

Modified LIATHE Rule by Including Inverse Hyperbolic Functions

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Abstract: The study aimed to evaluate indefinite integrals involving inverse hyperbolic functions using column integration. LIATHE (Logarithm, Inverse trigonometric, Algebraic, Trigonometric, Hyperbolic and Exponential) rule was modified by including inverse hyperbolic function that makes the meaning into (Logarithm, Inverse trigonometric/ Inverse hyperbolic, Algebraic, Trigonometric, Hyperbolic and Exponential). Applications of modified LIATHE rule on integrals of products that include inverse hyperbolic functions were illustrated. Horowitz's diagram was adopted and modified for better evaluation of the integrals. Inclusion of inverse hyperbolic functions in LIATHE rule made this expository study distinct from other studies in Calculus. Generalized formula that includes inverse hyperbolic functions of the form $\int \sinh^{-1}(ax) dx$, $\int \tanh^{-1}(ax) dx$, $\int \operatorname{sech}^{-1}(ax) dx$, $\int x^{n-1} \sinh^{-1} x^n dx$, $\int x^{n-1} \cosh^{-1} x^n dx$, $\int x^{n-1} \tanh^{-1} x^n dx$, $\int x^{n-1} \operatorname{sech}^{-1} x^n dx$, $\int e^{ax} \sinh^{-1} e^{ax} dx$, $\int e^{ax} \tanh^{-1} e^{ax} dx$, and $\int e^{ax} \operatorname{sech}^{-1} e^{ax} dx$ are derived.

Keywords: inverse hyperbolic functions, elementary functions, reduction formula, tabular integration by parts, LIATHE, and modified LIATHE rule

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1 Introduction

Integration and differentiation are the two parts of Calculus and being able to integrate any complicated given function is not so easy and requires experience as there are many techniques. One common method of integrating more complicated functions of a certain type is integration by parts. Integration by Parts is used to compute integral of the form, $\int f(x)g(x) dx$, typically one use u-v method and it can be expressed in the following form, $\int u dv = uv - \int v du$ [9].

In using the above formula, one has to determine which will be the u and the dv . Remember that there are five classes of elementary functions, recalled by the mnemonic LIATE, the acronym was first introduced in the work of Kasube [5]. Whichever function comes first in the following list should be u : **L** - Logarithmic functions: $\ln x, \log_{25} x$, etc., **I** - Inverse trigonometric functions: $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x$, etc., **A** - Algebraic functions: $x, x^3, 4x^{55}$, etc., **T** - Trigonometric functions: $\sin x, \cos x, \tan x$, etc., **E** - Exponential functions: $e^x, 2^x$, etc.

The above technique can also be applied in integrating inverse hyperbolic functions for these functions possess closely analogous properties with that of six inverse trigonometric functions. Inverse hyperbolic functions are the inverse functions of the hyperbolic functions. The inverse hyperbolic sine of

x , $\sinh^{-1} x$; inverse hyperbolic cosine of x , $\cosh^{-1} x$; inverse hyperbolic tangent of x , $\tanh^{-1} x$; inverse hyperbolic cotangent of x , $\coth^{-1} x$; inverse hyperbolic secant of x , $\operatorname{sech}^{-1} x$ and inverse hyperbolic cosecant of x , $\operatorname{csch}^{-1} x$ are the six inverse hyperbolic functions. Hyperbolic functions and their inverses occur in many problems in engineering and sciences, which made them as important as trigonometric and inverse trigonometric functions.

Robles [7] in his study, mentioned that hyperbolic functions are analogous to trigonometric functions or circular functions. Trigonometric functions are related to the circle, $x^2 + y^2 = 1$ whereas the hyperbolic functions are related to the standard hyperbola, $x^2 - y^2 = 1$. He showed the geometric similarity of trigonometric functions to hyperbolic functions and presented the generalized formula for the integration of hyperbolic functions. Since hyperbolic functions are closely similar to trigonometric functions so it is expected that their inverses are also analogous. The inverse trigonometric functions can be solved by integration by parts therefore the technique can be extended to inverse hyperbolic functions. Since inverse hyperbolic functions are also as important as inverse trigonometric functions, the researcher conducted a study that will expose the integration of product that involves inverse hyperbolic functions.

The researcher modified the LIATHE rule by including inverse hyperbolic functions. This paper attempted to expand the integration by parts technique to integrals involving inverse hyperbolic functions to further help the students solve those integrals. This study was only limited in modifying LIATHE to involve inverse hyperbolic functions in combinations with other elementary functions and generalizing a formula that include inverse hyperbolic functions which will help students in evaluating integral of products that involve inverse hyperbolic functions.

The researcher organized the paper as follows: the section 2 presents the modification of the LIATHE rule by including inverse hyperbolic functions in combination with other elementary functions such as algebraic, exponential, trigonometric, hyperbolic and logarithmic functions. While section 3 shows the derivation of some reduction formula with the use of tabular integration by parts and application of the modified LIATHE rule that includes inverse hyperbolic functions in combination with other elementary functions stated in section 2.

2 Modification of LIATHE Rule

Integration by parts formula, $\int u dv = uv - \int v du$. The given integral is divided into two parts. One part is u and the other is dv , so that the product of u and dv is the original integral. Then, obtain du by differentiating u and obtain v by integrating dv . The trick here is to let dv be something that is not only easy to integrate but does not get worse when one integrates it, and u is the rest of the equation which is something that gets better when being differentiated.

