Fourier-Sobolev Series in Continuous-Discrete Orthogonal Sobolev Polynomials

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Abstract: - The article studies Fourier series in continuous-discrete Sobolev spaces. The questions about the behavior of partial sums and linear means for Fourier series in orthonormal Sobolev polynomials $\{\hat{q}_n(x)\}(x \in [-1,1];\ n \in \mathbb{Z}_+)$ are considered. Results on the convergence of $\Lambda -$ summation methods uniformly and almost everywhere are obtained. The compact convergence of linear summation methods in the Sobolev spaces is studied. A consequence of the obtained results is linear summation methods for Fourier - Gegenbauer -Sobolev series in a discrete Sobolev space.

Key words: - Sobolev polynomials, linear means, Fourier series, summation methods, continuous-discrete space, Sobolev space, Sobolev polynomials, Gegenbauer -Sobolev polynomials

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1 Introduction

We consider sequences of polynomials which are orthogonal with respect to the Sobolev inner product defined on the set of polynomials $\mathbb{P}$ by

$$(\mathrm{f},\mathrm{g}) = \sum_{i=0}^{N} \int_{\mathbb{R}} f^{(i)}(x) g^{(i)}(x) d\mu_i(x), \qquad (1)$$

for some $N \geq 1$, where $(\mu_i\}_{i=o}^{N}$ are positive finite Borel measures. The interest of the study of orthogonal polynomials is justified by several reasons: the spectral theory for ordinary differential equations, the analysis of spectral methods in the numerical treatment of partial differential equations, the search of algorithms for computing Fourier-Sobolev series as well as the approximation to both a function and its derivative in terms of Sobolev orthogonal polynomials; the extension of Gauss quadrature formulas [1] – [7]. There exists a real-valued polynomial $h: \mathbb{R} \to \mathbb{R}$ satisfying

$$(hp, q) = (p, hq)(p, q \in \mathbb{P}),$$

if and only if each of measures $\mu_i(x)(1 \leq i \leq N)$ are purely atomic with a finite number of mass points [8].

2 Continual-discrete Sobolev polynomials

Let μ be a finite positive Borel measure on the interval $[-1,1]$ with infinitely many points at the support and let the points $a_k, a_k \in \mathbb{R}, k = 1, 2, \dots, m.$ For f and g in $L_\mu^2([-1,1])$ such that there exist the derivatives in a_k, we can introduce the inner product

$$(f, g) = \int_{-1}^{1} f(x) g(x) d\mu(x) + \\ + \sum_{k=1}^{m} \sum_{i=0}^{N_k} M_{k,i} f^{(i)}(a_k) g^{(i)}(a_k), \qquad (2)$$

where $\mu(\{a_k\}) = 0, M_{k,i} > 0$ $(i = 0,1, \dots, N_k;\ k = 1,2, \dots, m), \mu'(x) > 0\ a.e.$ Linear spaces with this inner product are called a "**continual-discrete Sobolev spaces".**

Continual-discrete Sobolev spaces are an interesting topic in many fields of mathematics and its applications [9, 10]. If we investigate the oscillation of a string loading with masses M_k at the points a_k and use the Fourier method for the corresponding Sturm-Liouville boundary value problem associated with the second-order partial differential equation, then the eigenvectors are orthogonal with respect to the inner product

$$(f, g) = \int_{-1}^{1} f(x) g(x) dx + \sum_{k=1}^{m} M_k f(a_k) g(a_k).$$

If we study the oscillation of girder, we get a fourth-order partial differential equation. The corresponding eigenfunctions are orthogonal with respect to the inner product involving equilibrium theory of plates strengthened by rods, was considered for the first time by S.P. Timoshenko as early as 1915 (he was a famous specialist in elastic theory).

In the future we will consider the following two cases

$$|\mathrm{a}_k| \leq 1 \text{ и } |\mathrm{a}_k| > 1\ (\mathrm{k} = 1,2, \dots, \mathrm{m}).$$

A) Case $|\mathrm{a}_k| \leq 1$ $(\mathrm{k} = 1,2, \dots, \mathrm{m})$ [11].

