot\sigma^2(\Delta Q) = \frac{1}{m_{NSM}^2\cdot\pi}\cdot\int_0^{+\infty}\left|B(\omega)\right|^2\cdot S_n(\omega)\cdot d\omega \quad (7.4)$$

$$\sigma^2(\gamma) = \frac{1}{m_{NSM}}\cdot\sigma^2(\Delta Q) = \frac{1}{m_{NSM}^2\cdot\pi}\cdot\int_0^{+\infty}\left|B(\omega)\right|^2\cdot S_n(\omega)\cdot d\omega$$

$$\sigma^2(\gamma) = \frac{1}{\pi}\cdot\int_0^{+\infty}\beta^4\cdot\omega_n^4\cdot\frac{1+4\zeta^2\cdot\beta^2}{\left(1-\beta^2\right)^2+4\zeta^2\cdot\beta^2}\cdot S_n(\omega)\cdot d\omega \quad (7.5)$$

From the above equations and the analytical form of the spectrum of the longitudinal vertical defects of the track we could effectively calculate the variance (mean square value) of the dynamic component of the Loads on the track panel due to the Non-Suspended Masses. All the results of measurements on track in the French railways network show that the *spectrum of vertical defects* in the longitudinal sense has the form [[24], [22]]:

$$S_n(\Omega) = \frac{A}{(B+\Omega)^3} \quad (7.6)$$

This implies that the mean square value or variance of the defects is given by:

$$\sigma^2(z) = \frac{1}{\pi}\cdot\int_0^{+\infty}\frac{A}{(B+\Omega)^3}\cdot d\Omega = \frac{A}{\pi}\cdot\int_0^{+\infty}\frac{1}{(x)^3}\cdot dx = -\frac{A}{2\pi}\left[\frac{1}{(B+\Omega)^2}\right]_0^{+\infty} \Rightarrow$$

$$\sigma^2(z) = -\frac{A}{2\pi}\cdot\left[\frac{1}{B^2+2B\Omega+\Omega^2}\right]_0^{+\infty} = -\frac{A}{2\pi}\left[0-\frac{1}{B^2}\right] \Rightarrow$$

$$\sigma^2(z) = \frac{1}{2\pi}\cdot\frac{A}{B^2} \quad (7.7)$$

If we examine only the much more severe case, for the case of the Non-Suspended Masses, of the defects of short wavelength, consequently large *Ω* – like the undulatory wear– then we can omit the term *B*, and, using Eqn. (6.8b 15b):

$$S_n(\Omega) = \frac{A}{\Omega^3} = \frac{A}{\frac{1}{V^3}\cdot\omega^3} = \frac{A\cdot V^3}{\omega^3} \quad (7.8)$$

The term *B* characterizes the defects with large wavelengths, for which the maintenance of track is effective, and when we examine this kind of defects term *B* should be taken into account.

From equations (6.9 16) and (7.8 24):

$$S_n(\omega) = \frac{1}{V}\cdot S(\Omega) = \frac{1}{V}\cdot\frac{A\cdot V^3}{\omega^3} = \frac{A\cdot V^2}{\omega_n^3\cdot\beta^3} \quad (7.8)$$

8 Input-Output Spectral Densities in Relation to the Recordings of the Measurements

8.1 Spectral Densities

If we have the system "Railway Vehicle-Railway Track" then the condition and the position in space of the rail running table is the excitation (Input) and the movement of the vehicle is the response (Output).

From Eqn. (6.4.15) [2024] we have (**8.1**):

$$S_u(\omega) = S_p(\omega)\int_{-\infty}^{+\infty}h(\theta_1)\cdot e^{-i\omega\cdot\theta_1}\cdot`d\theta_1\cdot\int_{-\infty}^{+\infty}h(\theta_2)\cdot e^{i\omega\cdot\theta_2}\cdot d\theta_2$$

that is, the Spectral Density of the Response (of the Track Recording Car too) $S_u(\omega)$ is related to the Spectral Density of the input (excitation, namely the track defects) by the Eqn. 8.1) $S_p(\omega)$ with the response $h(t-\overline{t})$ of the response to the unitary Dirac Impulse and its integral, since [[12], Eqn. 6.4.12-13)] the mean value of the response is:

$$\overline{u} = E[u] = E\left[\int_{-\infty}^{+\infty}h(\theta)\cdot P_i(t-\theta)\cdot d\theta\right] =$$
$$= \int_{-\infty}^{+\infty}h(\theta)\cdot E\left[P_i(t-\theta)\right]\cdot d\theta =$$
$$= \int_{-\infty}^{+\infty}h(\theta)\cdot\overline{P_i}\cdot d\theta = \overline{P_i}\int_{-\infty}^{+\infty}h(\theta)\cdot d\theta \quad (8.2)$$