The part dv need to be integrated twice, so select functions like $e^x dx$, $\sin x dx$, or $(x + a)^n dx$ as dv , because one can integrate these twice without too much difficulty. Meanwhile, select u as something that gets better like disappearing when being differentiated, then the scheme will work. For example, if you let $u = x$ then $du = dx$, which is helpful because the algebraic function x more or less disappears.

When determining what is u and dv in the integration by parts technique, a LIATE (Logarithmic, Inverse trigonometric, Algebraic, Trigonometric, and Exponential) rule provides a suggestion of how to do that. When applying the LIATE rule, choose u as the one furthest to the left in LIATE and choose dv as the one furthest to the right.

However, the researcher adapted the mnemonic LIATHE used in the study of Alcantara

[1] and modified the mnemonic by including inverse hyperbolic functions. The researcher presented (I) as inverse trigonometric/ inverse hyperbolic functions that makes the modified LIATHE into: Logarithmic, Inverse trigonometric/ inverse hyperbolic, Algebraic, Trigonometric, Hyperbolic and Exponential functions. The modification of LIATHE was possible because inverse trigonometric functions behave like that of inverse hyperbolic functions when differentiating and integrating.

Properties of the LIATHE functions could also help in determining what function is easy to differentiate (u part) and integrate (dv part). Logarithmic functions are one of the most preferable to differentiate because the results lead to algebraic functions. Inverse trigonometric functions are also one of the most preferable to differentiate because the results also lead to algebraic functions that are in rational or radical form. Since inverse hyperbolic function is analogous to inverse trigonometric function, therefore its differentiation also leads to algebraic functions. Logarithmic functions and inverse trigonometric/ hyperbolic functions are usually placed in the u part because there are no learn basic integration formulas for them. Algebraic/Polynomial function can be placed in either u or dv part because it can be integrated or differentiated. It is usually preferable to use in the u part since the results are also algebraic functions and eventually get down to constant while integrating repeatedly leads the exponents to infinity. Trigonometric functions differentiation gets the results into trigonometric functions too that is why these are one of the most preferable to integrate. Hyperbolic functions are analogous to trigonometric functions so the results of the differentiation are also similar. While exponential functions are the easiest to differentiate and also one of the most preferable to integrate since the results lead to exponential function [6],[8],[10].

2.1 Including Inverse Hyperbolic Functions in Combination with Algebraic Functions.

Consider the product $\int f(x)g(x) dx$, where $f(x)$ is an algebraic function and $g(x)$ is an inverse hyperbolic function.

In this integrand, the given algebraic function is the third letter in the acronym LIATHE and inverse hyperbolic is in the second letter. Applying modified LIATHE rule, it is appropriate to let $u =$ inverse hyperbolic and $dv =$ algebraic dx .

Example 1. Evaluate $\int x \tanh^{-1} x dx$

It is evident in the given integrand that x is an Algebraic function which is the third letter in the

LIATHE while $\tanh^{-1} x$ is an Inverse Hyperbolic function which is the second letter. Applying the LIATHE rule, let $u = \tanh^{-1} x$ and $dv = x dx$.

2.2 Including Inverse Hyperbolic Functions in Combination with Exponential Functions.

Consider the product $\int f(x)g(x) dx$, where $f(x)$ is an exponential function and $g(x)$ is an inverse hyperbolic function.

In this case, the given exponential function is the last letter in the acronym LIATHE and inverse hyperbolic is in the second letter. Applying modified LIATHE rule, it is appropriate to let $u =$ inverse hyperbolic and $dv =$ exponential dx .

Example 2. Evaluate $\int e^x \sinh^{-1} e^x dx$

It shows in the given integrand that e^x is an Exponential function which is the last letter in LIATHE while $\sinh^{-1} e^x$ is an Inverse hyperbolic function which is the second letter. Applying the LIATHE rule, let $u = \sinh^{-1} e^x$ and $dv = e^x dx$.

2.3 Including Inverse Hyperbolic Functions in Combination with Trigonometric Functions.

Consider the product $\int f(x)g(x) dx$, where $f(x)$ is a trigonometric function and $g(x)$ is an inverse hyperbolic function.

In this case, the given trigonometric function is in the fourth letter of the acronym LIATHE. Applying modified LIATHE rule, it is appropriate to let $u =$ inverse hyperbolic and $dv =$ trigonometric dx .

Example 3. Evaluate $\int \cos x \tanh^{-1}(\sin x) dx$

As shown in the given, the integrand is a product of Trigonometric function, $\cos x$, which is the fourth letter in LIATHE and Inverse hyperbolic function, $\tanh^{-1}(\sin x)$, which is the second letter. Applying the LIATHE rule, let $u = \tanh^{-1}(\sin x)$ and $dv = \cos x dx$.

2.4 Including Inverse Hyperbolic Functions in Combination with Hyperbolic Functions.

Consider the product $\int f(x)g(x) dx$ where $g(x)$ is a hyperbolic function and $f(x)$ is an inverse hyperbolic function.

In this case, the given hyperbolic function comes after trigonometric function and it is in the fifth letter of the acronym LIATHE. Applying the modified LIATHE rule, it is appropriate to let $u =$ inverse hyperbolic and $dv =$ hyperbolic dx .