Let N_k^* be the positive integer number defined by

$$N_k^* = \begin{cases} N_k + 1, & if\ N_k\ is\ odd, \\ N_k + 2, & if\ N_k\ is\ even, \end{cases}$$

$$w_N(x) := \prod_{k=1}^{m}(x-a_k)^{N_k^*}, N=\sum_{k=1}^{N} N_k^*;$$
$$w_{N+1}(x) = \int_{-1}^{x} w_N(t)dt.$$

B) Case $|a_k| > 1 (k=1,2,\dots,m)$ [12]:

$$w_N(x) = \prod_{k=1}^{m}(x-a_k)^{N_k+1}, N=\sum_{s=1}^{m}(N_s+1).$$

We will assume that all points a_k belong to the interval $(-\infty < a_s < -1, s=1,2,\dots,m)$; otherwise, we only have to change the corresponding factor $(x-a_k)$ by (a_k-x) in the definition of $w_N(x)$.

In order to study the Fourier series in the case $|a_s| > 1$ $(s=1,2,\dots,m)$ the condition $\mu'(x) > 0$ a.e is not sufficient for our purposes in what follows. Thus, we will consider the measure μ in the Szegö class:

$$\int_{-1}^{1} \log \mu'(x) \frac{dx}{\sqrt{1-x^2}} > -\infty.$$

Example: Jacobi weight $(1-x)^{\alpha}(1+x)^{\beta}$ $(\alpha,\beta > -1)$. Let $\{\hat{q}_n(x)\}$; $x \in [-1,1]$; $n \in \mathbb{Z}_+, \mathbb{Z}_+ = \{0,1,2,\dots\}$ be the sequence of polynomials of degree n with a positive leading coefficients orthonormal with respect to this inner product (2)

$$(\hat{q}_n, \hat{q}_m) = \delta_{n,m} \ (n,m \in \mathbb{Z}_+)$$

Orthonormal polynomials $\hat{q}_n(x)$ satisfy the following recurrence relations

$$w_{N+1}(x)\hat{q}_n(x) = \sum_{j=0}^{N+1} d_{n+j,j}\,\hat{q}_{n+j}(x) + \sum_{j=1}^{N+1} d_{n,j}\hat{q}_{n,n-j}(x) \quad (3)$$

$(n \in \mathbb{Z}_+; \hat{q}_{-j}=0, j=1,2,\dots; d_{n,s}=0, s>n)$.

Remarks. 1. If $\{\mathbf{e}_n\}_{n=0}^{\infty}$ is the orthonormal basis of a separate Hilbert space □ and J is Jacobi operator, then

$$\pi_{N+1}(J)\boldsymbol{e}_n = \sum_{j=0}^{N+1} d_{n+j,j}\,\boldsymbol{e}_{n+j} + \sum_{j=1}^{N+1} d_{n,j}\boldsymbol{e}_{n-j}$$

$(n \in \mathbb{Z}_+; \boldsymbol{e}_{-j}=0, j=1,2,\dots; d_{n,s}=0, s>n)$ is given by a generalized Jacobi matrix of order $(2N+3)$ [13].

2. Let us consider a special case of the inner product (2):

$$(f,g) := \int_{-1}^{1} f(x)g(x)d\mu(x) + \sum_{k=1}^{m}\sum_{i=0}^{N} M_{k,i} f^{(i)}(a_k)g^{(i)}(a_k)$$

$$= \int_{-1}^{1} f(x)g(x)d\mu(x) + \sum_{k=1}^{m} M_{k,0}f(a_k)g(a_k) + \sum_{k=1}^{m}\sum_{i=1}^{N} M_{k,i}f^{(i)}(a_k)g^{(i)}(a_k)$$

$$= \int_{-1}^{1} f(x)g(x)\,d\mu_0(x) + \sum_{i=1}^{N}\int_{-1}^{1} f^{(i)}(x)g^{(i)}(x)d\mu_i(x) = \sum_{i=0}^{N}\int_{-1}^{1} f^{(i)}(x)g^{(i)}(x)d\mu_i(x),$$

where $d\mu_0(x) = d\mu(x) + \sum_{k=1}^{m} M_{k,0}\delta(x-a_k)\,dx$, $d\mu_i(x) = \sum_{k=1}^{m} M_{k,i}\delta(x-a_k)\,dx$ $(i=1,2,\dots,N)$, and $\delta(x) - \delta$ Dirac's function. We obtain the classical Sobolev's inner product (1) with continuous and discrete measures.