Using Eqns. (7.6) and (8.1), we find Eqn. (**8.3**) below:

$$S_u(\omega) = H(-i\omega)\cdot H(i\omega)\cdot S_p(\omega) = \left|H(i\omega)\right|^2\cdot S_p(\omega)$$

where frequency ν has two parts one real and one imaginary.

However, *the spectral density function is real, it does not contain an imaginary part* (see [[12], paragraph 4]); the Spectral Density of the response (output) $S_u(\omega)$ is related to the Spectral Density of the excitation (input) $S_p(\omega)$ with a real number $|H(i\omega)|^2$. Consequently, if we find or measure the *Spectral Density of the input* we can calculate the *Spectral density of the output*.

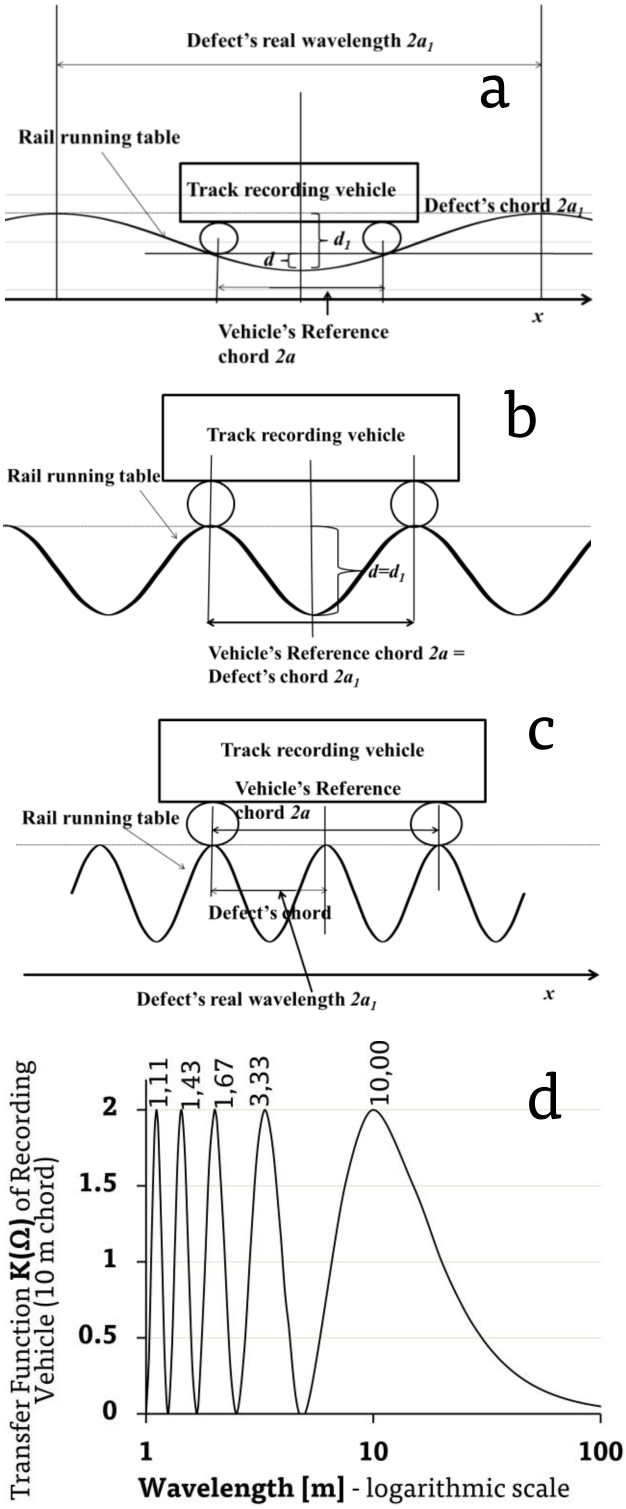

Figure 7. Measurement of a longitudinal vertical defect at a position x_i with a "*2a* long" chord/basis of measurements of the Recording Car and reliability of the recordings of the measurements: (**a**) defect's wavelength much longer than the measuring chord of the Car, (**b**) defect's wavelength equal to the measuring chord, (**c**) defect's wavelength shorter than the measuring chord and (**d**) The Recording Car's Transfer Function K(Ω) [cf. [11]].