Example 4. Evaluate $\int \sinh x \cosh^{-1}(\cosh x) dx$

It shows in the given integrand that $\sinh x$ is a Hyperbolic function which is the fifth letter in LIATHE while $\cosh^{-1}(\cosh x)$ is an Inverse hyperbolic function which is the second letter. Applying the LIATHE rule, let $u = \cosh^{-1}(\cosh x)$ and $dv = \sinh x dx$.

2.5 Including Inverse Hyperbolic Functions in Combination with Logarithmic Functions.

Consider the product $\int f(x)g(x) dx$ where $g(x)$ is a logarithmic function and $f(x)$ is an inverse hyperbolic function.

In this case, the given logarithmic function is the first letter in the acronym LIATHE. Applying the modified LIATHE rule, let $u =$ logarithmic and $dv =$ inverse hyperbolic dx . However, for this combination both logarithmic and inverse hyperbolic have no direct integration that is why some interchange L and I as dv part and use ILATHE rule.

Example 5. Evaluate $\int \ln x \cosh^{-1} x dx$

As shown in the given, the integrand consists of $\ln x$, a Logarithmic function which is the first letter in LIATHE and $\cosh^{-1} x$, an Inverse hyperbolic function which is the second letter. Both functions have no direct integration, so one cannot avoid the double integration in the dv part. In this case, both LIATHE and ILATHE rule can be used. Applying the LIATHE rule, let $u = \ln x$ and $dv = \cosh^{-1} x dx$ while applying the ILATHE rule let $u = \cosh^{-1} x dx$ and $dv = \ln x dx$.

3 Derivation of Some Reduction Formula with the Use of Tabular Integration by Parts and Application of Modified LIATHE Rule

Doing calculus problems which require integration by parts can be a lengthy and tedious process, even for someone with experience in finding the integrals of functions. Fortunately, for many of the most common types of integration by parts, there is a fast and simple shortcut available, the tabular method [1]. This method is sometimes known as column integration [2] or tabular integration by parts [3].

Horowitz [4] illustrates in his paper the tabular method in a diagram. He assumed throughout that F and G are smooth enough to allow repeated differentiation and integration, respectively.

Table 1. Tabular integration of $\int F(t)G(t) dt$

Column #1	Column #2
$+F$	G
$-F^1$	$\rightarrow G^{(-1)}$
$+F^2$	$\rightarrow G^{(-2)}$
$-F^3$	$\rightarrow G^{(-3)}$
$\vdots$	$\vdots$
$(-)^n F^n$	$G^{(-n)}$
$(-)^{(n+1)} F^{(n+1)}, \dots$	$\rightarrow G^{(-n-1)}$

The integral $\int F(t)G(t) dt$ is equal to the sum of the terms in table 1. Symbolically,

$$\begin{aligned} \int F(t)G(t) dt &= FG^{(-1)} - F^1 G^{(-2)} \\ &\quad + F^2 G^{(-3)} - \dots + (-)^n F^n G^{(-n-1)} \\ &\quad + (-)^{(n+1)} \int F^{(n+1)} G^{(-n-1)} dt \\ \int F(t)G(t) dt &= \sum_{k=0}^n (-)^k F^k G^{(-k-1)} \\ &\quad + (-)^{(n+1)} \int F^{(n+1)} G^{(-n-1)} dt \end{aligned}$$

The researcher also used trial and error method in the following combination of inverse hyperbolic functions for the derivation and application of modified LIATHE rule.

3.1 Including Inverse Hyperbolic Functions in Combination with Algebraic Functions

Example 6. Derive the formula of the given below, where n is any real number and $n \neq 0$

$$\begin{aligned} \int x^{n-1} \sinh^{-1} x^n dx \\ = \frac{x^n}{n} \sinh^{-1} x^n - \frac{n}{n} \int \frac{x^{2n-1}}{\sqrt{x^{2n} + 1}} dx \end{aligned}$$

Derivation: Applying the LIATHE rule, let $u = \sinh^{-1} x^n$ and $dv = x^{n-1} dx$. A table below can be formed using the column integration technique:

Table 2. Table for evaluating $\int x^{n-1} \sinh^{-1} x^n dx, n \neq 0, n \in \mathbb{R}$

Derivative	Integral	Product
$+ \sinh^{-1} x^n$	x^{n-1}	
$-\frac{nx^{n-1}}{\sqrt{x^{2n} + 1}}$	$\frac{x^n}{n}$	$\frac{x^n}{n} \sinh^{-1} x^n - \frac{n}{n} \int \frac{x^{2n-1}}{\sqrt{x^{2n} + 1}} dx$

From table 2, the integral part on the product column can be simplified and be solved using u-substitution method, let $u = x^{2n} + 1$ and $du = 2nx^{2n-1} dx$ that gives

$$\begin{aligned} \int x^{n-1} \sinh^{-1} x^n dx \\ = \frac{x^n}{n} \sinh^{-1} x^n - \frac{1}{2n} \int u^{-\frac{1}{2}} du \\ \int x^{n-1} \sinh^{-1} x^n dx = \frac{x^n}{n} \sinh^{-1} x^n - \frac{1}{2n} \left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right) \end{aligned}$$

simplify the right side of the equation and substitute u to x , therefore the result is

$$\begin{aligned} \int x^{n-1} \sinh^{-1} x^n dx \\ = \frac{x^n}{n} \sinh^{-1} x^n - \frac{\sqrt{x^{2n} + 1}}{n} + C. \end{aligned}$$

