Reliable Kalman Filter Prediction and Estimation Metrics

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Abstract: - In this paper we propose metrics for the reliability of Kalman filter prediction and estimation. These metrics depend on the known state and measurements noise covariances, the prediction or estimation error covariances and the measurements covariance. These metrics concern the time varying, the time invariant and the steady state Kalman filters. The proposed metrics are time varying since the measurements covariance is time varying. The closer the metrics are to zero, the better the prediction or the estimation. Examples are presented to test the proposed metrics.

Key-Words: - Kalman filter, Estimation, Prediction, Steady state, Metrics, State Space Systems, Measurement

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1 Introduction

Kalman Filter [1]-[2] is the most well-known estimation algorithm which plays a fundamental role in estimation theory, since it has been widely used in various applications: object detection and tracking [3], robotic applications [4], electric load estimation [5], weather forecasts [6], autonomous orbit determination [7], satellite orbit determination [8], power generation prediction [9], cases prediction of Covid-19 [10], control effectiveness estimation on airplanes [11], applications with time-correlated measurement errors [12], aircraft state estimation [13], GPS position estimation [14], estimation with unlimited sensing measurements [15], multi-target localization [16], stock price prediction [17].

The discrete time Kalman Filter is associated with discrete time state space systems of the form:

$$x(k+1) = F(k) \cdot x(k) + w(k) \quad (1)$$

$$z(k) = y(k) + v(k) = H(k) \cdot x(k) + v(k) \quad (2)$$

for $k = 0,1,\dots$

where is the discrete time, $x(k)$ is the state, $z(k)$ is the measurement, $y(k)$ is the output, $F(k)$ is the known transition matrix, $H(k)$ is the known output matrix, $w(k)$ and $v(k)$ are the state noise and the measurement noise, respectively.

It is known [2, chapter 2] that:

- $\{w(k)\}$ and $\{v(k)\}$ are individually zero mean, Gaussian process with known covariances $Q(k)$ and $R(k)$, respectively
- $\{x(k)\}$ and $\{z(k)\}$ are jointly Gaussian processes

The initial state $x(0)$ is a Gaussian random variable with known mean x_0 and covariance P_0.

Kalman Filter produces iteratively the state estimation $x(k|k)$ and the estimation error covariance $P(k|k)$ as well as the state prediction $x(k+1|k)$ and the prediction error covariance $P(k+1|k)$ using the measurements $z(k)$ till time k and the known Kalman Filter parameters: $F(k), H(k), Q(k), R(k)$.

The time varying Kalman filter is derived for time varying systems where the Kalman filter parameters $F(k), H(k), Q(k), R(k)$ are time varying:

time varying Kalman filter

$$K(k) = P(k|k-1) \cdot H^T(k) \cdot [H(k) \cdot P(k|k-1) \cdot H^T(k) + R(k)]^{-1}$$

$$x(k|k) = K(k) \cdot z(k) + [I - K(k) \cdot H(k)] \cdot x(k|k-1)$$

$$P(k|k) = [I - K(k) \cdot H(k)] \cdot P(k|k-1)$$

$$x(k+1|k) = F(k) \cdot x(k|k)$$

$$P(k+1|k) = Q(k) + F(k) \cdot P(k|k) \cdot F^T(k)$$

for $k = 0,1,\dots$

with initial conditions

$x(0|-1) = x_0, P(0|-1) = P_0$

$K(k)$ is the Kalman filter gain. I denotes the identity matrix. M^T denotes the transpose of matrix M. Note that the existence of the inverse of the matrices required to compute the Kalman filter gain, is ensured assuming that every covariance matrix $R(k)$ is positive definite; this has the significance that no measurement is exact.