In the case of the Track defects, it should be clarified that different wavelengths address different vehicles' responses depending on the length of the cord/base of measurement. This is of decisive importance for the wavelengths of 30 – 33 m, which are characteristic for very High Speed Lines ([1]=2022, [2]=2023). In the real tracks, the forms of the defects are random with wavelengths from few centimeters until 100 m. The defects constitute the "Input" in the system "Vehicle-Track" since the deflection y and the Action/Reaction R of each support point of the rail (and sleeper) are the "Output" or "Response" of the system. The accuracy of the measurements of the defects is of utmost importance for the calculation of the deflection y and the Reaction R; this accuracy, due to the bases of the measuring devices/vehicles, is fluctuating. Thus, we should pass from the space-time domain to the frequencies' domain through the Fourier transform, in order to use the power spectral density of the defects, especially for defects, with (long) wavelength, larger than the measuring base of the vehicle.

In the case of random defects then we do not use the function f(x) but its Fourier transform:

$$F(\Omega) = \int_{-\infty}^{+\infty} f(x) \cdot e^{-i\Omega x} \cdot dx \qquad (8.4)$$

In practice we don't know the function of real defects y(x) but the measured values f(x), from the recording vehicle [7]; we imply that [cf. [[19], 312]:

$$S_Z(\Omega) = S_{INPUT}(\Omega) = \frac{S_F(\Omega)}{\left|K(\Omega)\right|^2} = \frac{S_{OUTPUT}(\Omega)}{\left|K(\Omega)\right|^2} \qquad (8.5)$$

where $S_Z(\Omega)$ is the spectral density of the Fourier transform of the real defects (input in the track recording vehicle), $S_F(\Omega)$ is the spectral density of the Fourier transform of the measured values (output) and $K(\Omega)$ is a complex transfer function (of the Recording Vehicle/Car), called *frequency response function*, transforming the measured values of defects to the real values. For very High Speed Lines we should analyze the system "railway track – railway vehicle". The calculation of the spectrum of track defects is described in [[[20], §6] and [[28], 155-158].

Before we continue, please see Figure 7 for an example of a *Transfer Function* (let's name it *K(Ω)*) of the -suspension of a- *Track Recording Car* with (an ideal) *measurement basis of 10m* [10]. In real conditions the Track Recording Cars have more complicated -but fixed- measuring systems and, each time, the *Transfer Function of each Vehicle* should be mathematically calculated. We can explain the above (Eqns 8.5 and 7.8) in more details now: since we know -each time- the *Transfer Function* of any *Track Recording Car* and the spectrum of the longitudinal vertical defects along the Track (*as these defects have been recorded by it*), we can determine the *variance* of these defects and also the variance of the Dynamic Component of the acting Load of the wheel on the Track, due to these defects and the *probability of occurrence* of this Dynamic Component [[7], 7].

We can describe also this relation as [[21], 23]:

[power spectrum of the input] =
= [power spectrum of the output] /
/ [squared modulus of the transfer function].

A question arises: *How does Figure 7-Lower result*?

8.2 Measurements by Track Recording Cars - Calculating the Car's Transfer Function

Every point on each rail (at the rail running table) of a track (Figure 7-Upper (three schemas)) can be defined by its three coordinates: at the position x and the functions y(x), in the horizontal alignment, and z(x), in the vertical alignment. The measurements of the vertical defects are performed with track geometry cars, the contact cars, which are made by actual contact with the rails, movable feeler points (transducers) that touch the rails to measure the parameters, e.g. the profile. The cars use the position of the carbody, its yaw and roll and, consequently, their axles as the reference base for a relative measurement [[14], 678-679]. We examine the vertical defect, the dip between two bumps, of an oscillograph recording [[14], 684], with a reference chord of length *2a* (normally 10 m) and the reliability of the measurements. The length of d at the position x is:

