General Function Method in Periodic Boundary Value Problems on Thermoelastic Star Graphs

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Abstract: - Boundary value problems of thermoelasticity on graphs can be used to study various thermoelastic network structures under the action of various external forces and thermal heating or cooling. This makes it relevant to formulate and solve such problems on graphs of various structures. In this paper, using the method of generalized functions, a technique for calculating the thermoelastic state of star-type rods and rod structures is developed. Generalized solutions of non-stationary and stationary boundary value problems of thermoelasticity are constructed for various boundary conditions at the ends of the star graph and the generalized Kirchhoff condition at its common node. Regular integral representations of solutions to boundary value problems in analytical form are obtained. The obtained solutions allow modeling sources of forces and heat of various types, including using singular generalized functions. The developed technique allows solving a wide class of boundary value problems with local and coupled boundary conditions at the ends of graph edges and various transmission conditions at the nodes of not only a star graph, but also graphs of linear and mixed structure.

Keywords: — thermoelasticity, rod, boundary conditions, transmission condition, fundamental and generalized solutions, Fourier transform, resolving boundary equations, thermal conductivity, star graph.

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1. Introduction

Graph theory has wide applications in subjects such as economics, logistics, sociology, optimal control and navigation [1], [2]. The properties of graphs are also actively used to solve boundary value problems (BVPs) on network-like structures [3], [4], [5], [6], [7], [8], [9], [10].

With the development of mechanical engineering, complex multi-link rod structures operating under various thermal conditions began to be actively used. They are widely used in structural mechanics, mechanical engineering, robotics, and many other fields. An urgent scientific and technical task is to study the thermally stressed state of network systems for various purposes under dynamic and thermal influences, taking into account their thermoelastic properties under dynamic and thermal influences, including impact types. This is necessary to analyze the strength and reliability of such objects, determine safe operating modes, and prevent disasters.

Mathematical modeling of the thermodynamics of rod structures and the creation of information technologies based on it is one of the more effective and inexpensive methods for researching and designing such systems. Here boundary value problems of uncoupled thermoelasticity are considered on a star thermoelastic graph (Fig. 1), which can be used to study various mesh structures under conditions of volume and thermal heating (cooling).

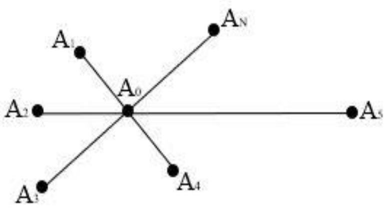

Fig. 1. Star graph

The novelty of the present work lies in the fact that a generalized function method is used to solve boundary value problems, leading to a differential equation solution with a singular right-hand side [11]. The solution is constructed as the convolution of the Green's function of the equation with the appropriate right-hand side. To determine the unknown boundary values of the solution and its derivatives on each segment, resolving boundary equations are constructed at the ends, employing the asymptotic properties of

Green's function and its derivative at zero. To construct a closed system of equations, the obtained algebraic equations for each edge of the graph are supplemented with transmission conditions at the node and linear boundary conditions at its ends. These conditions can be either locally or not locally connected. Thus, the proposed method applies to a wide range of BVPs, including those on mesh structures.

2. Statement of a Boundary Value Problem on a Thermoelastic Star Graph

We consider the periodic vibration of a thermo-elastic star graph with frequency ω. This graph contains N edges $S_j = (A_0, A_j)$ of the length $L_j\ (j = 1, 2, ..., N)$ with a common node A_0, $0 \le x \le L_j$. Amplitudes of displacement $u_j(x,t) = u_j(x)e^{-i\omega t}$ and temperature $\theta_j(x,t) = \theta_j(x)e^{-i\omega t}$ satisfy the equations [12]:

$$\frac{\partial^2 u_j}{\partial x^2} + \frac{\omega^2}{c_j^2} u_j - \tilde{\gamma}_j \frac{\partial \theta_j}{\partial x} + F_1^j(x) = 0, \quad (1)$$

$$\kappa_j \frac{\partial^2 \theta_j}{\partial x^2} + i\omega\theta_j - F_2^j(x) = 0. \quad (2)$$

Here γ_j, κ_j are the thermoelastic constants, $\tilde{\gamma}_j = \frac{\gamma_j}{\rho_j c_j^2}$, c_j is the velocity of elastic waves, ρ_j is the density of mass, $F_1^j(x,t) = F_1^j(x)e^{-i\omega t}$ is the longitudinal component of acting periodic force in a j-th edge of the graph; $F_2^j(x,t) = F_2^j(x)e^{-i\omega t}$ describes the power of heat sources on it.