The time invariant Kalman filter is derived for time invariant systems where all the Kalman filter parameters are constant, i.e. $F(k) = F, H(k) = H, Q(k) = Q, R(k) = R$:

time invariant Kalman filter
$$K(k) = P(k|k-1) \cdot H^T \cdot [H \cdot P(k|k-1) \cdot H^T + R]^{-1}$$
$$x(k|k) = K(k) \cdot z(k) + [I - K(k) \cdot H] \cdot x(k|k-1)$$
$$P(k|k) = [I - K(k) \cdot H] \cdot P(k|k-1)$$
$$x(k+1|k) = F \cdot x(k|k)$$
$$P(k+1|k) = Q + F \cdot P(k|k) \cdot F^T$$
for $k = 0,1, \ldots$
with initial conditions
$x(0|-1) = x_0, P(0|-1) = P_0$

For time invariant systems, it is well known [2] that if the signal process model is asymptotically stable, then there exists a unique steady state value P_p of the prediction error covariance matrix. In this case, the resulting steady state Kalman filter is derived:

steady state Kalman filter
$$x(k|k) = K \cdot z(k) + A \cdot x(k-1|k-1)$$
for $k = 1,2, \ldots$
with initial condition $x(0|0)$

The initial condition is calculated by:
$$x(0|0) = [I - K(0) \cdot H] \cdot x(0|-1) + K(0) \cdot z(0)$$
where
$$K(0) = P(0|-1) \cdot H^T \cdot [H \cdot P(0|-1) \cdot H^T + R]^{-1}$$
with $x(0|-1) = x_0, P(0|-1) = P_0$

The steady state Kalman filter coefficients K, A are computed off-line, by first solving the corresponding discrete time Riccati equation [2]:
$$P_p = Q + F \cdot P_p \cdot F^T - F \cdot P_p \cdot H^T \cdot \left[H \cdot P_p \cdot H^T + R\right]^{-1} \cdot H \cdot P_p \cdot F^T \quad (3)$$
and then calculating the steady state gain K:
$$K = P_p \cdot H^T \cdot \left[H \cdot P_p \cdot H^T + R\right]^{-1} \quad (4)$$
and then calculating
$$A = [I - K \cdot H] \cdot F \quad (5)$$
It is worth to note that the steady state value P_e of the estimation error covariance matrix is:
$$P_e = [I - K \cdot H] \cdot P_p \quad (6)$$

2 Prediction and Estimation Metrics

Let $\bar{z}(k)$ and $C(k)$ be the mean and the covariance of $\{z(k)\}$; this covariance can be computed by the measurements.

The difference between the state prediction and the real state is $x(k+1|k) - x(k)$.
Then the difference between the predicted output and the real output is:
$$y(k+1|k) - y(k) = H(k) \cdot [x(k+1|k) - x(k)]$$
$$= H(k) \cdot x(k+1|k) - H(k) \cdot x(k)$$
$$= H(k) \cdot x(k+1|k) - [z(k) - v(k)]$$
$$= H(k) \cdot x(k+1|k) + v(k) - z(k)$$
with mean
$$\mu(k+1|k) = H(k) \cdot x(k+1|k) - \bar{z}(k) \quad (7)$$
(since $\{v(k)\}$ is zero mean)
and covariance
$$\Sigma(k+1|k) = H(k) \cdot P(k+1|k) \cdot H^T(k) + R(k) - C(k) \quad (8)$$
Similarly, the difference between the state estimation and the real state is $x(k|k) - x(k)$.
Then the difference between the estimated output and the real output is:
$$y(k|k) - y(k) = H(k) \cdot [x(k|k) - x(k)]$$
$$= H(k) \cdot x(k|k) - H(k) \cdot x(k)$$
$$= H(k) \cdot x(k|k) - [z(k) - v(k)]$$
$$= H(k) \cdot x(k|k) + v(k) - z(k)$$
with mean
$$\mu(k|k) = H(k) \cdot x(k|k) - \bar{z}(k) \quad (9)$$
(since $\{v(k)\}$ is zero mean)
and covariance
$$\Sigma(k|k) = H(k) \cdot P(k|k) \cdot H^T(k) + R(k) - C(k) \quad (10)$$
It is clear that $\mu(k+1|k)$ and $\mu(k|k)$ depend on $H(k)$, the prediction and the estimation respectively and the mean of measurements. Also, $\Sigma(k+1|k)$ and $\Sigma(k|k)$ depend on $Q(k)$, $R(k)$, the prediction error covariance and the estimation error covariance respectively and the covariance of the measurements. Of course the prediction and the estimation error covariances depend on $Q(k)$, $R(k)$. Thus, $\Sigma(k+1|k)$ and $\Sigma(k|k)$ depend on $Q(k)$, $R(k)$, and the covariance of the measurements.