$$d = f(x) = \frac{1}{2}\left[z(x-a) + z(x+a)\right] - z(x) \quad (8.6)$$

This transformation cannot be easily reversed: If we know *z(x)* in every *x* easily *f(x)* can be derived, but if we know *f(x)* it is not easy to calculate *z(x)*. Thus we try to approach the matter for the case of a sinusoidal defect: $z(x) = b\cdot\sin\frac{2\pi}{\ell}x$, we derive:

$$f(x) = \frac{1}{2}\cdot\left[b\cdot\sin\frac{2\pi}{\ell}(x-a) + b\cdot\sin\frac{2\pi}{\ell}(x+a)\right] - b\cdot\sin\frac{2\pi}{\ell}x =$$
$$= \frac{1}{2}\cdot b\cdot\sin\frac{2\pi}{\ell}(x-a) + \frac{1}{2}\cdot b\cdot\sin\frac{2\pi}{\ell}(x+a) - b\cdot\sin\frac{2\pi}{\ell}x =$$
$$= \frac{1}{2}\cdot b\cdot(\sin\frac{2\pi}{\ell}x\cdot\cos\frac{2\pi}{\ell}a - \cos\frac{2\pi}{\ell}x\cdot\sin\frac{2\pi}{\ell}a + \sin\frac{2\pi}{\ell}x\cdot\cos\frac{2\pi}{\ell}a +$$
$$+\cos\frac{2\pi}{\ell}x\cdot\sin\frac{2\pi}{\ell}a) - b\cdot\sin\frac{2\pi}{\ell}x \Rightarrow$$

$$\Rightarrow f(x) = -b\cdot\sin\left(\frac{2\pi}{\ell}x\right)\cdot\left(1-\cos\frac{2\pi}{\ell}a\right) =$$
$$= -z(x)\cdot\underbrace{\left(1-\cos\frac{2\pi}{\ell}a\right)}_{K(\Omega)} = -z(x)\cdot K(\Omega) \quad (8.7)$$

with Ω=2π/ℓ. *K(Ω)=1-cos[(2π/ℓ)•a]* is a real function and it is the *Transfer Function*, permitting to pass from *z(x)* to *f(x)*. If the chord *2a* is the chord used as base from the track recording vehicle then *K(Ω)* is the transfer function of the vehicle. The track recording vehicles measure the defects, the track displacement, "under load", that is under their axle load, which is usually smaller than the maximum axle load of the line but enough for the measurement of the gaps under the seating surface of the sleepers, if any. Normally the chord used as reference base both for the vertical and horizontal defects is the 10 m length and the axles of the vehicle are used for that. Regarding the reliability of the measurements, three cases are distinguished:

(**a**) The vehicle's reference base *2a* (chord of 10 m) is smaller than the defect's wavelength $\ell=2a_1$ (Figure 7–a). In that case the measured defect's ordinate *f(x)=d* is much smaller than the real vertical defect's ordinate $z(x)=d_1$.

(**b**) The chord *2a* is equal to the defect's wavelength $\ell=2a_1$, and $f(x)=d=z(x)=d_1$ (Figure 7-b).

(**c**) The chord *2a* is larger than the defect's wavelength $\ell=2a_1$, with the reliability fluctuating. The most characteristic case happens when the defect's wavelength ($2a_1$) equal to ½ of the chord's length (*2a*) and the measured ordinate *f(x)=d=0* instead of the real defect's ordinate $z(x)=d_1$ (Fig. 7-c).

(**d**) In order to approach the matter of the reliability of the measured values, by the track recording vehicle, we examine its transfer function *K(Ω)* (Figure 7-d), presenting minimums, zero, for:

$$\frac{2\pi a}{\ell} = 2k\pi \Rightarrow \ell = \frac{a}{k} \quad (8.8)$$

with k integer and maximums equal to 2 for:

$$\frac{2\pi a}{\ell} = (1+2k)\cdot\pi \Rightarrow \ell = \frac{2a}{1+2k} \quad (8.9)$$

and in the case of a reference base (chord) of 10 m, that is a=10 m, then the values of K(Ω):