The thermoelastic stress in the rod is determined by Duhamel-Neumann law:

$$\sigma_j(x,t) = \rho_j c_j^2 u_{j,x}(x,t) - \gamma_j \theta_j(x,t) \quad (3)$$

where $u_{j,x}(x,t) \triangleq \frac{\partial u_j}{\partial x}$.

Here we pose the following boundary value problem (BVP). Amplitudes of displacements and temperature are known at the ends of the graph: for all $j = 1, ..., N$

$$\begin{aligned} u_j\left(L_j, t\right) &= w_2^j(\omega)e^{-i\omega t} \\ \theta_j(L_j, t) &= \theta_2^j(\omega)e^{-i\omega t} \end{aligned} \quad (4)$$

We enumerate by index 1 the point $x=0$ and by index 2 the point $x = L_j$ $\left(x_1 = 0,\ x_2 = L_j\right)$ and boundary displacements and temperature $\left(w_1^j, \theta_1^j, w_2^j, \theta_2^j\right)$ at the ends of segments.

The following continuity conditions and generalized Kirchhoff conditions are specified in the common node A_0 of the graph:

$$\begin{aligned} w_1^1 &= w_1^2 = ... = w_1^N, \\ \theta_1^1 &= \theta_1^2 = ... = \theta_1^N, \end{aligned} \quad (5)$$

$$\sum_{j=1}^N \lambda_j E_j p_1^j = P(\omega), \quad \sum_{j=1}^N \kappa_j q_1^j = G(\omega). \quad (6)$$

$$p_1^j(\omega) = \left.\frac{\partial u_j}{\partial x}\right|_{x=0}, \quad p_2^j(\omega) = \left.\frac{\partial u_j}{\partial x}\right|_{x=L_j}$$

$$q_1^j(\omega) = \left.\frac{\partial \theta_j}{\partial x}\right|_{x=0}, \quad q_2^j(\omega) = \left.\frac{\partial \theta_j}{\partial x}\right|_{x=L_j}$$

We need to find the solution of the boundary periodic value problem of uncoupled thermoelasticity on this star graph.

3. Generalized Solution of Thermal Boundary Value Problems on an Graph Segment

To determine the solution on the graph at first we consider BVP on graph segment by use the general function method. For this we consider the BVP for heat equation (2) on the segment $[0, L]$ in the space of generalized functions of slow growth $S'(R^2) = \{\hat{f}(x,t),\quad (x,t) \in R^2\}$ [13]. To do this, we introduce a regular generalized function (we mark it with a cap):

$$\hat{\theta}(x,t) = \begin{cases} \theta(x,t), (x,t) \in D^- \\ 0, \quad x \notin D^- \end{cases},$$

where $\theta(x,t)$ is the solution of BVP, $D^{-}=[0,L]\times[0,\infty)$. It can be represented in the form

$$\hat{\theta}(x,t)=\theta(x,t)H(L-x)H(x)H(t). \qquad (7)$$

Here $H(x)$ is Heaviside step function.

To construct the equation for $\hat{\theta}(x,t)$ in $S'(R^2)$, we find generalized derivatives of $\hat{\theta}(x,t)$:

$$\frac{\partial\hat{\theta}}{\partial x}=\frac{\partial\theta}{\partial x}H(L-x)H(x)H(t)-\\ -\theta_2(t)\delta(L-x)H(t)+\theta_1(t)\delta(x)H(t),$$

$$\frac{\partial^2\hat{\theta}}{\partial x^2}=\frac{\partial^2\theta}{\partial x^2}H(L-x)H(x)H(t)-\\ -q_2(t)\delta(L-x)H(t)+q_1(t)\delta(x)H(t)+\\ +\theta_2(t)\delta'(L-x)H(t)+\theta_1(t)H(t)\delta'(x),$$

$$\frac{\partial\hat{\theta}}{\partial t}=\frac{\partial\theta}{\partial t}H(L-x)H(x)H(t)+\\ +\theta_0(x)H(L-x)\delta(t),$$

where $\delta\left(x\right)$ is singular generalized delta - function, $q_j(t)=\left.\frac{\partial\theta}{\partial x}\right|_{x=x_j}, \quad j=1,2.$