3. Example. Discrete symmetric Gegenbauer-Sobolev polynomials

$$(f,g)_{\alpha} = \int_{-1}^{1} f(x)g(x)w_{\alpha}(x)dx + M[f(1)g(1)+f(-1)g(-1)] + N[f'(1)g'(1)+f'(1)g'(-1)] (M \geq 0, N \geq 0),$$

$$w_{\alpha}(x) = (1-x^2)^{\alpha} (\alpha > -1)$$

$$\left\{\hat{q}_n^{(\alpha)}(x) \equiv \hat{q}_n^{(\alpha)}(x; M, N)\right\} (n \in \mathbb{Z}_+, x \in [-1,1]),$$

$$({}_n^{(\alpha)}, \hat{q}_m^{(\alpha)})_{\alpha} = \delta_{n,m} \ (n, m \in \mathbb{Z}_+).$$

They were introduced by H. Bavinck, Y.J. Meijer ([14,15]) and have been investigated in the following articles [16]-[22] and so on.

Some of their properties differ from the properties of classical orthonormal Gegenbauer polynomials $p_n^{\alpha}(x)$:

1. For n large enough, the polynomials $\hat{q}_n^{(\alpha)}(x; M, N), N > 0$, positive exactly $(n-2)$ different, real and simple zeros belonging to the interval (-1,1); the two remainder zeros are outside of the interval being one positive and the other one negative; all roots of $p_n^{\alpha}(x)$ in $(-1,1)$;

2. $p_n^{\alpha}(x)$ are eigenfunctions of the differential operators of a second-order. Polynomials $\hat{q}_n^{(\alpha)}(x; M, N)$ are eigenfunctions of the linear differential operator usually infinite degree:

if $\alpha = 0, 1, 2, \dots.$ $M > 0, N > 0$: degree $4\alpha + 10$;
$M > 0, N = 0$: $2\alpha + 4$;
$M = 0, N > 0$: $2\alpha + 8$; these degrees are least;

3. $\left|\hat{q}_n^{(\alpha)}(\pm 1)\right| \approx n^{-\alpha-\frac{3}{2}}, |\{\widehat{q}_n^{(\alpha)}\}'(\pm 1)| \approx n^{-\alpha-\frac{3}{2}}$,

$$|\widehat{\{q_n^{(\alpha)}\}}'(\pm 1)| \approx n^{-\alpha-\frac{7}{2}},$$

$$|p_n^{\alpha}(\pm 1)| \approx n^{\alpha+\frac{1}{2}}, \alpha > -\frac{1}{2};$$

4. $(x^3-3x)\hat{q}_n^{(\alpha)}(x) = a_{n+3}\hat{q}_{n+3}^{(\alpha)}(x) + b_{n+1}\hat{q}_{n+1}^{(\alpha)}(x) + b_n\hat{q}_{n-1}^{(\alpha)}(x) + a_n\hat{q}_{n-3}^{(\alpha)}(x)$