Example 7. Derive the formula of the given below, where n is any real number and $n \neq 0$

$$\begin{aligned} \int x^{n-1} \cosh^{-1} x^n dx \\ = \frac{x^n}{n} \cosh^{-1} x^n \\ - \frac{n}{n} \int \frac{x^{2n-1}}{\sqrt{x^{2n} - 1}} dx \end{aligned}$$

Derivation: Applying the LIATHE rule, let $u = \cosh^{-1} x^n$ and $dv = x^{n-1} dx$. A table below can be formed using the column integration technique:

Table 3. Table for evaluating $\int x^{n-1} \cosh^{-1} x^n dx, n \neq 0, n \in \mathbb{R}$

Derivative	Integral	Product
$+\cosh^{-1} x^n$	x^{n-1}	
$-\frac{nx^{n-1}}{\sqrt{x^{2n}-1}}$	$\frac{x^n}{n}$	$\frac{x^n}{n} \cosh^{-1} x^n$ $-\frac{n}{n} \int \frac{x^{2n-1}}{\sqrt{x^{2n}-1}} dx$

From table 3, the integral part on the product column can be simplified and be solved using u-substitution method, let $u = x^{2n} - 1$ and $du = 2nx^{2n-1} dx$ that gives

$$\begin{aligned} \int x^{n-1} \cosh^{-1} x^n dx &= \frac{x^n}{n} \cosh^{-1} x^n - \frac{1}{2n} \int u^{-\frac{1}{2}} du \\ \int x^{n-1} \cosh^{-1} x^n dx &= \frac{x^n}{n} \cosh^{-1} x^n - \frac{1}{2n} \left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right) \end{aligned}$$

simplify the right side and substitute u to x, therefore the result is

$$\begin{aligned} \int x^{n-1} \cosh^{-1} x^n dx &= \frac{x^n}{n} \cosh^{-1} x^n - \frac{\sqrt{x^{2n}-1}}{n} + C. \end{aligned}$$

Example 8. Derive the formula of the given below, where n is any real number and $n \neq 0$

$$\begin{aligned} \int x^{n-1} \tanh^{-1} x^n dx &= \frac{x^n}{n} \tanh^{-1} x^n - \frac{n}{n} \int \frac{x^{2n-1}}{1-x^{2n}} dx \end{aligned}$$

Derivation: Applying the LIATHE rule, let $u = \tanh^{-1} x^n$ and $dv = x^{n-1} dx$. A table below can be formed using the column integration technique:

Table 4. Table for evaluating $\int x^{n-1} \tanh^{-1} x^n dx, n \neq 0, n \in \mathbb{R}$

Derivative	Integral	Product
$+\tanh^{-1} x^n$	x^{n-1}	
$-\frac{nx^{n-1}}{1-x^{2n}}$	$\frac{x^n}{n}$	$\frac{x^n}{n} \tanh^{-1} x^n$ $-\frac{n}{n} \int \frac{x^{2n-1}}{1-x^{2n}} dx$

From table 4, the integral part on the product column can be simplified and be solved using u-substitution method, let $u = 1 - x^{2n}$ and $du = -2nx^{2n-1} dx$ that gives

$$\begin{aligned} \int x^{n-1} \tanh^{-1} x^n dx &= \frac{x^n}{n} \tanh^{-1} x^n + \frac{1}{2n} \int \frac{du}{u} \\ \int x^{n-1} \tanh^{-1} x^n dx &= \frac{x^n}{n} \tanh^{-1} x^n + \frac{1}{2n} \ln|u| \end{aligned}$$

substitute u to x, therefore the result is

$$\begin{aligned} \int x^{n-1} \tanh^{-1} x^n dx &= \frac{x^n}{n} \tanh^{-1} x^n \\ &+ \frac{1}{2n} \ln|1-x^{2n}| + C. \end{aligned}$$

Example 9. Derive the formula of the given below, where n is any real number and $n \neq 0$

$$\begin{aligned} \int x^{n-1} \operatorname{sech}^{-1} x^n dx &= \frac{x^n}{n} \operatorname{sech}^{-1} x^n \\ &+ \frac{n}{n} \int \frac{x^n}{x\sqrt{1-x^{2n}}} dx \end{aligned}$$

Derivation: Applying the LIATHE rule, let $u = \operatorname{sech}^{-1} x^n$ and $dv = x^{n-1} dx$. A table below can be formed using the column integration technique:

Table 5. Table for evaluating $\int x^{n-1} \operatorname{sech}^{-1} x^n dx, n \neq 0 \in \mathbb{R}$

Derivative	Integral	Product
$+\operatorname{sech}^{-1} x^n$	x^{n-1}	
$-\frac{nx^{n-1}}{x\sqrt{1-x^{2n}}}$	$\frac{x^n}{n}$	$\frac{x^n}{n} \operatorname{sech}^{-1} x^n$ $+\frac{n}{n} \int \frac{x^n}{x\sqrt{1-x^{2n}}} dx$

From table 5, the integral part on the product column can be simplified into

$$\begin{aligned} \int x^{n-1} \operatorname{sech}^{-1} x^n dx &= \frac{x^n}{n} \operatorname{sech}^{-1} x^n + \int \frac{x^{n-1} dx}{\sqrt{1-x^{2n}}} \end{aligned}$$

using integral that yields inverse sine $\sqrt{1-u^2}$, let $u^2 = x^{2n}$, $u = x^n$ and $du = nx^{n-1} dx$, therefore the result is

$$\int x^{n-1} \operatorname{sech}^{-1} x^n dx$$

$$= \frac{x^n}{n} \operatorname{sech}^{-1} x^n + \frac{1}{n} \sin^{-1} x^n + C.$$

Note: The process used in evaluating the integrals in Example 8 and Example 9 can also be applied to evaluate the integrals of products concerning $x^{n-1} \coth^{-1} x^n$ and $x^{n-1} \operatorname{csch}^{-1} x^n$.