It is evident that the closer the quantities $\Sigma(k+1|k)$ and $\Sigma(k|k)$ are to zero, the better the prediction or the estimation. Thus, the quantities $\Sigma(k+1|k)$ and $\Sigma(k|k)$ can be used as metrics for how close to the real state the prediction or the estimation are.

Then, it is evident that:
- the closer the quantity $H(k) \cdot P(k+1|k) \cdot H^T(k) + R(k)$ is to $C(k)$, the better the prediction,
- the closer the quantity $H(k) \cdot P(k|k) \cdot H^T(k) + R(k)$ is to $C(k)$, the better the estimation.

Then we are able to propose the following metrics for the reliability of Kalman Filter prediction and estimation:

a) **metric for prediction**

$$\|\Sigma(k+1|k)\| = \left\|H(k)\cdot P(k+1|k)\cdot H^T(k)+R(k)-C(k)\right\| \quad (11)$$

It is desired for $\|\Sigma(k+1|k)\|$ to be close to zero or for $\left\|H(k)\cdot P(k+1|k)\cdot H^T(k)+R(k)\right\|$ to be close to $\|C(k)\|$

b) **metric for estimation**

$$\|\Sigma(k|k)\| = \left\|H(k)\cdot P(k|k)\cdot H^T(k)+R(k)-C(k)\right\| \quad (12)$$

It is desired for $\|\Sigma(k|k)\|$ to be close to zero or for $\left\|H(k)\cdot P(k|k)\cdot H^T(k)+R(k)\right\|$ to be close to $\|C(k)\|$

Here $\|\cdot\|$ denotes the Frobenius norm (Euclidian norm).

These metrics are valid for time invariant Kalman Filter as well. In the time invariant case all the Kalman Filter parameters are constant: $F(k)=F, H(k)=H, Q(k)=Q, R(k)=R$. Thus we can use $H(k)=H$ and $R(k)=R$ in (7), (8), (9), (10) and in (11), (12).

In the steady state case, these metrics remain also valid. In fact (8), (10) take the form:

$$\Sigma(k+1|k) = H\cdot P_p\cdot H^T+R-C(k) \quad (13)$$

$$\Sigma(k|k) = H\cdot P_e\cdot H^T+R-C(k) \quad (14)$$

where P_p and P_e are the steady state prediction and estimation error covariances, respectively.

Then the metrics (11), (12) take the form:

$$\|\Sigma(k+1|k)\| = \left\|H\cdot P_p\cdot H^T+R-C(k)\right\| \quad (15)$$

It is desired for $\|\Sigma(k+1|k)\|$ to be close to zero or for $\left\|H\cdot P_p\cdot H^T+R\right\|$ to be close to $\|C(k)\|$

and

$$\|\Sigma(k|k)\| = \left\|H\cdot P_e\cdot H^T+R-C(k)\right\| \quad (16)$$

It is desired for $\|\Sigma(k|k)\|$ to be close to zero or for $\left\|H\cdot P_e\cdot H^T+R\right\|$ to be close to $\|C(k)\|$

3 Examples

In this section, two examples are presented.

Example1. Random walk.

A scalar system which represents the random walk movement is assumed in this example.

The system parameters are:

$F=H=Q=R=1$

The initial conditions are:

$x_0=1, P_0=1$

The time invariant Kalman filter and the steady state Kalman filter are applicable.

The steady state Kalman filter coefficients are:

$K=0.618$

$A=0.382$

The steady state prediction error covariance and the steady state estimation error covariance are:

$P_p=1.618$

$P_e=0.618$

Figure 1 presents the real state and the state estimation using time invariant Kalman filter and steady state Kalman filter.

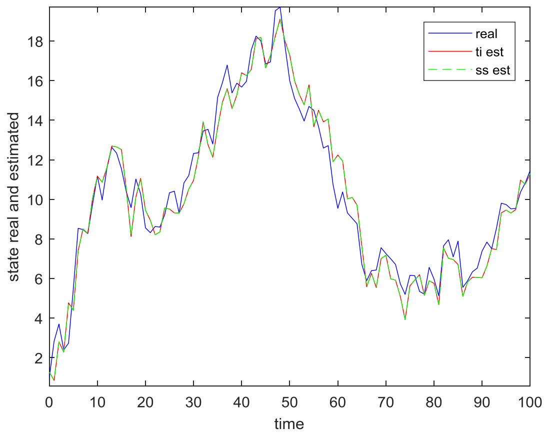

Fig 1. Example 1. Real state and state estimation

Figure 2 presents the metris using time invariant Kalman filter.