$$\frac{f(x)}{z(x)} = \frac{d}{d_1} = 0, \tag{8.10}$$

$$for \quad \ell = 5m, \; 2.5m, \; 1.67m, \; 1{,}25m, \; 1{,}00m \quad and$$

as depicted in Figure 7-d. The 10 m chord is very important because it includes the wavelengths of the vehicles' hunting, but for larger wavelengths the measured values *f(x)=d* are smaller than the real ordinates *z(x)=*d_1 (Figure 7-d). It should be clarified that different wavelengths address different vehicles' responses depending on the cord (different from 10 m). This is of decisive importance for the wavelengths of 30 – 33 m, that are characteristic for very High Speeds [[[1], 30], [18], [[6], 342]]: for 30 m, K(Ω)=0.5 and for 50 m K(Ω)=0.2. In any case, for each vehicle the reference chord should be used instead of the 10 m chord. *The table of the values up to 100m are presented in **ANNEX 1**, at the end of this article*.

We should underline that the above arise from mathematical analysis; the reason for that is presented in the next paragraph 9. The calculation of the Spectral Density of the real vertical defects of Track (input), which is a continuous (unknown) function and results from the measurement performed by a Track Recording Car, can be accurately calculated after the calculation of the Spectral Density of the (known) measurements performed by the Track Recording Car (output).

9 Physical Meaning of the Fourier Transform or Why do We Use the Fourier Transform?

9.1 Physical Meaning of Fourier Transform

Figure 8 presents the real Physical Meaning of the Fourier Transform of one *function y = f(x)* at the space-time domain: e.g. the deflection of Track under the Load of a circulating wheel *y = f(x)*, where *x* is defined as the distance between the beginning of the axis *x* (the beginning of the measurements $x_0 = 0$ at time *t = 0*) and $x_i = V \bullet t_i$, at $t = t_i$, with $y_i = f(x_i)$, with the speed *V* considered constant. The random function y = f(x) is decomposed, as it is described in the paragraph 3 above, in an infinite sum of infinite series of harmonic functions.

In the case of a function *f(x, t)*, its Fourier Transform is:

$$F(\nu) = \int_{-\infty}^{+\infty} f(x,t) \cdot e^{-2\pi i \nu t} dt \tag{9.1}$$

where, *F(v)* and *f(x, t)* represent the same physical quantity, but in a different representation. If we consider *f(x, t)*, the representative point moves in the (space-time) domain. If we consider *F(v)*, the representative point moves in the frequencies domain.

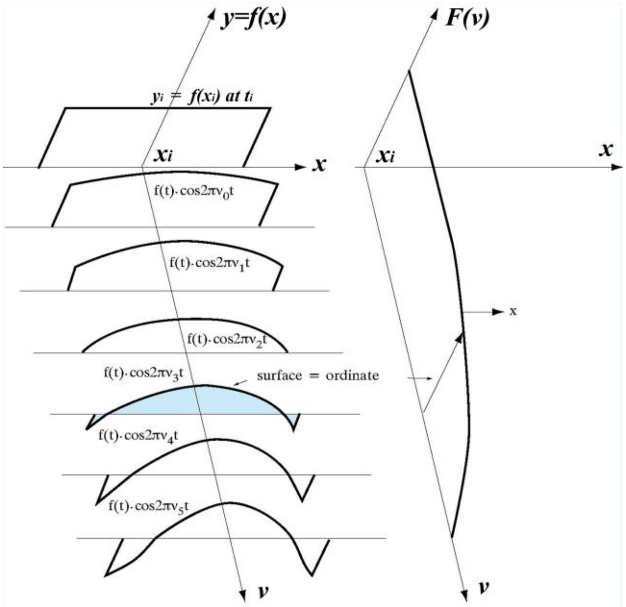

Figure 8. *Physical meaning of the Fourier transform*: (Left) at the plane *x, y* (*space-time domain*), the random function *f(y)* -a function of *x* which is dependent on time (e.g.: $x_i = V \bullet t_i$)-, at a time-instant t_i, *x* has a value x_i *at* t_i and $y=f(x_i)$, while in the -perpendicular to the *x, y plane*- axis of frequencies *v*, the function *y = f(x)* can be "decomposed" in an infinite sum of harmonic functions. (Right) The Fourier Transform transforms the *time-space function y=f(x)* to a function in the frequencies domain: *F(v)* (*the Fourier Transform of the function f(y)*) and *v*; the surface (area) of each function $f(t) \cdot cos2\pi v_i t$ of the left schema, which is shown sliced at each frequency v_i, is the ordinate of the right schema at the position of each frequency vi in the frequency domain plane *F(v), v* [By the author, *who updated the Figures in* [[13], 160], [[12], p.13, Fig. 10]; ***cf.*** [[17], I:23], [[4], 19-20]].