Example 10. Derive the formula of the given below, where a is any real number and $a \neq 0$

$$\int \sinh^{-1}(ax) dx$$

$$= x \sinh^{-1} ax - a \int \frac{x}{\sqrt{a^2 x^2 + 1}} dx$$

Derivation: Applying the LIATHE rule, let $u = \sinh^{-1}(ax)$ and $dv = 1 dx$. A table below can be formed using the column integration technique:

Table 6. Table for evaluating $\int \sinh^{-1}(ax) dx$, $a \neq 0, a \in \mathbb{R}$

Derivative	Integral	Product
$+ \sinh^{-1}(ax)$	1	
$-\frac{a}{\sqrt{a^2 x^2 + 1}}$	x	$x \sinh^{-1} ax$ $- a \int \frac{x}{\sqrt{a^2 x^2 + 1}} dx$

From table 6, the integral part on the product column can be solved using u-substitution method, let $u = a^2 x^2 + 1$ and $du = 2a^2 x dx$, that gives

$$\int \sinh^{-1}(ax) dx = x \sinh^{-1}(ax) - \frac{a}{2a^2} \int u^{-\frac{1}{2}} du$$

$$\int \sinh^{-1}(ax) dx = x \sinh^{-1}(ax) - \frac{1}{2a} \left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right)$$

simply and substitute u to x , therefore the result is

$$\int \sinh^{-1} ax dx = x \sinh^{-1} ax - \frac{1}{a} \sqrt{a^2 x^2 + 1} + C.$$

Example 11. Derive the formula of the given below, where a is any real number and $a \neq 0$

$$\int \tanh^{-1}(ax) dx$$

$$= x \tanh^{-1}(ax)$$

$$- a \int \frac{x}{1 - a^2 x^2} dx$$

Derivation: Applying the LIATHE rule, let $u = \tanh^{-1}(ax)$ and $dv = 1 dx$. A table below can be formed using the column integration technique:

Table 7. Table for evaluating $\int \tanh^{-1}(ax) dx$, $a \neq 0, a \in \mathbb{R}$

Derivative	Integral	Product
$+ \tanh^{-1}(ax)$	1	
$-\frac{a}{1 - a^2 x^2}$	x	$x \tanh^{-1}(ax)$ $- a \int \frac{x}{1 - a^2 x^2} dx$

From table 7, the integral part on the product column can be solved using u-substitution method, let $u = 1 - a^2 x^2$ and $du = -2a^2 x dx$, that gives

$$\int \tanh^{-1}(ax) dx = x \tanh^{-1}(ax) + \frac{a}{2a^2} \int \frac{du}{u}$$

$$\int \tanh^{-1}(ax) dx = x \tanh^{-1}(ax) + \frac{1}{2a} \ln|u|$$

substitute u to x , therefore the result is

$$\int \tanh^{-1}(ax) dx = x \tanh^{-1}(ax)$$

$$+ \frac{1}{2a} \ln|1 - a^2 x^2| + C.$$

Example 12. Derive the formula of the given below, where a is any real number and $a \neq 0$

$$\int \operatorname{sech}^{-1}(ax) dx$$

Derivation: Applying the LIATHE rule, let $u = \operatorname{sech}^{-1}(ax)$ and $dv = 1 dx$. A table below can be formed using the column integration technique:

Table 8. Table for evaluating $\int \operatorname{sech}^{-1}(ax) dx$, $a \in \mathbb{R}$

Derivative	Integral	Product
$+ \operatorname{sech}^{-1}(ax)$	1	
$-\frac{a}{ax\sqrt{1 - a^2 x^2}}$	x	$x \operatorname{sech}^{-1}(ax)$ $-\frac{a}{a} \int \frac{x}{x\sqrt{1 - a^2 x^2}} dx$

From table 8, the integral part on the product column can be simplified and be solved using integral that yields inverse sine $\sqrt{1 - u^2}$, let $u^2 = a^2 x^2$, $u = ax$ and $du = a dx$, that gives

$$\int \operatorname{sech}^{-1}(ax) dx = x \cosh^{-1}(ax) - \frac{1}{a} \sin^{-1} u$$

substitute u , therefore the result is

$$\int \operatorname{sech}^{-1}(ax) dx$$

$$= x \cosh^{-1}(ax) - \frac{1}{a} \sin^{-1}(ax) + C.$$

Note: The process used in deriving a generalized formula in Example 10, Example 11 and Example 12 can also be applied to derive a formula for arccosh (ax), arccoth (ax) and arccsch ax.

