

A note on various special calculations between the Ramp function and Caputo-type derivatives

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Abstract - This special note provides preliminary information on the ramp (signal) function, which plays a significant role in engineering and science, and offers advantages for the efficient use of various computer-aided programs in different types of (mathematical) computations. Following this, we will present the fundamentals of certain Caputo-type calculations, which are also associated with a form of fractional calculus of arbitrary order. Furthermore, several specific computations involving the Caputo-type derivatives of the ramp function will be performed, and the key results obtained from these computations will be discussed.

Key-Words: The ramp (signal) function, Caputo type calculations, Fractional calculus of arbitrary order, Applications of the Caputo-type derivatives, Special functions.

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1 Some information on the Ramp (Signal) Function

The Ramp function is a mathematically important concept that has a part in an significant role in nearly all fields of science and engineering. Commonly, it is denoted as $r := r(x)$ and it is a real-valued function whose graph resembles a ramp.

In mathematical analysis, the variable (or parameter) x is known as the *independent* variable, while r (or $r(x)$) is referred to as the *dependent* variable of the function. This special function is also typically defined by the following form given by

$$r(x) = \begin{cases} x & \text{when } x \geq 0 \\ 0 & \text{when } x < 0 \end{cases}, \quad (1)$$

whose graph is below:

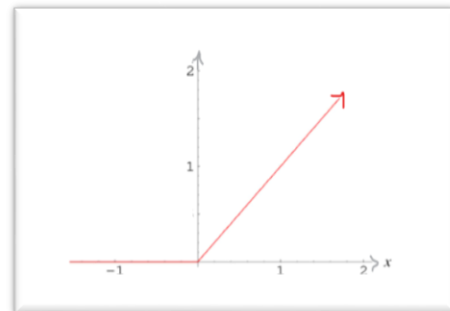


Fig 1: The graph of the Ramp function

The Ramp (Signal) function produces linearly increasing (or decreasing) output values over the (independent) variable x , with a specified slope. In addition, when the variable x represents time, the starting time of the ramp can be optionally defined. The slope and starting time can be either constant or variable, depending on the application.

Furthermore, the Ramp function can be represented by multiple equivalent definitions. The term “ramp” is also used to describe various

functions derived through scaling and shifting. In general, a common example is defined as: “0 for negative input values and equal to the input for non-negative values.” This special case is known as the *unit ramp function*, which is usually denoted by $u := u(x)$. It has a slope of 1 and starts at $x = 0$.

In particular, in the written literature, this special function is commonly referred to as the positive part function. At the same time, in the field of machine learning, it is generally known as the ReLU (Rectified Linear Unit) activation function, or simply as a rectifier, drawing an analogy to half-wave rectification in electrical engineering and related fields. Furthermore, in statistics, *when* used as a likelihood function, it forms the basis of the Tobit model. The function is known by various names across different disciplines and has numerous applications in both mathematics and engineering, depending on the context.

Moreover, various differentiable variants of the Ramp function have been developed to accommodate specific modeling requirements in different fields.

As is well known in mathematics and applied sciences, especially when analyzing the relationship between dependent and independent variables through differentiation or integration, the Ramp function has important derivative properties. Specifically, the first derivative of the Ramp function is the Heaviside step function, also called the unit step function, while its second derivative corresponds to the Dirac delta distribution (or δ -distribution), also known as the unit impulse.

These two special functions are typically denoted by $H(x)$ and $\delta(x)$, respectively, and can be defined by the following mathematical expressions:

$$u(x) = \begin{cases} 1 & \text{when } x \geq 0 \\ 0 & \text{when } x < 0 \end{cases} \quad (2)$$

and

$$\delta(x) = \begin{cases} 0 & \text{when } x > 0 \\ 0 & \text{when } x < 0 \end{cases} \quad (3)$$

We note here that, especially in mathematical analysis (and its comprehensive applications), the Dirac delta function (or $\delta(x)$ distribution), also known as the unit impulse function, is a generalized function (or distribution) defined over the set of real numbers. It is identically zero everywhere except at the origin, yet its integral

over the entire real line equals one. Informally, it is also denoted as

$$\delta(x) = \begin{cases} 0 & \text{when } x > 0 \\ \infty & \text{when } x < 0 \end{cases} \quad (4)$$

Since no ordinary (classical) function satisfies these properties, a rigorous formulation of the delta “function” requires the use of limits or, more formally, the frameworks of measure theory and distribution theory.