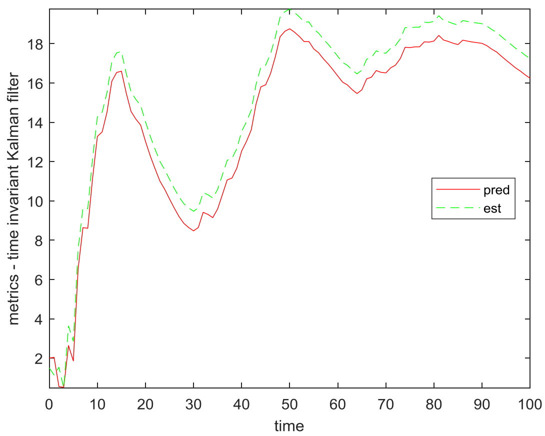

Fig. 2. Example 1. Metris using time invariant Kalman filter

It is evident that:

$\|\Sigma(k+1|k)\| = \left\|H\cdot P(k+1|k)\cdot H^T+R-C(k)\right\|$

is not close to zero

and

$\|\Sigma(k|k)\| = \left\|H\cdot P(k|k)\cdot H^T+R-C(k)\right\|$

is not close to zero

Figure 3 presents the norms $\|C(k)\|, \|H \cdot P(k|k) \cdot H^T + R\|, \|H \cdot P(k+1|k) \cdot H^T + R\|$ which are related to the metrics, using time invariant Kalman filter.

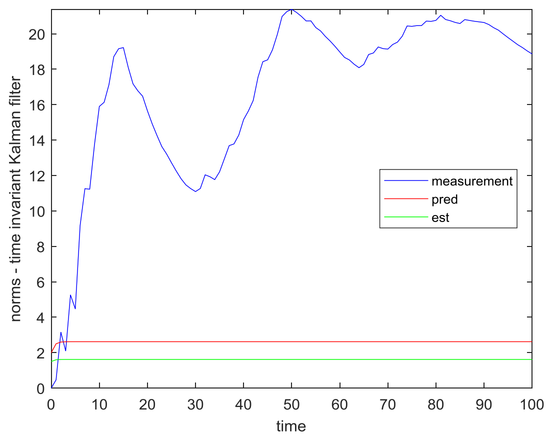

Fig. 3. Example 1. Norms related to the metrics using time invariant Kalman filter

It is evident that:
$\|H \cdot P(k+1|k) \cdot H^T + R\|$ is not close to $\|C(k)\|$
and
$\|H \cdot P(k|k) \cdot H^T + R\|$ is not close to $\|C(k)\|$

Figure 4 presents the metris using steady state Kalman filter.

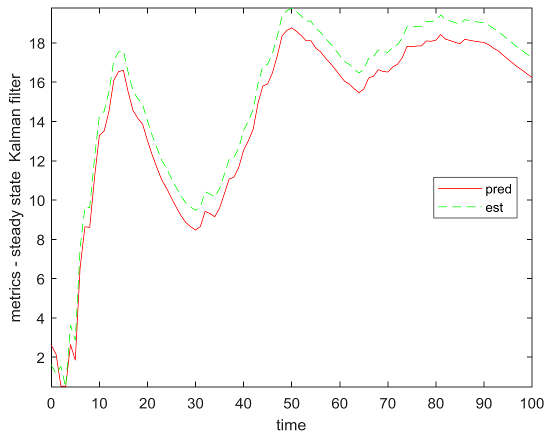

Fig. 4. Example 1. Metris using steady state Kalman filter

It is evident that
$\|\Sigma(k+1|k)\| = \|H \cdot P_p \cdot H^T + R - C(k)\|$
is not close to zero
and
$\|\Sigma(k|k)\| = \|H \cdot P_e \cdot H^T + R - C(k)\|$
is not close to zero

Figure 5 presents the norms $\|C(k)\|, \|H \cdot P_e \cdot H^T + R\|, \|H \cdot P_p \cdot H^T + R\|$ which are related to the metrics, using steady state Kalman filter.