$(n \in \mathbb{Z}_+;\ \hat{q}_{-s}^{(\alpha)}(x)=0,\quad s = 1,2,\dots;\ a_n = 0,\quad n = 0,1,2;\ b_0 = 0)$,

$$a_n = \frac{1}{8} + \frac{C_1}{n} + O\left(\frac{1}{n^2}\right),$$

$$b_n = -\frac{9}{8} + \frac{C_2}{n} + O\left(\frac{1}{n^2}\right).$$

This recurrence relation has the lowest order. Orthonormal polynomials $p_n^{\alpha}(x)$ satisfy a three-term recurrence relation

$$xp_n^{\alpha}(x) = c_{n+1}p_{n+1}^{\alpha}(x) + d_np_n^{\alpha}(x) + c_np_{n-1}^{\alpha}(x)(n \in \mathbb{Z}_+),$$

$$p_{-1}^{\alpha}(x) = 0,\ \ p_0^{\alpha}(x) = constant,\ \ c_0 = 0,$$

with

$$c_n = \tfrac{1}{2} + O\left(\tfrac{1}{n^2}\right), d_n = O\left(\tfrac{1}{n^2}\right)\ (n \to \infty).$$

4 Fourier-Sobolev series in continual-discrete Sobolev spaces

Let $\{\hat{q}_n(x), n \in \mathbb{Z}_+\,; x \in [-1,1]\}$ be the sequence of orthonormal polynomials of degree n with a positive leading coefficient:

$$(\hat{q}_n, \hat{q}_m) = \delta_{n,m}\ (n, m \in \mathbb{Z}_+),$$

$$\hat{q}_n(x) = k(\hat{q}_n)x^n + r(\hat{q}_n)x^{n-1} + \cdots, k(\hat{q}_n) > 0.$$

Denote by $\mathfrak{R}_p$ $(1 \le p < \infty)$, $\mathfrak{R}_1 = \mathfrak{R}$, the set of functions

$$\mathfrak{R}_p = \left\{\begin{matrix} f, \int_{-1}^{1}|f(x)|^p d\mu(x) < \infty;\ f^{(i)}(a_k)\ \ exist \\ i = 0,1,2,\dots,N_k\,, -\infty < a_k < \infty (k = 1,2,\dots,m)\end{matrix}\right\}.$$

For each $f \in \mathfrak{R}$ we form a Fourier-Sobolev series $f(x) \sim \sum_{k=0}^{\infty} c_k(f)\,\hat{q}_k(x)$,

$$c_k(f) = (f, \hat{q}_k)\ (k \in \mathbb{Z}_+;\ x \in [-1,1]). \qquad (4)$$

We consider the trilinear T-regular method of summability defined by the matrix

$$\Lambda = \{\lambda_k^{(n)}, k = 0,1,\dots,n,n+1; \lambda_0^{(n)} = 1, \lambda_{n+1}^{(n)} = 0, n \in \mathbb{Z}_+\}. \qquad (5)$$

Matrix Λ is called T - regular, if the following conditions are valid:

a) $lim_{n\to\infty}\lambda_k^{(n)} = 1$ ($k \in \mathbb{Z}_+$is fixed);

b) $\sum_{k=0}^{n}\left|\lambda_k^{(n)} - \lambda_{k+1}^{(n)}\right| \le C\ (n \in \mathbb{Z}_+)$.

For example: Cesaro means $(C, \alpha > 0)$, Voronoj - Nörlund means, Riesz's means and so on. For every $f \in \mathfrak{R}$ we introduce Λ-means

$$\mathbb{U}_nf(x;\Lambda) := \sum_{k=0}^{n}\lambda_k^{(n)}\,c_k(f)\hat{q}_k(x)\ (x \in [-1,1]). \qquad (6)$$

Compact convergence in a normed function space H on a set F is called convergence in the metric of the space H on any compact subset K of F (in the case of a space of continuous function $C([-1,1])$ with a uniform metric, the topology of uniform convergence on compact subsets).

We study the following problem:to investigate Λ – summability of a given Fourier-Sobolev series, that is, to obtain conditions for orthonormal system $\{\hat{q}_n\}_{n=0}^{\infty}$ and the entries of $\Lambda - matrix$ (5), for which the relations

$$\lim_{m\to\infty}\mathbb{U}_nf(x;\Lambda) = f(x), \qquad (7)$$

$$\lim_{n\to\infty}\sum_{l=0}^{n}\lambda_l^{(n)}\,c_l(f)\hat{q}_l^{(i)}(a_k) = f^{(i)}(a_k) \qquad (8)$$

$$(i = 0,1,2,\dots,N_k; k = 1,,\dots,m)$$

hold at the point $x \in (-1,1)$, almost everywheree and in the topology of compact convergence in the spaces $C([-1,1])$ and $W_{\omega}^{p}([-1,1])$.