Figure 8 depicts the mapping -*in the space-time domain*- of the function *y = f(x)* where *x* is dependent on time *t* and its Fourier Transform *F(v)* in the frequencies domain and the relation between the two graphs: at the plane *x, y* (*space-time domain*, Figure 8-Left), the random function *f(y)* -a function of *x* which is dependent on time (e.g.: $x = V \bullet t$)-, at a time-instant t_i, *x* has a value x_i *at* t_i and $y=f(x_i)$, while in the -perpendicular to the *x, y plane*- axis of frequencies *v*, the function *y = f(x)* can be "decomposed" in an infinite sum of harmonic functions. The Fourier Transform transforms the *time-space function y=f(x)* to a function in the frequencies domain (Figure 8-Right): *F(v)* (*the Fourier Transform of the function f(y)*) and *v*; the surface (area) of each function $f(t)\cdot cos2\pi v_i t$ of the left schema, which is shown sliced at each frequency v_i, is the ordinate of the right schema at the position of each frequency vi in the frequency domain plane *F(v), v*.

Figure 8 clearly presents that if we know the Spectral Density of the defects of the Track, we know the Power of the signal at each position x_i, for each frequency v_i.

When we search for the value of *F(v)* for a value v_i of *v*, this means that we are searching through the entire history (and future) of *f(x, t)* for what corresponds to the frequency v_i. This corresponds to "infinitely selective filtering" (in French "*filtrage infiniment selectif*"). Such filtering is not physically feasible [[[17], I:12-13, see also chapt.VI. 7]; cf. [[21], 270, 274]]. Therefore, we cannot know *F(v)* with perfect location on the frequency axis. Similarly, if we want to find *f(x, t)* from *F(v)*, we must know the spectrum for all frequencies up to infinity, and the formula shows that the same infinitely selective filtering operation (see below) is involved, with the time and frequency variables being swapped. This means that to perfectly know the value of *f(x, t)* at a time t_i, one must have an *infinite bandwidth*. All of this is simply another form of *the* (Heisenberg's) *uncertainty principle* that expresses the impossibility for the human observer to grasp reality without distorting it or making it somehow "*fuzzy*" [cf. [[17], I:12-13]; [[4], 19-20]].

9.2 Linear Systems

Any system can be viewed as a transducer (Figure 9-upper), with the cause *f(t)* as its input and the effect *g(t)* as its output or response; *g(t)* is uniquely determined in terms of *f (t)*.

$$g(t)=T\lfloor f(t)\rfloor \qquad (9.2)$$

A system is called linear if: with $g_1(t)$ the output to $f_1(t)$, $g_2(t)$ the output to $f_2(t)$, and a_1 and a_2 two arbitrary constants, the output to $a_1f_1(t) + a_2f_2(t)$ is given by $a_1g_1(t) + a_2g_2(t)$. Using the notation of (9.2) and introducing L for linearity, we can express the above definition by (**Eqn. 9.3**):

$$L\lfloor a_1\cdot f_1(t)+a_2\cdot f_2(t)\rfloor=a_1\cdot L\lfloor f_1(t)\rfloor+a_2\cdot L\lfloor f_2(t)\rfloor$$

The importance of the Fourier integral in the analysis of linear systems is due to the fact that, if the input is an exponential $e^{i\omega t}$, then the output is also an exponential proportional to the input.

$$L\lfloor e^{i\omega t}\rfloor=k\cdot e^{i\omega t} \qquad (9.4)$$

If the Dirac Impulse is the input the System's response is h(t) [[12], §5]. The Fourier Transform of h(t) of a Linear System (figure 9-Lower) is called *System Function* (**Eqn.9.5**):

$$H(\omega)=\int_{-\infty}^{+\infty}h(t)\cdot e^{-i\omega t}\cdot dt=R(\omega)+i\cdot X(\omega)=A(\omega)e^{i\varphi(\omega)}$$