The equation (2) in $S'(R^2)$ has the next form for $\hat{\theta}(x,t)$:

$$\frac{\partial\hat{\theta}}{\partial t}-\kappa\frac{\partial^2\hat{\theta}}{\partial x^2}=\hat{F}_2(x,t)+\\ +\kappa q_2(t)\delta(L-x)H(t)-\kappa q_1(t)\delta(x)H(t)-\\ -\kappa\theta_2(t)\delta'(L-x)H(t)-\\ -\kappa\theta_1(t)\delta'(x)H(t)+\theta_0(x)H(L-x)H(x)\delta(t). \qquad (8)$$

Note that the right side of this equation includes all initial and boundary temperature $\theta_j(t)$ and heat flows $\Pi_j(t)=\kappa q_j(t)$ (j=1, 2).

According to the theory of generalized functions [13], the solution of Eq. (8) can be represented as a convolution of fundamental solution of heat equation (2) with the right-hand side of this equation:

$$\hat{\theta}(x,t)=\hat{F}_2(x,t)*U(x,t)+\\ +\kappa q_2(t)H(t)\underset{t}{*}U(L-x,t)-\\ -\kappa q_1(t)H(t)\underset{t}{*}U(x,t)-\\ -\kappa\theta_2(t)H(t)\underset{t}{*}U_{,x}(L-x,t)-\\ -\kappa\theta_1(t)H(t)\underset{t}{*}U_{,x}(x,t)+\\ +\theta_0(x)H(L-x)H(x)\underset{x}{*}U(x,t). \qquad (9)$$

We denote $\hat{F}_2(x,t)=F_2(x,t)H(x)H(L-x)H(t)$.

Here $U(x,t)$ is the fundamental solution of the heat equation (1) by $F(x,t)=\delta(x,t)=\delta(x)\delta(t)$. It decays at ∞ and has the form [13]:

$$U(x,t)=\frac{1}{\sqrt{2\pi\kappa t}}\exp(-x^2/4\kappa t)H(t), \qquad (10)$$

$U_{,x}(x,t)=\frac{\partial U}{\partial x}$. If $F(x,t)$ is a regular function, then relation (9) can be represented in the next integral form:

$$\theta(x,t)H(L-x)H(x)H(t)=\\ =H(t)\int_0^t d\tau\int_{-\infty}^{+\infty}U\left(x-y,t-\tau\right)F_2(y,\tau)dy+\\ +\kappa H(x)H(t)\int_0^t q_2(t-\tau)U(L-x,\tau)d\tau-\\ -\kappa H(L-x)H(t)\int_0^t U(x-y,t-\tau)q_1(\tau)d\tau-\\ -\kappa H(x)H(t)\int_0^t\theta_2(t-\tau)U_{,x}(L-x,\tau)d\tau-\\ -\kappa H(L-x)H(t)\int_0^t U_{,x}(x,t-\tau)\theta_1(\tau)d\tau+\\ +\int_0^L U(x-y,t)\theta_0(y)H(L-y)H(y)dy. \qquad (11)$$

Formula (11) determines the temperature inside a segment by known temperature and heat flows at its ends and is very useful for engineering applications.

However, for correctly posed boundary value problems, out of 4 boundary functions on the right side of formula (11), only 2 are known. To determine two unknown boundary functions, resolving boundary equations should be constructed using boundary conditions at the ends of the segment.