Originally introduced by the renowned physicist Paul Dirac, the delta function has become a fundamental concept in physics and engineering, particularly for representing point masses and instantaneous impulses. It is called the “delta function” because it serves as the continuous counterpart to the Kronecker delta, which is defined on a discrete domain and takes only the values 0 and 1.

For some time, the delta function lacked a formal mathematical justification. This issue was resolved by Laurent Schwartz, who developed the theory of distributions, within which the delta function is rigorously defined as a linear functional acting on a space of test functions.

For more detailed information on those special functions (and some of their applications) topics mentioned above, readers are referred to the key studies listed in the mentined references, [1], [2], [11], [14], [16], [21].

2 A Brief Introduction to Some Caputo-Type Calculations

As is well known, various type derivatives and their applications are inalienable mathematical tools that frequently appear in the mathematical literature and are also involved in a wide variety of calculations. These tools play a crucial role in nearly all fields of science and engineering, particularly in mathematics and physics. Of course, under favorable conditions, these logical calculations may also take the form of fractional derivatives (and integrals) of arbitrary order, commonly referred to as arbitrary fractional-order operations.

Such fractional-order calculations, namely the mentioned type derivatives, in particular, continue to find valuable and sustainable applications across both scientific and engineering disciplines, [7].

Also see the earlier studies in the references, [8], [9].

One notable category of these special calculations involves the well-known Caputo-type approach, specifically fractional-order derivatives. In this second part of the investigation, we will first present some fundamental information about this particular type of derivative. In special, the essential concepts of the Caputo fractional derivative(s), used for various computations involving fractional-order operations such as integrals and derivatives, were originally introduced by M. Caputo in the context of solving certain boundary value problems arising in the theory of viscoelasticity, [12].

Let us now try to introduce the relevant concepts in a simple and concise manner.

First of them, let the well-known notations $\mathbf{R}$, $\mathbf{R}^+$ and $\mathbf{N}^+$ denote the set of real numbers, the set $[0, \infty)$ and the set $\{1, 2, 3, \dots\}$, respectively.

Definition 1. Let $f := f(x)$ be a continuous (or piecewise continuous) function defined from $\mathbf{R}^+$ to $\mathbf{R}$. Its Caputo type fractional derivative of order α is denoted by

$${}^C D^\alpha [f(x)] \equiv D^\alpha [f(x)] \equiv D^\alpha [f],$$

and is also defined by

$$D^\alpha [f(x)] = \frac{1}{\Gamma(v-\alpha)} \int_0^x \frac{f^{(v)}(t)}{(x-t)^{\alpha-v+1}} dt, \quad (5)$$

where

$$v-1 \leq \alpha < v \quad \text{and} \quad v \in \mathbf{N}^+, \quad (6)$$

and the familiar notation $\Gamma(\cdot)$ denotes the Gamma function with the parameter a defined by

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt \quad (a > 0). \quad (7)$$

As is well known, the notation $f^{(v)}(t)$ denotes the v th ordinary derivative of the related function f . Namely, for $f := f(t)$,

$$f^{(v)}(t) \equiv \frac{d^v f}{dt^v} \quad (8)$$

for all $v := 1, 2, 3, \dots$.

In the same time, the Gamma function defined by (7) represents an improper integral. This integral converges for all real numbers except zero and the negative integers. We also denote the related set of the real numbers as

$$\mathbf{R}^* := \mathbf{R} - \{0, -1, -2, -3, \dots\}.$$

Although this special investigation is primarily concerned with positive real numbers, the graphical relationship between all real numbers in the special set $\mathbf{R}^*$ and the Gamma function is presented just below.