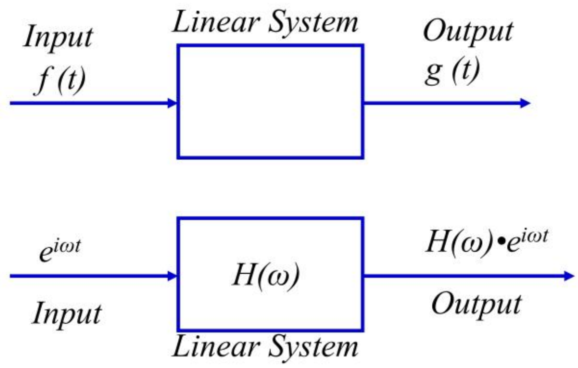

Figure 9. Linear System: (upper) function *f(x,t)* as input and *g(x,t)* as output; (lower) exponential function as input and output the exponential function multiplied by the *Transfer Function* of the Linear System.

and is often specified by its attenuation a(ω) and phase shift or phase lag θ(ω), defined by (**Eqn. 9.6**):

$$a(\omega)=-\ln A(\omega),\ \theta(\omega)=-\varphi(\omega),\ H(\omega)=e^{-a(\omega)}e^{-i\theta(\omega)}$$

From the inversion formula of Fourier Transform [[19], p.11, Eqn.2.17] we have:

$$h(t)=\frac{1}{\pi}\cdot\int_{0}^{\infty}A(\omega)\cdot\cos\left[\omega t-\theta(\omega)\right]d\omega \qquad (9.7)$$

With *F(ω)* and *G(ω)* the Fourier Transforms of the input *f(t)* (i.e. *f(x,t)*) and the output *g(t)*, we obtain (due to the property of the Linear System as time invariant):

$$G(\omega) = F(\omega) \bullet H(\omega) \qquad (9.8)$$

and the g(t) can be written:

$$g(t)=\frac{1}{2\pi}\cdot\int_{-\infty}^{+\infty}F(\omega)\cdot H(\omega)\cdot e^{i\omega t}d\omega \qquad (9.9)$$

9.3 Filtering

The term filter is used to describe Linear Systems whose amplitude characteristic *A(ω)* is negligible in certain parts of the frequency axis. If *A(ω)* is small in some sense for $\omega > \omega_c$, then the filter is called lowpass, and ω_c its cutoff frequency [Papoulis 1962, 94]. A bandpass filter is a system whose amplitude characteristic *A(ω)* has significant values only in an interval not containing the origin. The response *g(t)* to an arbitrary input *f(t)* is determined from (Eqn. 9.9). In a number of important cases, however, special techniques lead to simple results [[[19], 94, 120]; cf. [[21], 270-276]].

The *infinitely selective filtering* of § 9.1 represents the frequencies that the filter will let through and the frequencies that it will attenuate (eliminate); the greater this attenuation, the more selective the filter [29], that is the *infinitely selective filter* does not permit but only one frequency v_i to pass and attenuates all the other frequencies; this is not physically feasible [[17], 22-23] or "in practice we cannot quite achieve this degree of 'perfect suppression' since this is an unrealistic demand (that is we could not, in general, synthesize an electrical filter to succeed this, but, there are practical designs which provide a close approximation to this type of filter" [[21], 270-271].

10 Accuracy of the Estimation of the Spectral Density of the Recording(s) of the Measurements of the Track

As it has been presented previously, the (Power) Spectral Density of a random signal (e.g. the recording of the measurement(s) of the defects of a Railway Track) at a point x_i at the space-time domain (Figure 8) includes the entire history (the future is included also) of *f(x, t)* for what corresponds to the frequency v_i. According to Fourier's Theory any arbitrary (random) function, even one with a finite number of discontinuities, could be represented as (decomposed to) an *infinite summation of sine and cosine terms*. Furthermore, the Spectral Density has the important property that ***an approximation consisting of a given number of terms achieves*** (since, after a number of few terms, the next terms do not contribute significantly) ***the minimum mean square error between the (real) signal and the approximation***. Consequently, since we can measure the (Power) Spectral Density of the output -*that is of the recording of the measured values by the Track Recording Vehicle*-, then the (Power) Spectral Density of the input -*that is of the real (vertical) defects of the Track*- can be accurately calculated from the output Spectral Density and the Vehicle's Transfer Function *K(Ω)*; *K(Ω)* is well, mathematically, defined for every Car/Vehicle. Hence, *there is not any problem of accuracy in the calculation of the Spectral Density of the output (recorded measurements) and from that of the calculated value of the Spectral Density of the real Track's Defects*.