4. Solving of Heat Boundary Value Problem in Fourier Transformation Space in Time

To construct the resolving system of equations, we use Fourier transformation in time:

$$\bar{\theta}(x,\omega) = \mathrm{F}\left[\hat{\theta}(x,t)\right] = $$

$$= H(x)H(L-x)\int_0^\infty \theta(x,t)e^{i\omega t}dt, \qquad (12)$$

$$\hat{\theta}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\theta}(x,\omega)e^{-i\omega t}d\omega.$$

To define Fourier transform of generalized solution (9) we use the property of Fourier transform of convolution [12]:

$$\begin{aligned}\hat{\theta}(x,\omega) &= \bar{F}_2(x,\omega)\underset{x}{*}\bar{U}(x,\omega) + \\ &+\theta_0(x)H(L-x)H(x)\underset{x}{*}\bar{U}(x,\omega) + \\ &+\kappa\bar{q}_2(\omega)H(x)\bar{U}(L-x,\omega) - \\ &-\kappa\bar{q}_1(\omega)H(L-x)\bar{U}(x,\omega) - \\ &-\kappa\bar{\theta}_2(\omega)H(x)\bar{U},_x(L-x,\omega) - \\ &-\kappa\bar{\theta}_1(\omega)H(L-x)\bar{U},_x(x,\omega).\end{aligned} \qquad (13)$$

Here a variable under a sign of convolution shows the convolution only over this variable $\left(\underset{x}{*}\right)$. The integral representation of Eq. (13) has the form:

$$\begin{aligned}&\bar{\theta}(x,\omega)H(L-x)H(x)H(\omega) = \\ &= H(x)\int_0^L \bar{U}\left(x-y,\omega\right)F_2(y,\omega)dy + \\ &+\kappa H(x)\int_0^L \bar{U}(x-y,\omega)\theta_0(y)dy + \\ &+\kappa\bar{q}_2(\omega)H(x)\bar{U}(L-x,\omega) - \\ &-\kappa\bar{q}_1(\omega)H(L-x)\bar{U}(x,\omega) - \\ &-\kappa\bar{\theta}_2(\omega)H(x)\bar{U},_x(L-x,\omega) - \\ &-\kappa\bar{\theta}_1(\omega)H(L-x)\bar{U},_x(x,\omega).\end{aligned} \qquad (14)$$

Fourier transform of Green's function of heat equation is equal to

$$\bar{U}(x,\omega) = -\frac{\sin\left(k|x|\right)}{2k\kappa}, \qquad (15)$$

where $k = (1+i)\sqrt{\dfrac{\omega}{2\kappa}}$. It satisfies the equation:

$$\frac{d^2\bar{U}}{dx^2} + i\omega\kappa^{-1}\bar{U} = \delta(x).$$

Its derivative has the gap in point x=0 and equal to

$$\bar{U},_x(x,\omega) = -\frac{\operatorname{sgn} x}{2\kappa}\cos(\kappa|x|),$$

$$\operatorname{sgn} x = \begin{cases} 1, & x>0, \\ -1, & x<0. \end{cases}$$

There are next symmetry conditions:

$$\begin{aligned}&\bar{U}(x,\omega) = \bar{U}(-x,\omega), \\ &\bar{U},_x(\pm 0,\omega) = \mp\frac{1}{2\kappa}.\end{aligned} \qquad (16)$$

We use these properties for solving BVP.

5. Resolving Equations of Boundary Value Problems

To find unknown boundary functions, we pass in relation (17) to the limit at $x \to 0+\varepsilon, \quad \varepsilon > 0$:

$$\bar{\theta}_1(\omega) = \lim_{\varepsilon\to 0}\bar{\theta}(0+\varepsilon,\omega) = \bar{F}(x,\omega)\underset{x}{*}\bar{U}(x,\omega)\Big|_{x=0} +$$
$$+\theta_0(x)H(L-x)H(x)\underset{x}{*}\bar{U}(x,\omega)\Big|_{x=0} +$$
$$+\kappa\bar{q}_2(\omega)H(x)\bar{U}(L-0-\varepsilon,\omega) -$$
$$-\kappa\bar{q}_1(\omega)H(L-x)\bar{U}(0+\varepsilon,\omega) -$$
$$-\kappa\bar{\theta}_2(\omega)H(x)\bar{U}_x(L-0-\varepsilon,\omega) -$$
$$-\kappa\bar{\theta}_1(\omega)H(L-x)\bar{U}_{,x}(0+\varepsilon,\omega).$$