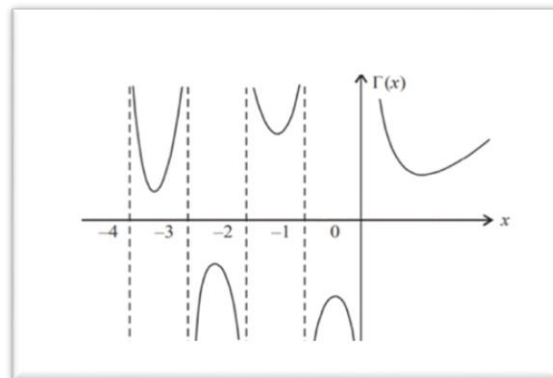


Fig 2: The graph of the Gamma Function

Furthermore, one of them relates directly to the assertion given by

$$\mathbf{B}(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad (a, b \in \mathbf{R}^+), \quad (9)$$

where the notation $\mathbf{B}(\cdot, \cdot)$ expresses the familiar Beta function defined by

$$\mathbf{B}(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt. \quad (10)$$

In special, the essential identity presented in (9) immediately yields that

$$\mathbf{B}(a, b) = \frac{(a-1)!(b-1)!}{(a+b-1)!}, \quad (11)$$

where $a, b \in \mathbf{N}^+$.

It is important to emphasize here that these special functions are highly suitable for, and widely used in, the fields of science and engineering, particularly in mathematics and physics. The calculations required by the equation (5) are especially useful, as the special functions defined by the equation (10) cannot be expressed in terms of elementary functions.

Furthermore, the wide range of results that can be obtained using the function given by simple form in (7) makes it particularly valuable. For those and some their applications, see more information, [10], [11], [12], [13].

In the same time, since they are integrated into scientific software packages such as Mathematica and MATLAB, they also play a crucial role in a wide range of numerical computations. For more

information, see the scientific investigations in the references of this special investigation, [14], [15], [16], [17], [18].

In short, under appropriate conditions, as it can be easily perceived the existence of Caputo-type calculations, *which* are fractional derivatives of arbitrary order for a suitable function f , depends on the existence of the special integral defined by (5).

Notably, under the condition specified in equation (6), the special function introduced in equation (7), namely the Gamma function plays a fundamental and indispensable role in the definition of the Beta function given in the equation (10), as well as in various derivatives of arbitrary fractional order.

Therefore, the fact that these mentioned functions possess broad and versatile identities and properties greatly expand the scope and applicability of the aforementioned fractional-order type calculations.

Now, according to our special information above, based on the specific details provided in items (7) through (11), it is worth emphasizing the numerous potential implications of the definition given in item (5), as outlined in Definition 1.

- (i) Using fundamental mathematical information and also applying a change of variable such as

$$x - t = ux \quad (12)$$

to the special definition given in (5), the integral then takes the following useful form:

$$\begin{aligned} \mathbf{D}^\alpha [f(x)] &= \frac{x^{v-\alpha}}{\Gamma(v-\alpha)} \\ &\times \int_0^1 \frac{f^{(v)}(x(1-u))}{u^{\alpha-v+1}} du. \end{aligned} \quad (13)$$

- (ii) The Caputo-type derivative possesses the linearity property. For two suitable functions $f_i := f_i(x)$ ($i = 1, 2$), it can be easily shown that the following essential result holds:

$$\begin{aligned} \mathbf{D}^\alpha [c_1 f_1 + c_2 f_2] \\ = c_1 \mathbf{D}^\alpha [f_1] + c_2 \mathbf{D}^\alpha [f_2], \end{aligned} \quad (14)$$

where c_1 and c_2 any scalars. In the same time, this fundamental property can also be extended to n functions f_i , where $i = 1, 2, \dots, n$. Therefore, to verify this, it is sufficient to use the main definition presented in (5)

along with basic mathematical (integral) knowledge.