However, the following issue has been arised, in general: "the essence of the spectral estimation problem is captured by the following informal formulation: *from a **finite record** of a stationary data sequence, estimate how the total power is distributed over frequency*" [[26], 1].

However, the Recordings of the Measurements of a Track Section (e.g. Domokos-Larissa) are performed by the Track Recording Car *all over the whole length of this Track Section* and not to a part of its length, namely, it is not a sample (partial in length) recording. The *critical remark on this case* -based on the reality in Railways- is: the recordings of the measurements are performed *all over the whole length of a Track Section* -from which recordings, we derive the spectral density of this Track Section- and not to a "*single finite segment of the 'signal'*" [='Track Section defects']. Only in the latter case an "aliasing issue" -as described in [[[21], 389], [[3], 1]]- could be introduced, and not in the real case of the measurements of a Railway Track.

11 Synopsis - Conclusions

In this article, which is a quasi-Part II of [12], we examined the relation between the Power Spectral Density of the output -that is recordings of the measurements of the longitudinal vertical defects along the Railway Track, which were performed by a Track Recording Car (a Linear System)- and the Power Spectral Density of the input (real defects of the Track).

Since the calculation of the Spectral Density of the recordings of the measurements can be performed easily from the results of these recordings, we can calculate the Spectral Density of the input (real defects of the Track) through the *Transfer Function K(Ω)* of the Car/Vehicle, which can also be calculated easily for each Car/Vehicle (as it is described in an example presented in the present article).

Acknowledgement

I want to specially thank professor T.P. Tassios for his questions and suggestions -after the publication of [12]-, which led me to investigate the issue of the Spectral Density of measurements and that of the real (vertical) defects of the Track and present here the accurate and reliable mathematical relation beyond the description of the relevant article [12]. In any case, any existing mistakes in the present article are exclusively mine.

ANNEX 1

Values of the Transfer Function K(Ω) of a Track Recording Car with 10m measurement chord (basis)

Wavelenghts ℓ [m]	K(Ω)=1-cos(2πa/ℓ)	log(ℓ)
100	0.048943402	2
95	0.054182668	1.9777
90	0.060307278	1.9542
85	0.067527658	1.9294
80	0.076120341	1.9031
75	0.086454398	1.8751
70	0.099030968	1.8451
65	0.114543785	1.8129
60	0.133974375	1.7782
55	0.158746206	1.7404
50	0.190982694	1.699
45	0.233955178	1.6532
40	0.29289275	1.6021
35	0.376509605	1.5441
30	0.499999234	1.4771
25	0.690981996	1.3979
20	0.999998673	1.301
15	1.499998468	1.1761
10	2	1
7.5	1.500003064	0.8751
5	1.40831E-11	0.699
4.2	0.634653094	0.6232
3.9	1.200019027	0.5911
3.6	1.766039705	0.5563
3.333333333	2	0.5229
3.1	1.758763698	0.4914
2.9	1.161791026	0.4624
2.7	0.402849292	0.4314
2.5	5.63323E-11	0.3979
2.35	0.304799456	0.3711
2.18	1.270388329	0.3385
2.05	1.927497612	0.3118
2	2	0.301
1.9	1.677291847	0.2788
1.8	0.826366341	0.2553
1.7	0.067533409	0.2304
1.666666667	1.26808E-10	0.2218
1.6	0.292881492	0.2041
1.54	0.979584235	0.1875
1.48	1.721943688	0.1703
1.428571429	2	0.1549
1.38	1.715122942	0.1399
1.34	1.116974989	0.1271
1.3	0.431952052	0.1139
1.25	2.25329E-10	0.0969
1.21	0.325724025	0.0828
1.17	1.147123037	0.0682
1.13	1.890363888	0.0531
1.111111	2	0.0458
1.09	1.853780304	0.0374
1.06	1.206003116	0.0253
1.03	0.390251233	0.0128
1	3.52077E-10	0

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