3.2 Including Inverse Hyperbolic Functions in Combination with Exponential Functions

Example 13. Derive the formula of the given below, where a is any real number and $a \neq 0$

$$\begin{aligned} \int e^{ax} \sinh^{-1}(e^{ax}) dx \\ = \frac{e^{ax}}{a} \sinh^{-1}(e^{ax}) \\ - \frac{a}{a} \int \frac{e^{2ax}}{\sqrt{e^{2ax} + 1}} dx \end{aligned}$$

Derivation: Applying the LIATHE rule, let $u = \sinh^{-1}(e^{ax})$ and $dv = e^{ax} dx$. A table below can be formed using the column integration technique:

Table 9 . Table for evaluating $\int e^{ax} \sinh^{-1}(e^{ax}) dx$, $a \neq 0$, $a \in \mathbb{R}$

Derivative	Integral	Product
$+ \sinh^{-1}(e^{ax})$	e^{ax}	
$-\frac{ae^{ax}}{\sqrt{e^{2ax} + 1}}$	$\frac{e^{ax}}{a}$	$\frac{e^{ax}}{a} \sinh^{-1}(e^{ax}) - \frac{1}{a} \int \frac{e^{2ax}}{\sqrt{e^{2ax} + 1}} dx$

From table 9, the integral part on the product column can be simplified and solved using u-substitution method, let $u = e^{2ax} + 1$ and $du = 2ae^{2ax} dx$, that gives

$$\begin{aligned} \int e^{ax} \sinh^{-1}(e^{ax}) dx \\ = \frac{e^{ax}}{a} \sinh^{-1}(e^{ax}) - \frac{1}{2a} \int u^{-\frac{1}{2}} du \\ \int e^{ax} \sinh^{-1}(e^{ax}) dx \\ = \frac{e^{ax}}{a} \sinh^{-1}(e^{ax}) - \frac{1}{2a} \left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right) \end{aligned}$$

simplify the right side of the equation and substitute u to x, therefore the result is

$$\begin{aligned} \int e^{ax} \sinh^{-1}(e^{ax}) dx \\ = \frac{e^{ax}}{a} \sinh^{-1}(e^{ax}) - \frac{1}{a} \sqrt{e^{2ax} + 1} \\ + C. \end{aligned}$$

Example 14. Derive the formula of the given below, where a is any real number and $a \neq 0$

$$\begin{aligned} \int e^{ax} \tanh^{-1}(e^{ax}) dx \\ = \frac{e^{ax}}{a} \tanh^{-1}(e^{ax}) \\ - \frac{a}{a} \int \frac{e^{2ax}}{1 - e^{2ax}} dx. \end{aligned}$$

Derivation: Applying the modified LIATHE rule, let $u = \tanh^{-1} e^{ax}$ and $dv = e^{ax} dx$. A table below can be formed using the modified column integration technique:

Table 10. Table for evaluating $\int e^{ax} \tanh^{-1}(e^{ax}) dx$, $a \neq 0$, $a \in \mathbb{R}$

Derivative	Integral	Product
$+ \tanh^{-1}(e^{ax})$	e^{ax}	
$-\frac{ae^{ax}}{1 - e^{2ax}}$	$\frac{e^{ax}}{a}$	$\frac{e^{ax}}{a} \tanh^{-1}(e^{ax}) - \frac{1}{a} \int \frac{e^{2ax}}{1 - e^{2ax}} dx$

From table 10, the integral part on the product column can be simplified and solved using u-substitution method, let $u = 1 - e^{2ax}$ and $du = -2ae^{2ax} dx$, that gives

$$\begin{aligned} \int e^{ax} \tanh^{-1}(e^{ax}) dx \\ = \frac{e^{ax}}{a} \tanh^{-1}(e^{ax}) + \frac{1}{2a} \int \frac{du}{u} \\ \int e^{ax} \tanh^{-1}(e^{ax}) dx \\ = \frac{e^{ax}}{a} \tanh^{-1}(e^{ax}) + \frac{1}{2a} \ln|u| \end{aligned}$$

substitute u to x, therefore the result is

$$\begin{aligned} \int e^{ax} \tanh^{-1}(e^{ax}) dx \\ = \frac{e^{ax}}{a} \tanh^{-1}(e^{ax}) \\ + \frac{1}{a} \ln|1 - e^{2ax}| + C. \end{aligned}$$

Example 15. Derive the formula of the given below, where a is any real number and $a \neq 0$

$$\begin{aligned} \int e^{ax} \operatorname{sech}^{-1}(e^{ax}) dx \\ = \frac{e^{ax}}{a} \operatorname{sech}^{-1}(e^{ax}) \\ + \frac{a}{a} \int \frac{e^{ax}}{\sqrt{1 - e^{2ax}}} dx. \end{aligned}$$

Derivation: Applying the LIATHE rule, let $u = \operatorname{sech}^{-1}(e^{2x})$ and $dv = e^{2x} dx$. A table below can be formed using the column integration technique:

Table 11. Table for evaluating $\int e^{ax} \operatorname{sech}^{-1}(e^{ax}) dx$, $a \neq 0, a \in \mathbb{R}$

Derivative	Integral	Product
$+ \operatorname{sech}^{-1}(e^{ax})$	e^{ax}	
$-\frac{ae^{ax}}{e^{ax}\sqrt{1 - e^{2ax}}}$	$\frac{e^{ax}}{a}$	$\frac{e^{ax}}{a} \operatorname{sech}^{-1}(e^{ax}) + \frac{a}{a} \int \frac{e^{ax}}{\sqrt{1 - e^{2ax}}} dx$