- (iii) The index law for the Caputo-type derivative of arbitrary order also holds. Namely, in consideration of the aforementioned definition in (1), for all α and β , it can be also shown that

$$\mathbf{D}^\alpha [f] \cdot \mathbf{D}^\beta [f] = \mathbf{D}^{\alpha+\beta} [f], \quad (15)$$

where $v - 1 \leq \alpha < v$ and $v - 1 \leq \beta < v$ when $v \in \mathbf{N}^+$. To verify this, it is sufficient to use classical integral knowledge within the framework of the definition given in (5).

- (iv) If $\alpha := 0$ is chosen in Definition 1, then $v := 1$ necessarily follows, which leads to the main implication given by

$$\begin{aligned} \mathbf{D}^0 [f] &= \frac{1}{\Gamma(1-0)} \int_0^x \frac{f^{(1)}(t)}{(x-t)^{0-1+1}} dt \\ &= \frac{1}{\Gamma(1)} \int_0^x f'(t) dt \\ &= f(x), \end{aligned} \quad (16)$$

that is, that the Caputo derivative (of the related function f) of order $\alpha := 0$ exists when $t > 0$.

- (v) When considering Definition 1, since

$$v - 1 \leq \alpha < v,$$

it follows that $\alpha := v - 1$, where v is any natural number. This also gives us the equivalent assertions given by

$$\begin{aligned} \mathbf{D}^{v-1} [f] &= \frac{1}{\Gamma(v-v+1)} \int_0^x \frac{f^{(v)}(t)}{(x-t)^{v-1-v+1}} dt \\ &= \frac{1}{\Gamma(1)} \int_0^x f^{(v)}(t) dt \\ &= f^{(v-1)}(x), \end{aligned} \quad (17)$$

which also shows that the Caputo derivative (of the related function f) of integer order exists. Naturally, it can be also expressed as in the equivalent form:

$$\mathbf{D}^n [f] \equiv f^{(n)}(x), \quad (18)$$

for all $n \in \mathbf{N}_0 := \mathbf{N}^+ \cup \{0\}$.

3 Some Results, Implications and Applications

In this section, some fundamental results will first be obtained by considering the Caputo-type calculations (derivatives) presented in the second section. Then, based on these obtained results, the discussion will proceed towards certain

implications and some of their possible applications, including the special functions introduced in the first section.

By taking into consideration of the definition defined in (5) (or, the equivalent form given by (11)), the following remark consisting of Caputo type derivative(s) of any constant function can be first determined.

Remark 1. Let $f := f(x) \equiv A$. In view of the definition in (12), for all $n \in \mathbf{N}^+$, all ordinary derivatives of the constant function f :

$$f^{(n)}(t) \equiv \frac{d^n}{dt^n}(A) = 0. \quad (19)$$

Hence, according to the definition of the Caputo type derivative(s), we first have

$$\mathbf{D}^\alpha[A] = 0 \quad (20)$$

for some α with $n - 1 \leq \alpha < n$.

Remark 2. As we have emphasized before, the information between (7) and (10) facilitates the computation of the integrals in (5) (or (10)). As clearly shown in the Fig 2, the fact that both special functions are defined on $\mathbf{R}^*$ significantly broadens the scope of Caputo-type derivative(s) for the power function given by

$$f(x) = x^q \quad (q \in \mathbf{R}^*). \quad (21)$$

For our next goals, let us now focus on the fractional derivative of the corresponding power function.

a) Let $q \leq v - 1$ and $v, q \in \mathbf{N}^+$. Then, the mentioned definition easily presents us the pending result:

$$\mathbf{D}^\alpha[x^q] = 0 \quad (v - 1 \leq \alpha < v). \quad (22)$$

b) Let $q > v - 1$, $v \in \mathbf{N}^+$ and $q \in \mathbf{R}^+$. Then, by taking cognizance of