Next, we move the last term to the left side and, taking into account the right limit of $\bar{U}_{,x}(x,\omega)$ at zero (16) , we obtain the next equation on left end of the segment:

$$\frac{1}{2}\bar{\theta}_1(\omega) = \bar{F}(x,\omega)\underset{x}{*}\bar{U}(x,\omega)\Big|_{x=0} +$$
$$+\theta_0(x)H(L-x)H(x)\underset{x}{*}\bar{U}(x,\omega)\Big|_{x=0} +$$
$$+\kappa\bar{q}_2(\omega)H(x)\bar{U}(L,\omega) -$$
$$-\kappa\bar{q}_1(\omega)\bar{U}(0,\omega) - \kappa\bar{\theta}_2(\omega)H(x)\bar{U}_{,x}(L,\omega)$$

Similarly, we consider the limit at $x \to L-\varepsilon, \varepsilon > 0.$

$$\bar{\theta}_2(\omega) = \lim_{\varepsilon\to 0}\bar{\theta}(L-\varepsilon,\omega) =$$
$$= \bar{F}(x,\omega)\underset{x}{*}\bar{U}(x,\omega)\Big|_{x=L} +$$
$$\theta_0(x)H(L-x)H(x)\underset{x}{*}\bar{U}(x,\omega)\Big|_{x=L} -$$
$$-\kappa\bar{q}_1(\omega)\bar{U}(L-\varepsilon,\omega) -$$
$$-\kappa\bar{\theta}_1(\omega)\bar{U}_{,x}(L-\varepsilon,\omega) - \kappa\bar{\theta}_2(\omega)H(x)\bar{U}_{,x}(\varepsilon,\omega)$$

We move the last term to the left side, and obtain the second boundary equation:

$$\frac{1}{2}\bar{\theta}_2(\omega) = \bar{F}(x,\omega)\underset{x}{*}\bar{U}(x,\omega)\Big|_{x=L} +$$
$$+\theta_0(x)H(L-x)H(x)\underset{x}{*}\bar{U}(x,\omega)\Big|_{x=L} -$$
$$-\kappa\bar{q}_1(\omega)\bar{U}(L,\omega) - \kappa\bar{\theta}_1(\omega)\bar{U}_{,x}(L,\omega)$$

Let us formulate the obtained results in the form of this theorem.

Theorem 1. *The Fourier time transformants of boundary functions of boundary value problems satisfy the system of linear algebraic equations of the form:*

$$\begin{bmatrix} 0,5 & 0 \\ \kappa\bar{U}_{,x}(L,\omega) & \kappa\bar{U}(L,\omega) \end{bmatrix}\begin{bmatrix} \bar{\theta}_1(\omega) \\ \bar{q}_1(\omega) \end{bmatrix} +$$
$$+\begin{bmatrix} \kappa\bar{U}_{,x}(L,\omega) & -\kappa\bar{U}(L,\omega) \\ 0,5 & 0 \end{bmatrix}\begin{bmatrix} \bar{\theta}_2(\omega) \\ \bar{q}_2(\omega) \end{bmatrix} = \quad (17)$$
$$= \begin{bmatrix} \bar{Q}_1(0,\omega) \\ \bar{Q}_2(L,\omega) \end{bmatrix},$$

where

$$\bar{Q}_1(0,\omega) = \bar{F}(x,\omega)\underset{x}{*}\bar{U}(x,\omega)\Big|_{x=0} +$$
$$+\theta_0(x)H(L-x)H(x)\underset{x}{*}\bar{U}(x,\omega)\Big|_{x=0},$$
$$\bar{Q}_2(L,\omega) = \bar{F}(x,\omega)\underset{x}{*}\bar{U}(x,\omega)\Big|_{x=L} +$$
$$+\theta_0(x)H(L-x)H(x)\underset{x}{*}\bar{U}(x,\omega)\Big|_{x=L}.$$

This system makes possibility to solve BVP for any given two boundary functions of temperature and heat flow at the ends of a segment of four boundary functions.