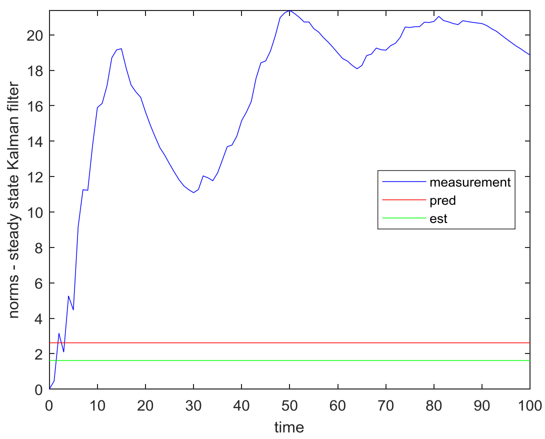

Fig. 5. Example 1. Norms related to the metrics using steady state Kalman filter

It is evident that
$\|H \cdot P_p \cdot H^T + R\|$ is not close to $\|C(k)\|$
and
$\|H \cdot P_e \cdot H^T + R\|$ is not close to $\|C(k)\|$

Example 2. Constant estimation.
In this example the Kalman filter is used to estimate a random constant [18]. Consider the real constant $c = 0.35$.
The system parameters are:
$F = 1, H = 1, Q = 10^{-5}, R = 10^{-2}$
The initial conditions are:
$x_0 = 0.1, P_0 = 0.1$
The time invariant Kalman filter and the steady state Kalman filter are applicable.
The steady state Kalman filter coefficients are:
$K = 0.03112673$
$A = 0.96887327$
The steady state prediction error covariance and the steady state estimation error covariance are:
$P_p = 0.00032127$
$P_e = 0.00031127$

Figure 6 presents the real state and the state estimation using time invariant Kalman filter and steady state Kalman filter.

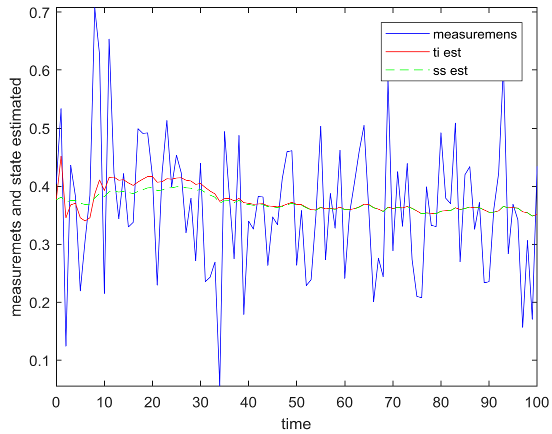

Fig. 6. Example 2. Real state and state estimation

Figure 7 presents the metris using time invariant Kalman filter.

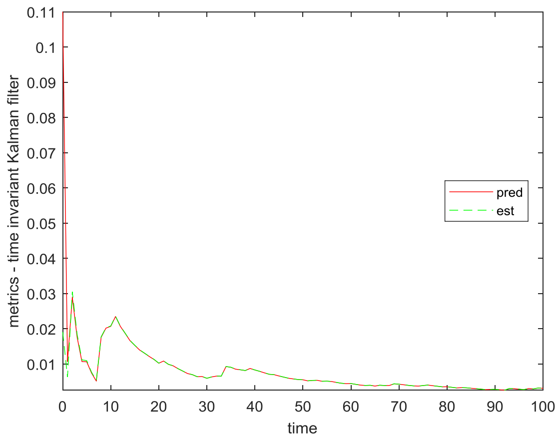

Fig 7. Example 2. Metris using time invariant Kalman filter

It is evident that:

$\|\Sigma(k+1|k)\| = \|H \cdot P(k+1|k) \cdot H^T + R - C(k)\|$

is close to zero

and

$\|\Sigma(k|k)\| = \|H \cdot P(k|k) \cdot H^T + R - C(k)\|$

is close to zero

Figure 8 presents the norms $\|C(k)\|, \|H \cdot P(k|k) \cdot H^T + R\|, \|H \cdot P(k+1|k) \cdot H^T + R\|$ which are related to the metrics, using time invariant Kalman filter.