$$\begin{aligned} \frac{d^v}{dt^v}(x^q) &\equiv (x^q)^{(v)} \\ &= q(x^{q-1})^{(v-1)} \\ &= q(q-1)(x^{q-2})^{(v-2)} \\ &= \dots \\ &= q(q-1) \cdot \dots \cdot (q-v+1)x^{q-v} \\ &= \frac{\Gamma(q+1)}{\Gamma(q-v+1)}x^{q-v}, \end{aligned} \quad (23)$$

and in view of the Gamma function in (7), the Beta function in (10), the private information

presented by (5) together with (12) and (13), the equivalent results given by

$$\begin{aligned} \mathbf{D}^\alpha[x^q] &= \frac{1}{\Gamma(v-\alpha)} \int_0^x \frac{(t^q)^{(v)}}{(x-t)^{\alpha-v+1}} dt \\ &= \frac{1}{\Gamma(v-\alpha)} \int_0^x \left(\frac{\Gamma(q+1)}{\Gamma(q-v+1)} t^{q-v} \right. \\ &\quad \left. \times (x-t)^{v-\alpha-1} \right) dt \\ &= \frac{\Gamma(q+1)}{\Gamma(v-\alpha)\Gamma(q-v+1)} x^{q-\alpha} \\ &\quad \times \int_0^1 u^{v-\alpha-1} (1-u)^{q-v} du \\ &= \frac{\Gamma(q+1)}{\Gamma(v-\alpha)\Gamma(q-v+1)} x^{q-\alpha} \\ &\quad \times B(v-\alpha, q-v+1) \\ &= \frac{\Gamma(q+1)}{\Gamma(v-\alpha)\Gamma(q-v+1)} \\ &\quad \times \frac{\Gamma(v-\alpha)\Gamma(q-v+1)}{\Gamma(q-\alpha+1)} x^{q-\alpha} \\ &= \frac{\Gamma(q+1)}{\Gamma(q-\alpha+1)} x^{q-\alpha} \end{aligned} \quad (24)$$

is then determined, where α is any real number such that $v - 1 \leq \alpha < v$ with $v \in \mathbf{N}^+$.

The result presented in Remark 2 is a general finding that plays a significant role in formalizing both integer-order and fractional-order Caputo-type derivatives. In particular, based on the information provided by the graph given by Fig 2, it is clear that the Beta function given in (10) will be also defined for all real numbers for which the Gamma function in (7) is defined. Therefore, all possible fractional-order Caputo-type derivatives will certainly be feasible. As some more examples, by setting $q := 2$ in the general result in (24), the Caputo-type derivatives of the elementary function $f(x) = x^2$ can be easily obtained as the form given by

$$\begin{aligned} \mathbf{D}^\alpha[x^2] &= \frac{x^{1-\alpha}}{\Gamma(1-\alpha)} \int_0^x \frac{f^{(1)}(t)}{(x-t)^{\alpha-1+1}} dt \\ &\equiv \frac{x^{1-\alpha}}{\Gamma(1-\alpha)} \int_0^1 \frac{f^{(1)}(x(1-u))}{u^{\alpha-1+1}} du \\ &= \frac{\Gamma(3)}{\Gamma(3-\alpha)} x^{2-\alpha} \\ &= \frac{2!}{\Gamma(3-\alpha)} x^{2-\alpha} \\ &= \frac{2}{\Gamma(3-\alpha)} x^{2-\alpha}, \end{aligned} \quad (25)$$

where $v - 1 \leq \alpha < v$ for all $v \in \mathbf{N}^+$ ($v < 3$).

As special cases of the result obtained in (25), when choosing the possible values of the parameter v as $v := 1$ and $v := 2$ in the mentioned

result (25), for the related function $f(x) = x^2$, the Caputo-type derivatives of orders $\alpha = 1/2$ and $\alpha = 3/2$ can be easily received as in the forms:

$$\begin{aligned} \mathbf{D}^{1/2}[x^2] &= \frac{2}{\Gamma(3-1/2)} x^{2-1/2} \\ &= \frac{2}{\Gamma(5/2)} x^{3/2} \\ &= \frac{2}{\Gamma(5/2)} x\sqrt{x} \end{aligned} \quad (26)$$

and

$$\begin{aligned} \mathbf{D}^{3/2}[x^2] &= \frac{2}{\Gamma(3-3/2)} x^{2-3/2} \\ &= \frac{2}{\Gamma(3/2)} \sqrt{x}, \end{aligned} \quad (27)$$

respectively.