To solve all temperature BVPs, it is convenient to consider the extended system of equations in the form of matrix equation:

$$\mathbf{A}(\omega)\times\mathrm{B}(\omega) = \mathrm{C}(\omega), \quad (18)$$

where

$$\mathbf{A}(\omega) = \begin{pmatrix} 0,5 & 0 & \kappa\bar{U}_{,x}(L,\omega) & -\kappa\bar{U}(L,\omega) \\ \kappa\bar{U}_{,x}(L,\omega) & \kappa\bar{U}(L,\omega) & 0,5 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix},$$

$$B(\omega) = \left(\bar{\theta}_1(\omega), \bar{q}_1(\omega), \bar{\theta}_2(\omega), \bar{q}_2(\omega)\right),$$
$$\mathrm{C}(\omega) = (\bar{Q}_1(0,\omega), \bar{Q}_2(L,\omega), \bar{b}_3(\omega), \bar{b}_4(\omega)).$$

The last two equations in the system (18) are determined by boundary conditions at the ends of the segment, which are known for BVP:

$$\begin{bmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \begin{bmatrix} \bar{\theta}_1(\omega) \\ \bar{q}_1(\omega) \end{bmatrix} + \\ + \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} \bar{\theta}_2(\omega) \\ \bar{q}_2(\omega) \end{bmatrix} = \begin{bmatrix} \bar{b}_3(\omega) \\ \bar{b}_4(\omega) \end{bmatrix}. \tag{19}$$

By given coefficients a_{ij} and right-hand side $b_i(\omega)$, we have four equations (18) for definition of four boundary functions. The solution of Eqs (18) has the form:

$$B(\omega) = \mathbf{A}^{-1}(\omega) \times C(\omega), \tag{20}$$

where $\mathbf{A}^{-1}(\omega)$ is the inverse matrix of $\mathbf{A}(\omega)$.

So, all boundary functions are defined, therefore, the Fourier transform for solving the boundary value problem is constructed. Using the inverse Fourier transform (12), we obtain the original $\theta(x,t)$ on the segment [0, L].

Note that the resolving system of equations (18) makes it possible to solve all boundary value problems for the heat equation with local and associated linear boundary conditions that determine the coefficients a_{ij} (19) of the matrix $\mathbf{A}(\omega)$. This does the Method of Generalized Functions universal for solving similar boundary value problems for differential equations.

In a similar manner, in the articles [14], [15] the solutions of boundary value problems for the D'Alembert wave equation on a segment are constructed.

6. Presentation of Amplitudes of Displacement and Temperature at Edges of a Star Graph

Let us use the solution (12) of BVP in Fourier transformant space on the segment and Eqs (17) and similar formulas for wave equation [14] for construction the periodic solution with frequency ω on the line graph.

The amplitudes of temperature at S_j (j=1,…,N) is equal to

$$\begin{aligned} \theta_j(x,\omega) &= F_2^j(x,\omega) \underset{x}{*} U_2^j(x,\omega) + \\ &+\kappa_j q_2^j(\omega) H(x) U_2^j(L_j - x,\omega) - \\ &-\kappa_j q_1^j(\omega) H(L_j - x) U_2^j(x,\omega) - \\ &-\kappa_j \theta_2(\omega) H(x) U_{2,x}^j(L_j - x,\omega) - \\ &-\kappa_j \theta_1(\omega) H(L_j - x) U_{2,x}^j(x,\omega) \end{aligned} \tag{21}$$

The amplitudes of displacements at edges S_j (j=1,…, N) is equal to

$$\begin{aligned} u_j(x,\omega) H(L_j - x) H(x) &= \mathrm{P}^j(x,\omega) - \\ &-c_j^2 p_1^j(\omega) H(L_j - x) U_1(x,\omega) - \\ &-c_j^2 w_1^j(\omega) H(L_j - x) U_{1,x}(x,\omega) + \\ &+c_j^2 p_2^j(\omega) H(x) U_1(L_j - x,\omega) + \\ &+c_j^2 \bar{w}_2(\omega) H(x) U_{1,x}(L_j - x,\omega) \end{aligned} \tag{22}$$

where

$$U_1^j(x,\omega) = -\frac{1}{2\omega\, c_j} \sin\left(\lambda_j |x|\right), \\ \lambda_j(\omega) = \frac{\omega}{c_j}; \tag{23}$$

$$U_2^j(x,\omega) = -\frac{\sin\left(k_j |x|\right)}{2k_j(\omega)\kappa_j}, \\ k_j(\omega) = (1+i)\sqrt{\frac{\omega}{2\kappa_j}}. \tag{24}$$

If the boundary functions are known:

$$\mathbf{B}1^j(\omega) = \left(w_1^j, p_1^j, w_2^j, p_2^j\right), \\ \mathbf{B}2^j(\omega) = \left(\theta_1^j, q_1^j, \theta_2^j, q_2^j\right),$$

then formulas (21) and (22) determine the temperature and displacement on each edge of the graph.