The interest in orthogonal Sobolev polynomials and Fourier -Sobolev series is related to a number of problems of Function Theory, Functional Analysis, Quantum Mechanics, Mathematical Physics and Computing [1, 9]. In the classical book [24],[25], we can find the point of view of partial differential equations (see also [23, 26]).

5 Partial sums of the Fourier-Sobolev series

Define by

$$S_nf(x) = \sum_{k=0}^{n}c_k(f)\hat{q}_k(x) = \int_{-1}^{1}f(t)\,D_n(t;x)d\mu(t)$$

$$+\sum_{k=1}^{m}\sum_{i=0}^{N_k}M_{k,i}f^{(i)}(a_k)D_n^{(i)}\,(a_k;x), (x \in [-1,1]).$$

the partial sums of the Fourier-Sobolev series (4), where

$$D_n(t;x) = \sum_{k=0}^{n}\hat{q}_k(t)\hat{q}_k(x)\ \ (n \in \mathbb{Z}_+\,; t, x \in [-1,1])$$

is Dirichlet's kernel. The point $x \in (a,b)$ is called a Lebesgue point of a function f, if the following relation

$$\lim_{h\to 0}\frac{1}{2h}\int_{x-h}^{x+h}|f(t) - f(x)|d\mu(t) = 0$$

holds. As is known, the set of the Lebesgue points of $f \in L_{\mu}^{1}(a,b)$ is situated μ- almost-everywhere $x \in (a,b)$.

Define by

$$\mathcal{E}_m = \begin{cases} (-1,1)\setminus \cup_{s=1}^{m}\{a_s\}, if\ |a_s| \le 1\ (s=1,2,\dots,m), \\ (-1,1), \qquad if\ |a_s| > 1\ (s=1,2,\dots,m). \end{cases}$$

Theorem 1. *Assume that* $f \in \Re$ *and*

$$\int_{-1}^{1} |f(x)|h(x)d\mu(x) < \infty,$$

$$\int_{-1}^{1} h(x)d\mu(x) < \infty,$$

where h(x) is continuous in $\mathcal{E}_m$ *majorant of* $\hat{q}_n(x)$ *:*

$$|\hat{q}_n(x)| \le h(x)\ (x \in \mathcal{E}_m; n \in \mathbb{Z}_+). \qquad (9)$$

Then the following results hold:

(i) At each Lebesgue point $x \in \mathcal{E}_m$ *of the function* f*, the partial sums* $S_n f(x)$ *of the Fourier – Sobolev series (4) satisfy*

$$S_n f(x) = o_x(1)\, ln(n+1)\ (n \to \infty).$$

(ii) Let function f be a continuous on $[-1,1]$ *and the measure μ is absolutely continuous*

$$d\mu(x) = \omega(x)dx, \omega(x) > 0 \qquad (10)$$

is continuous $(x \in \mathcal{E}_m)$*. Then uniformly on every compact subset K of* $\mathcal{E}_m$ *the relation*

$$S_n f(x) = o(1)\, ln(n+1)\ (n \to \infty)$$

holds.

The statements (i)-(ii)are preserved in the case $|a_k| > 1 (k = 1,2,\dots,m)$, if the measure μ satisfies the Szegö condition. The issues of convergence of Fourier series for systems of continual-discrete orthogonal polynomials and their special cases were studied in the articles [11], [12], [27]-[36].

6 Λ-means of the Fourier-Sobolev series

Define by

$$\mathbb{U}_n f(x;\Lambda) := \sum_{k=0}^{n} \lambda_k^{(n)} c_k(f)\hat{q}_k(x)$$

$$(n \in \mathbb{Z}_+; x \in [-1,1])$$

the Λ -means of Fourier series (4).