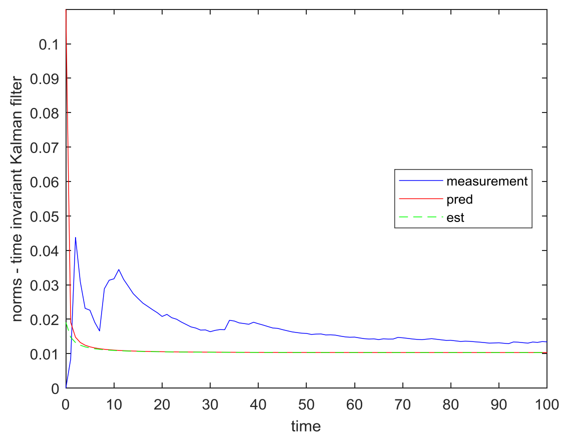

Fig. 8. Example 2. Norms related to the metrics using time invariant Kalman filter

It is evident that:

$\|H \cdot P(k+1|k) \cdot H^T + R\|$ is close to $\|C(k)\|$

and

$\|H \cdot P(k|k) \cdot H^T + R\|$ is close to $\|C(k)\|$

Figure 9 presents the metris using steady state Kalman filter.

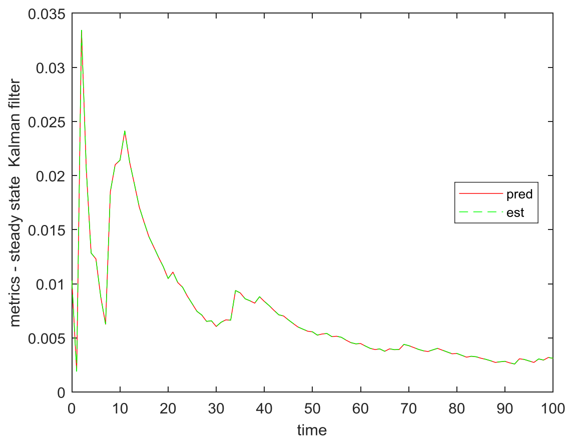

Fig. 9. Example 2. Metris using steady state Kalman filter

It is evident that

$\|\Sigma(k+1|k)\| = \|H \cdot P_p \cdot H^T + R - C(k)\|$

is close to zero

and

$\|\Sigma(k|k)\| = \|H \cdot P_e \cdot H^T + R - C(k)\|$

is close to zero

Figure 10 presents the norms $\|C(k)\|, \|H \cdot P_e \cdot H^T + R\|, \|H \cdot P_p \cdot H^T + R\|$ which are related to the metrics, using steady state Kalman filter.

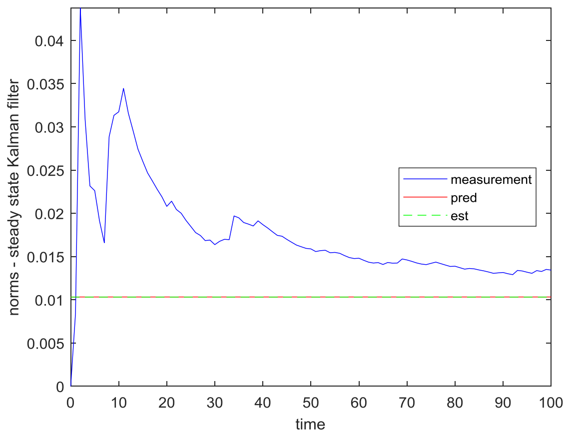

Fig. 10. Example 2. Norms related to the metrics using steady state Kalman filter

It is evident that
$\|H \cdot P_p \cdot H^T + R\|$ is close to $\|C(k)\|$
and
$\|H \cdot P_e \cdot H^T + R\|$ is close to $\|C(k)\|$

4 Conclusion

In conclusion, we proposed metrics for the reliability of Kalman Filter prediction and estimation, in the sense that they show how close to the real state is the state prediction and the state estimation. The proposed metrics depend on the known state and measurements noise covariances and the known measurements covariance. The closer the metrics are to zero, the better the prediction or the estimation.

The proposed metrics concern the time varying, the time invariant and the steady state Kalman filters. The proposed metrics are time varying since the measurements covariance is time varying. Examples are presented to test the proposed metrics.

A subject of future research is to propose metrics for the Finite Impulse Response (FIR) form of the steady state Kalman filter.

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