At first, we solve the BVP for temperature on the star graph.

7. Solution of BVP on the Heat Star Graphs

There are next equations of connection boundary temperature at S_j edges of the heat graph:

$$\mathbf{A}2^j(\omega)\times\mathbf{B}2^j(\omega)=\mathbf{F}2^j(\omega), \qquad (25)$$

$$\mathbf{A}2^j(\omega)=$$

$$\begin{bmatrix} 1 & 0 & -\cos(k_jL_j) & \dfrac{\sin(k_jL_j)}{k_j(\omega)} \\ -\cos(k_jL_j) & -\dfrac{\sin(k_jL_j)}{k_j(\omega)} & 1 & 0 \end{bmatrix}$$

$$\mathbf{F}2^j(\omega)=$$

$$=2\left(F_2^j \underset{x}{*} U_2^j\Big|_{x=0}, \quad F_2^j \underset{x}{*} U_2^j\Big|_{x=L_j} \right)$$

The resolving system of equations for the Dirichlet boundary value problem on a thermal star graph with N edges has the form:

$$\mathbf{A}2(\omega)\times\mathbf{B}2(\omega)=\mathbf{F}(\omega), \qquad (26)$$

where

$$\mathbf{B}2(\omega)=\left(\mathbf{B}2_1,\mathbf{B}2_2,...,\mathbf{B}2_N\right),$$

$$\mathbf{F}2(\omega)=$$

$$\left\{\mathbf{F}2^1,...,\mathbf{F}2^N;T^1,...,T^N;\underbrace{0,...,0}_{N-1};G(\omega)\right\},$$

$$T^j=\theta_j(L_j)$$

Here the matrices have the following dimensions: $[\mathbf{A}2]_{4\mathrm{N}\times 4\mathrm{N}}$, $[\mathbf{B2}(\omega)]_{4\mathrm{N}\times 1}$, $[\mathbf{F}2(\omega)]_{4\mathrm{N}\times 1}$. The first 2N lines along the diagonal $\mathbf{A}2$ contain connection matrices for each edge of this graph $\mathbf{A}2_j$. The remaining elements are zero.

The next N-1 rows of the matrix $\mathbf{A}2$ contain the continuity conditions $(4)_2$. The last row of the matrix contains the Kirchhoff condition $(5)_2$ at the node A_0 of the star graph.

The solution to algebraic equations (10) has the form:

$$\mathbf{B}2(\omega)=\mathbf{A}2^{-1}\times\mathbf{F}2(\omega). \qquad (27)$$

After determining the unknown edge and nodal functions $\mathbf{B}2(\omega)$, using formulas (21) we determine the temperature on the every edge of the graph. ($j=1,..., N$)

The boundary value problem on the thermal graph has been solved.

8. Solution of BVP on the Elastic Star Graphs

There are next equations of connection boundary temperature at S_j edges of elastic graph [16]:

$$\mathbf{A}1^j\times\mathbf{B}1^j=\mathbf{C}1^j(\omega), \qquad (28)$$

$$\mathbf{A}1^j(\omega)=$$

$$\begin{bmatrix} 1 & 0 & -\cos(\lambda_jL_j) & \dfrac{\sin(\lambda_jL_j)}{\lambda_j(\omega)} \\ -\cos(\lambda_jL_j) & \dfrac{-\sin(\lambda_jL_j)}{\lambda_j(\omega)} & 1 & 0 \end{bmatrix}$$

$$\mathbf{F}1^j(\omega)=$$

$$-2\left(\left(F_1^j-\tilde{\gamma}_j\partial_x\theta_j\right)\underset{x}{*}U_1^j\Big|_{x=0}, \quad \left(F_1^j-\tilde{\gamma}_j\partial_x\theta_j\right)\underset{x}{*}U_1^j\Big|_{x=L_j}\right),$$