Theorem 2. *Suppose that an orthonormal polynomial system* $\{\hat{q}_n(x)\}_{n=0}^{\infty}$ *has a majorant* $h(x)$ *(see (9)) and the recurrence coefficients (3) satisfy*

$$\sum_{j=1}^{N+1} j \sum_{l=0}^{N+1} \sum_{s=0}^{\infty} (|d_{s+j,s+j+l} - d_{s+l,s+j+l}| + |d_{s,s+j} - d_{s+l,s}|) < \infty$$

$$\sum_{j=1}^{N+1} j \sum_{l=0}^{N+1} \sum_{s=0}^{\infty} (|d_{s+j,s+j-l} - d_{s+l,s+j-l}| + |d_{s,s+j} - d_{s-l,s}|) < \infty, \qquad (11)$$

where the constant $C > 0$ *does not depend on* $n \in \mathbb{Z}_+$*, and for the entries of T-regular matrix Λ the following estimate*

$$\max_{0 \le k \le n} |\lambda_k^{(n)}| + \sum_{k=0}^{n} \frac{(k+1)(n-k+1)}{n+1} ln \frac{3(n+k+1)}{n-k+1} |\Delta^2 \lambda_k^{(n)}| \le C\ (n \in \mathbb{Z}_+) \qquad (12)$$

holds, where $\Delta^2 \lambda_k^{(n)} = \lambda_k^{(n)} - 2\lambda_{k+1}^{(n)} + \lambda_{k+2}^{(n)}$ $(k = 0,1,\dots,n; n \in \mathbb{Z}_+)$*. Then the following statements are valid:*

(i) Let f be a function $f \in \Re_p, 1 \le p < \infty$*, be satisfy*

$$\begin{aligned} &\int_{-1}^{1} |f(x)|^p h^p(x) d\mu(x) < \infty, \\ &\int_{-1}^{1} h^p(x) d\mu(x) < \infty. \end{aligned} \qquad (13)$$

Then at every Lebesgue point $x \in \mathcal{E}_m$ *of function* f*, the Λ - means* $\mathbb{U}_n f(x;\Lambda)$ *of the Fourier-Sobolev series converges to* $f(x)$*, that is, the relation (7) holds.*

(ii) If, an addition, the measure μ satisfies the condition (10) and the function f is continuous on $[-1,1]$*, then the relation (7) holds uniformly on all compact subsets K* $\subset (-1,1)$*.*

(iii) Let f be a function $f \in \Re_p, 1 \le p < \infty$*, be satisfy (13) and there exist the derivatives* $f^{(i)}(a_k)$ *for* $i = 0,1,\dots,N_k, k = 1,2,\dots,m$*, with*

$$\sup_{n \in \mathbb{Z}_+} \sum_{j=0}^{n} \left|\hat{q}_j^{(i)}(a_k)\right| < \infty \qquad (14)$$

then the relation (8) holds.

The statements (i)-(ii) are preserved in the case $|a_k| > 1 (k = 1,2,\dots,m)$, if the measure μ satisfies the Szegö condition.

Define by

$W_\omega^p(F) = \{f, \|f\|_{W_\omega^p(F)} < \infty,$

$\|f\|_{W_\omega^p(F)} = \|f\|_{L_\omega^p(F)} + \sum_{k=1}^{m} \sum_{i=0}^{N_k} M_{k,i} |f^{(i)}(a_k)|^p\},$

where $(1 \le p < \infty)$ the subset $F \subset (-1,1)$.