From table 11, the integral part on the product column can be simplified into

$$\begin{aligned} \int e^{ax} \operatorname{sech}^{-1}(e^{ax}) dx \\ = \frac{e^{ax}}{a} \operatorname{sech}^{-1}(e^{ax}) \\ + \int \frac{e^{ax}}{\sqrt{1 - e^{2ax}}} dx \end{aligned}$$

using integral that yields inverse sine $\sqrt{1 - u^2}$, let $u^2 = e^{2ax}$, $u = e^{ax}$ and $du = ae^{ax} dx$, therefore the result is

$$\begin{aligned} \int e^{ax} \operatorname{sech}^{-1}(e^{ax}) dx \\ = \frac{e^{ax}}{a} \operatorname{sech}^{-1}(e^{ax}) \\ + \frac{1}{a} \sin^{-1}(e^{ax}) + C. \end{aligned}$$

Note: The process used in deriving a generalized formula in Example 13, Example 14 and Example 15 can also be applied for $e^{ax} \cosh^{-1}(e^{ax})$, $e^{ax} \coth^{-1}(e^{ax})$ and $e^{ax} \operatorname{csch}^{-1}(e^{ax})$.

3.3 Including Inverse Hyperbolic Functions in Combination with Trigonometric Functions

Consider **Example 3** $\int (\cos x) \tanh^{-1}(\sin x) dx$

Solution: Applying LIATHE rule, let $u = \tanh^{-1}(\sin x)$ and $dv = \cos x dx$. A table below can be formed using the column integration technique:

Table 12. Table for evaluating $\int (\cos x) \tanh^{-1}(\sin x) dx$

Derivative	Integral	Product
$+ \tanh^{-1}(\sin x)$	$\cos x$	
$-\frac{\cos x}{1 - \sin^2 x}$	$\sin x$	$(\sin x) \tanh^{-1}(\sin x) - \int \frac{\sin x \cos x}{1 - \sin^2 x} dx$

From table 12, the integral part on the product column can be simplified and solved using trigonometric identity $\sin^2 x + \cos^2 x = 1$, that gives

$$\begin{aligned} \int (\cos x) \tanh^{-1}(\sin x) dx \\ = (\sin x) \tanh^{-1}(\sin x) \\ - \int \frac{\sin x \cos x}{\cos^2 x} dx \end{aligned}$$

simplify the right side of the equation

$$\begin{aligned} \int (\cos x) \tanh^{-1}(\sin x) dx \\ = (\sin x) \tanh^{-1}(\sin x) \\ - \int \frac{\sin x}{\cos x} dx \end{aligned}$$

The integral on the right side of the equation can be simplified using trigonometric identity $\sin x / \cos x = \tan x$,

$$\begin{aligned} \int (\cos x) \tanh^{-1}(\sin x) dx \\ = (\sin x) \tanh^{-1}(\sin x) \\ - \int \tan x dx \end{aligned}$$

integrate $\tan x$, then the result is

$$\begin{aligned} \int (\cos x) \tanh^{-1}(\sin x) dx \\ = (\sin x) \tanh^{-1}(\sin x) - \sec^2 x \\ + C. \end{aligned}$$

3.4 Including Inverse Hyperbolic Functions in Combination with Hyperbolic Functions

Consider **Example 4**
 $\int (\operatorname{csch} x \coth x) \sinh^{-1}(\operatorname{csch} x) dx$

Solution: Applying the LIATHE rule, let $u = \sinh^{-1}(\operatorname{csch} x)$ and $dv = \operatorname{csch} x \coth x dx$. A table below can be formed using the column integration technique:

Table 13. Table for evaluating $\int (\operatorname{csch} x \coth x) \sinh^{-1}(\operatorname{csch} x) dx$

Derivative	Integral	Product
$+ \sinh^{-1}(\operatorname{csch} x)$	$\operatorname{csch} x \coth x$	
$-\frac{\operatorname{csch} x \coth x}{\sqrt{1 + \operatorname{csch}^2 x}}$	$-\operatorname{csch} x$	$-(\operatorname{csch} x \coth x) \sinh^{-1}(\operatorname{csch} x) - \int \frac{\operatorname{csch}^2 x \coth x}{\sqrt{1 + \operatorname{csch}^2 x}} dx$

From table 13, the integral part on the product column can be simplified using hyperbolic identity $\coth^2 x - \operatorname{csch}^2 x = 1$,

$$\begin{aligned} & \int (\operatorname{csch} x \coth x) \sinh^{-1}(\operatorname{csch} x) dx \\ &= -(\operatorname{csch} x \coth x) \sinh^{-1}(\operatorname{csch} x) \\ & - \int \frac{\operatorname{csch}^2 x \coth x}{\sqrt{\coth^2 x}} dx \end{aligned}$$

simplify the right side of the equation,

$$\begin{aligned} & \int (\operatorname{csch} x \coth x) \sinh^{-1}(\operatorname{csch} x) dx \\ &= -(\operatorname{csch} x \coth x) \sinh^{-1}(\operatorname{csch} x) \\ & - \int \operatorname{csch}^2 x dx \\ & \int (\operatorname{csch} x \coth x) \sinh^{-1}(\operatorname{csch} x) dx \\ &= -(\operatorname{csch} x \coth x) \sinh^{-1}(\operatorname{csch} x) \\ & + \coth x + C. \end{aligned}$$