$$\lambda_j(\omega)=\omega/c_j.$$

The resolving system of equations for the Dirichlet boundary value problem on an elastic star graph with N edges has the form:

$$\mathbf{A}1(\omega)\times\mathbf{B}1(\omega)=\mathbf{F}(\omega), \qquad (29)$$

where

$$\mathbf{B}1(\omega)=\left(\mathbf{B}1_1,\mathbf{B}1_2,...,\mathbf{B}1_N\right),$$

$$\mathbf{F}1(\omega)=$$

$$\left\{\mathbf{F}1^1,...,\mathbf{F}1^N;d^1,...,d^N;\underbrace{0,...,0}_{N-1};P(\omega)\right\},$$

$$d^1=u_j\left(L_j\right)$$

Here the matrices have the following dimensions: $[\mathbf{A}1]_{4N\times 4N}$, $[\mathbf{B}1(\omega)]_{4N\times 1}$, $[\mathbf{F}1(\omega)]_{4N\times 1}$. The first $2N$ lines along the diagonal $\mathbf{A}1$ contain the connection matrices $\mathbf{A}1_j$ for each edge of this graph. The remaining elements are zero. The next *N-1* rows of the matrix $\mathbf{A}1$ contain the continuity conditions $(4)_2$. The last row of the matrix contains the Kirchhoff condition $(5)_2$ at the node A_0.

The solution to algebraic equations (29) has the form:

$$\mathbf{B}1(\omega) = \mathbf{A}1^{-1} \times \mathbf{F}1(\omega)\,. \qquad (30)$$

After determining the unknown boundary and nodal functions $\mathbf{B}1(\omega)$, using formulas (7), we determine the temperature on any edge of the graph.

The boundary value problem on the elastic graph has been solved.

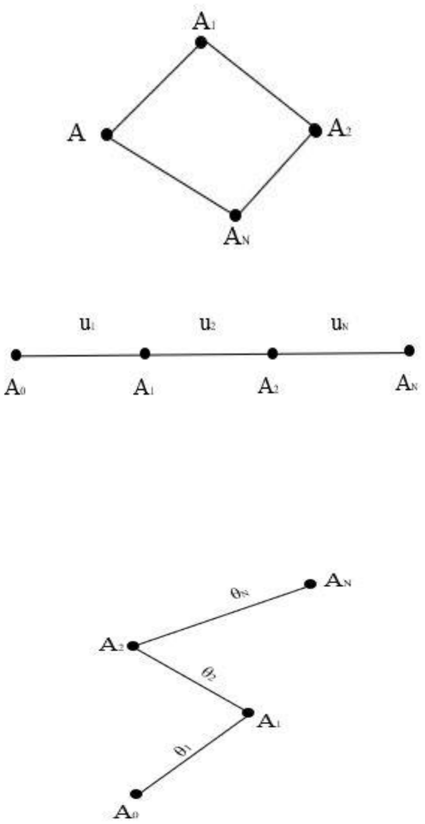

Figure 2. The Graphs

9. Conclusion

Note that the constructed solutions in the space of Fourier transforms describe the thermoelastic state of the graph under stationary oscillations with a fixed oscillation frequency. Therefore, these solutions can be used to construct a solution to time-periodic boundary value problems. To do this, it is sufficient to expand the boundary conditions and external effects into Fourier series in time and use the solutions constructed here for each harmonic of this series

The technique developed here for solving boundary value problems of thermoelasticity on a star graph can be extended to graphs of linear and mixed structure, including closed ones (Fig. 2). The determining factor here is the equations for the connection of boundary functions for temperature, displacements, heat fluxes and stresses at the ends of each link of the graph, which are presented in this article. To construct a matrix of the resolving system of equations, one should add conditions at the ends of the graph and transmission conditions, such as those considered, or others that take into account the orientation of the edges of the graph in space.

The developed method allows solving a wide class of boundary value problems with local and related boundary conditions at the ends of the graph and various transmission conditions at its nodes and should find application in the design of network systems and rod structures.

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