Moreover, if one considers the property of the Gamma function like the familiar factorial form given by

$$\Gamma(n+1) = n\Gamma(n) \quad (28)$$

and the computational result given by

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad (29)$$

the fractional order calculations determined as in (26) and (2.7) can be easily redesigned as the special assertions given by

$$\begin{aligned} \mathbf{D}^{1/2}[x^2] &= \frac{2}{\Gamma(\frac{5}{2})} x\sqrt{x} \\ &= \frac{2}{\frac{3}{2}\Gamma(\frac{3}{2})} x\sqrt{x} \\ &= \frac{2}{\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})} x\sqrt{x} \\ &= \frac{8}{3\sqrt{\pi}} x\sqrt{x} \end{aligned} \quad (30)$$

and

$$\begin{aligned} \mathbf{D}^{3/2}[x^2] &= \frac{2}{\Gamma(\frac{3}{2})} x^{1/2} \\ &= \frac{2}{\frac{1}{2}\Gamma(\frac{1}{2})} \sqrt{x} \\ &= \frac{4}{\sqrt{\pi}} \sqrt{x}, \end{aligned} \quad (31)$$

respectively.

Naturally, of course, the values of the corresponding fractional derivatives of the function $f(x) = x^2$ at a point such as $x_0 = 1$ will be as follows:

$$\begin{aligned} \mathbf{D}^{1/2}[x^2] \Big|_{x=1} &= \frac{8}{3\sqrt{\pi}} x\sqrt{x} \Big|_{x=1} \\ &= \frac{8}{3\sqrt{\pi}} \end{aligned}$$

$$\cong 1,504506 \quad (32)$$

and

$$\begin{aligned} \mathbf{D}^{3/2}[x^2] \Big|_{x=1} &= \frac{4}{\sqrt{\pi}} \sqrt{x} \Big|_{x=1} \\ &= \frac{4}{\sqrt{\pi}} \\ &\cong 2,256699. \end{aligned} \quad (33)$$

We would like to highlight some important information here. The specific calculations between (30) and (33) can be repeated as many times as desired, at the discretion of interested researchers. The calculations presented in (32) and (33) can also be compared with the values of the first-second derivatives of the relevant functions evaluated at the point $x_0 = 1$. Naturally, the related calculations can be extended to various derivatives of different orders as well. For example, one can refer to certain results in some special researches, [19], [20], [21].

4 Conclusion and Some Suggestions

This special research, which consists of four chapters, provides some basic information in the first chapter, including the ramp (signal) function, which plays a very different role in engineering and science, and its special implications, such as those in (2)-(4).

The second chapter introduces the Caputo-type calculation, one of the calculations related to various fractional-order derivatives, and some special functions that are quite related to it. Relevant information is found in (5) and (11). This chapter also includes some additional special calculations related to Caputo-type calculations, including highlights and conclusions, in (12) through (18).

The third chapter addresses and highlights some special calculations related to the mentioned Caputo-type derivatives that may be important for the purpose of this research. Relevant information is also found in (19) and (33).

In particular, each of the three special sections above includes specific information and references related to the objectives of this research, and some goals are highlighted for researchers.

Notably, the specific calculations between (21) and (24), obtained using Remark 2, can facilitate the determination of more complex fractional metric calculations. In other words, *when* solving rather complex integrals generated by the functions involved in the integrals given in (5) or (13), applying the information provided by Remark 2, followed by the appropriate function expansion, will greatly simplify the solution. In many respects, we believe that the earlier studies presented in the studies, [22], [23], [24], [25] and [25], will also be valuable resources for interested researchers.