3.5 Including Inverse Hyperbolic Functions in Combination with Logarithmic Functions

Consider **Example 5** $\int \ln x \cosh^{-1} x dx$

Solution: Applying the LIATHE rule, let $u = \ln x$ and $dv = \cosh^{-1} x dx$. A table below can be formed using the column integration technique:

Table 14. Table for evaluating $\int \ln x \cosh^{-1} x dx$

Derivative	Integral	Product
$+ \ln x$	$\cosh^{-1} x$	
$-\frac{1}{x}$	$\int \cosh^{-1} x dx$	$\ln x \left(\int \cosh^{-1} x dx \right) - \int \frac{1}{x} \left(\int \cosh^{-1} x dx \right) dx$

From table 14, the integration of the second entry on the integral column is

$$\int \cosh^{-1} x dx = x \cosh^{-1} x - \sqrt{x^2 - 1}$$

Then the product column can be simplified into

$$\begin{aligned} & \int \ln x \cosh^{-1} x dx \\ &= \ln x (x \cosh^{-1} x - \sqrt{x^2 - 1}) \\ & - \int \left(\cosh^{-1} x - \frac{\sqrt{x^2 - 1}}{x} \right) dx \end{aligned}$$

As shown the integrand on the right side of the equation can be separated into

$$\begin{aligned} & \int \ln x \cosh^{-1} x dx \\ &= \ln x (x \cosh^{-1} x - \sqrt{x^2 - 1}) \\ & - \int \cosh^{-1} x dx + \int \frac{\sqrt{x^2 - 1}}{x} dx \end{aligned}$$

Then take their integration separately, the first integrand can be solved using integration by parts,

$$\int \cosh^{-1} x dx = x \cosh^{-1} x - \sqrt{x^2 - 1} + C$$

while the second integrand can be solved using trigonometric substitution,

$$\int \frac{\sqrt{x^2 - 1}}{x} dx = \sqrt{x^2 - 1} - \sec^{-1} x + C$$

Now combine the solutions,

$$\begin{aligned} & \int \ln x \cosh^{-1} x dx \\ &= \ln x (x \cosh^{-1} x - \sqrt{x^2 - 1}) \\ & - x \cosh^{-1} x + \sqrt{x^2 - 1} \\ & + \sqrt{x^2 - 1} - \sec^{-1} x \end{aligned}$$

Simplify and it gives the results,

$$\begin{aligned} & \int \ln x \cosh^{-1} x dx \\ &= (x \ln x - x) \cosh^{-1} x \\ & + \sqrt{x^2 - 1} (\sqrt{x^2 - 1} - \ln x) \\ & - \sec^{-1} x + C \end{aligned}$$

As shown from the results above whatever dv part will be, the solution is still harder and needs a lot of patients in solving. In this case, LIATHE rule can be interchange into ILATHE since logarithmic and inverse hyperbolic both have no direct integration.

Conclusion and Recommendation

The LIATHE rule is modified where letter (I) is presented as inverse hyperbolic functions instead of only inverse trigonometric functions. The modified LIATHE rule that includes inverse hyperbolic function is applied and the following generalized formula

$\int \sinh^{-1}(ax) dx, \int \tanh^{-1}(ax) dx, \int \operatorname{sech}^{-1}(ax) dx$
where $a \in \mathbb{R}$,

$\int x^{n-1} \sinh^{-1} x^n dx, \int x^{n-1} \cosh^{-1} x^n dx,$
 $\int x^{n-1} \tanh^{-1} x^n dx, \int x^{n-1} \operatorname{sech}^{-1} x^n dx$ where
 $n \in \mathbb{R}$, and $\int e^{ax} \sinh^{-1}(e^{ax}) dx,$
 $\int e^{ax} \operatorname{sech}^{-1}(e^{ax}) dx, \int e^{ax} \tanh^{-1}(e^{ax}) dx$, where
 $a \in \mathbb{R}$, are derived. Further studies to derive other algorithm in Mathematics that involve inverse hyperbolic functions is recommended.

References

- [1] Alcantara, E. C. (2015). Integrals of Products of LIATHE Functions Using Tabular Integration by Parts. *Asia Pacific Journal of Multidisciplinary Research*, Vol. 3, No. 1.
- [2] Dence, J. and Dence, T. (2003). Column Integration and Series Representations. *The College Mathematics Journal*, 34(2).
- [3] Emeje, M. and Onalo, S. (2016). Tabular Method of Integration by Parts and Some of Its Striking Applications. *International Journal of Public Administration and Management Research (IJPAMR)*. Volume 3, Number 3.
- [4] Horowitz, D. (1990). Tabular Integration by Parts. *The College Mathematics Journal* 21, 307–311.
- [5] Kasube, H. (1983). A Technique for Integration by Parts. *The American Mathematical Monthly* 90, 210–211.
- [6] Leithold, L. (2002). *The Calculus* 7. Singapore: Pearson Education Asia Pte. Ltd.
- [7] Robles, R. (2014). Generalized Formulas for Evaluating Hyperbolic Integrals. Unpublished Masteral Thesis, University of Batangas.
- [8] Stewart, J. (2008). *Calculus Early Transcendental*, 6th ed. Brooks Cole Cengage Learning.
- [9] Stolyarov II, G. (2007) “How to Solve Problems of Integration by Parts Using the Tabular Method. *The Rational Argumentator*, A Journal for Western Man.
- [10] Thomas, G. (2004). *Calculus Early Transcendental*. Addison-Wesely, Earson.