Theorem 3. *Let a polynomial system* $\{\hat{q}_n(x)\}_{n=0}^{\infty}$ *satisfy the conditions (9)-(11), (14) and*

$$\|h\|_{L_\omega^q([-1,1])} < \infty \left(1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1\right) \qquad (15)$$

1. Then for any function $f \in W_\omega^p([-1,1])$*, satisfying (13), for Λ-means (6) we have*

$$\|U_n f(x)\|_{W_\omega^p(K)} \le C_p \|f\|_{W_\omega^p([-1,1])} < \infty$$

on an arbitrary compact subset $K \subset \mathcal{E}_m$*, where the constant* $C_p > 0$ *does not depend on* $n \in \mathbb{Z}_+$ *and the function* f*.*

2. (Compact Λ −summability) For any function $f \in W_\omega^p([-1,1])$*, satisfying (13), on all compact subset* $K \subset \mathcal{E}_m$ *there is a relation:*

$$\|U_n f(x;\Lambda) - f(x)\|_{W_\omega^p(K)} \to 0\ (n \to \infty).$$

The statements 1, 2 are preserved in the case $|a_k| > 1\ (k = 1,2,\dots,m)$, if the measure μ satisfies the Szegö condition.

The following propositions p lay an important role in the study of linear summation methods. The kernel of Λ -means has the form

$$K_n(t,x;\Lambda) = \sum_{k=0}^{n} \lambda_k^{(n)} \ \hat{q}_k(t)\hat{q}_k(x) \ (\mathrm{x} \in \mathcal{E}_{\mathrm{m}}, n \in \mathbb{Z}_+).$$

Let us introduce the function

$$\widetilde{K}_n(t,x;\Lambda) = \frac{K_n(t,x;\Lambda)}{h(t)h(x)} \ (t,\mathrm{x} \in \mathcal{E}_{\mathrm{m}};\ n \in \mathbb{Z}_+).$$

Nonnegative function $G_n^*(t,x)$ $(t \in (a,b), x \in [\alpha,\beta] \subset (a,b);\ n \in \mathbb{Z}_+)$ is called "a humpbacked majorant" for the sequence $G_n(t,x)$ in the variable t at the point x if the following conditions are satisfied:

(1) For all $n \in Z_+$ *and* $t \in (a,b), x \in [\alpha,\beta]$
$$|G_n(t,x)| \le G_n^*(t,x);$$
(2) For fixed $n \in Z_+, x \in [\alpha,\beta]$ the function $G_n^*(t,x)$ is nondecreasing on (a,x) and nonincreasing on (x,b).

Lemma 1. *Let a orthonormal polynomial system $\{\hat{q}_n(x)\}_{n=0}^{\infty}$ satisfy conditions (9)-(11) and for the entries of T-regular matrix Λ (5) the estimate (12) holds. Then for the function $\widetilde{K}_n(t,x;\Lambda)$ there is "a humpbacked majorant" for which the following estimate is valid*

$$\int_{-1}^{1} \widetilde{K}_n^*(t,x;\Lambda)\, h(t)\omega(t)dt \le C \ (x \in \mathrm{K};\ n \in \mathbb{Z}_+),$$

where the constant $C > 0$ does not depend on $x \in \mathrm{K}$ and $n \in \mathbb{Z}_+$.

Lemma 2 *Let a orthonormal polynomial system $\{\hat{q}_n(x)\}_{n=0}^{\infty}$ satisfy conditions (9) and (14).*
If

$$\int_{-1}^{1} |f(t)| h(t) d\mu(t) < \infty, \int_{-1}^{1} h(t) d\mu(t) < \infty,$$

is satisfied for a function $f \in \mathfrak{R}$, then the following relation is valid

$$\sum_{n=0}^{\infty} c_n(f)\, \hat{q}_n^{(i)}(a_k) = f^{(i)}(a_k)$$
$$(k = 1, \dots, m;\ i = 0,1,\dots,N_k).$$

The statement is preserved in the case $|a_k| > 1 (k = 1,2,\dots,m)$, if the measure μ satisfies the Szegö condition.

Remarks 1. Fourier-Gegenbauer polynomial system $\{\hat{q}_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ satisfy the conditions of Theorems 1–3 and Lemmas 1-2.

2. The Cesaro mean (C,α) $(\alpha > 0)$ and the Berstein – Rogosinski mean (see [38]) satisfy the condition (12).

7 Open problems

1. To investigate the order of approximation of function by linear means of Fourier-Sobolev series.
2. To conduct experimental research in this topic.
3. Study multiple Fourier-Sobolev series.

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