As final words, with the help of the indicated calculations, this special can be finished with some Caputo-type fractional derivatives related to the ramp (signal) function. For those, we want to focus on the most general form of the ramp function. If the linearity property in (14) is taken into consideration, Remark 1 is used and $q = 1$ is selected in Remark 2, various Caputo type derivatives of fractional order α for the function being of

$$r_{a,b}(x) = a + bx \quad (a, b \in \mathbf{R} \text{ with } b \neq 0)$$

can be then determine as the elementary form given by

$$\begin{aligned} \mathbf{D}^\alpha [r_{a,b}(x)] &= \mathbf{D}^\alpha [a + bx] \\ &= \mathbf{D}^\alpha [a] + \mathbf{D}^\alpha [bx] \\ &= \mathbf{D}^\alpha [a] + b\mathbf{D}^\alpha [x] \\ &= 0 + b \frac{\Gamma(1+1)}{\Gamma(1-\alpha+1)} x^{1-\alpha} \\ &= \frac{b \Gamma(2)}{\Gamma(2-\alpha)} x^{1-\alpha} \\ &= \frac{b}{\Gamma(2-\alpha)} x^{1-\alpha} \end{aligned} \quad (34)$$

for some (suitable) values of the parameter α . This elementary result also gives us a great number of special results when choosing the different values of α . As two special examples, in light of the information in (24) and (34), *when* one selects the special values given by

$$\alpha = \frac{1}{2} \quad \text{and} \quad \alpha = \frac{1}{3},$$

the followings:

$$\begin{aligned} \mathbf{D}^{1/2} [r_{0,1}(x)] &= \mathbf{D}^{1/2} [0 + 1x] \\ &= \mathbf{D}^{1/2} [x] \\ &= \frac{1}{\Gamma(2-\frac{1}{2})} x^{1-1/2} \\ &= \frac{1}{\Gamma(\frac{3}{2})} \sqrt{x} \\ &= \frac{1}{\frac{1}{2} \Gamma(\frac{1}{2})} \sqrt{x} \\ &= \frac{2}{\Gamma(\frac{1}{2})} \sqrt{x} \end{aligned} \quad (35)$$

and

$$\begin{aligned} \mathbf{D}^{1/3} [r_{1,2}(x)] &= \mathbf{D}^{1/3} [1 + 2x] \\ &= 2\mathbf{D}^{1/3} [x] \\ &= \frac{1}{\Gamma(2-\frac{1}{3})} x^{1-1/3} \\ &= \frac{1}{\Gamma(\frac{5}{3})} \sqrt[3]{x^2} \\ &= \frac{1}{\frac{2}{3} \Gamma(\frac{2}{3})} \sqrt[3]{x^2} \\ &= \frac{3}{2 \Gamma(\frac{2}{3})} \sqrt[3]{x^2} \end{aligned} \quad (36)$$

can be easily obtained. Here, when using the special value in (29) for the mentioned result (35), the following special results:

$$\begin{aligned} \mathbf{D}^{1/2} [r_{0,1}(x)] &= \frac{2}{\sqrt{\pi}} \sqrt{x} \\ &\cong 1,128379 \sqrt{x} \end{aligned}$$

and

$$\begin{aligned} \mathbf{D}^{1/2} [r_{0,1}(x)] \Big|_{x=1} &= \frac{2}{\sqrt{\pi}} \\ &\cong 1,128379. \end{aligned}$$

are also achieved. However, we cannot easily perform similar calculations for the special result given in (35). As is well known, while the evaluation of the value of $\Gamma(1/2)$ can be done using basic mathematical knowledge, computing $\Gamma(1/3)$ is not straightforward. As emphasized in Section 2, such values can also be calculated by using a variety of available computer software programs or packages, such as C, C++, Python, Mathematica, MATLAB, and others. As is known, these complex calculations are frequently used in science and engineering, particularly in applied mathematics, physics, and statistics. For numerical evaluations like these, as well as for the special

cases presented in the references given by (1), (2), (25)-(33), (35), and (36), a range of commonly used basic programs or specialized mathematical software tools offer considerable convenience. Further investigation is left to the individual efforts of interested researchers.

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