Defining Zero According to the Definition of the Golden Ratio

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Abstract: - The Golden Ratio φ is conventionally taken for its positive value in the algebra form of $\frac{1+\sqrt{5}}{2}$, while its negative value of $\frac{1-\sqrt{5}}{2}$ from the quadratic solution from the definition, $\frac{a+b}{a} = \frac{a}{b} = \varphi,\ a > b > 0$, has not traditionally drawn much attentions from scientists and researchers, and is usually denoted as the negative inverse of the positive value in the form $-\frac{1}{\varphi}$. From the quadratic definition of ϕ, I define $0 = \left(\lim_{n\to\infty} a^{2-n} - \frac{b^2}{a^n} - a^{1-n}b\right) \cap \left(\lim_{n\to\infty} \frac{a^2}{b^n} - b^{2-n} - \frac{a}{b^{n-1}}\right) = \left(\lim_{n\to\infty} \frac{1}{a^{n-2}} - \frac{1}{\phi a^{n-2}} + \frac{1}{\phi a^n}\right) \cap \left(\lim_{n\to\infty} \frac{\phi^2}{b^{n-2}} - \frac{1}{b^{n-2}} - \frac{\phi}{b^{n-1}}\right) = \emptyset, a > b > 0$, whereby the number ϕ is taken multivalued and the Zermelo–Fraenkel set theory is applied. And then with the relation $\phi^2 = \phi + 1$, I adopt an infinite deduction to summarize the function $\lim_{n\to\infty}(\phi^2 - \phi)^{\frac{1}{n}}$ being either a constant, which I argue against, or a multivalued recurring sequence. The derivative of the series has led to a metric space with the value of e expressed in a complex number form in a unit circle. The results corroborate with the Euler–Mascheroni constant and Gamma Function, and I hypothesize their further extensions to the whole number line.

Key-Words: - Integration of irrational numbers; transcendental relations of the Golden Ratio; a set-theoretical definition of zero; metric unit of the Riemann Zeta Function; Golden Ratio at infinity; Golden Ratio series
List of Abbreviations: GR, Golden Ratio; RH, Riemann Hypothesis; RZF, Riemann Zeta Function; ZFC, Zermelo–Fraenkel set theory with the axiom of choice

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1 Introduction

The research is inspired by the intuition and subsequent speculation that the Riemann Zeta Function (RZF) $\int_{\varphi}^{e} \zeta(x)$ is an expression for the negative value of the Golden Ratio (GR) φ, and then I devised the proof with the definition of zero with the formalism of the GR with the incorporation of the Zermelo–Fraenkel set theory with the axiom of choice included (ZFC), seen in Eq. (1) [1]:

$$0 = \left(\lim_{n\to\infty} a^{2-n} - \frac{b^2}{a^n} - a^{1-n}b\right) \cap \left(\lim_{n\to\infty} \frac{a^2}{b^n} - b^{2-n} - \frac{a}{b^{n-1}}\right) \equiv \left(\lim_{n\to\infty} \frac{1}{a^{n-2}} - \frac{1}{\varphi^2 a^{n-2}} + \frac{1}{\varphi a^n}\right) \cap \left(\lim_{n\to\infty} \frac{\varphi^2}{b^{n-2}} - \frac{1}{b^{n-2}} - \frac{\varphi}{b^{n-1}}\right) = \emptyset, a > b > 0 \quad (1)$$

2 Problem Formulation

With the separated notation ϕ for the positive value of the GR, φ for the negative value, and **Phi** for the multivalue from the definition formalism unchanged, Knott [2] noted several intrinsic properties of the GR:

$$\text{Phi}^2 = \text{Phi} + 1 \quad (2)$$

and

$$-(\phi - 1) = \varphi. \quad (3)$$

And I adopt a slightly different interpretation for the purpose that does not necessarily lead to Fibonacci numbers in Eq. (4),

$$\varphi\phi = e^{i\pi} = \text{Phi} - \text{Phi}^2. \quad (4)$$

In the following of the article, unless explicitly specified, I take the notation ϕ for the multivalued derivative from the GR formalism, and φ for the convention of the positive numerical value.

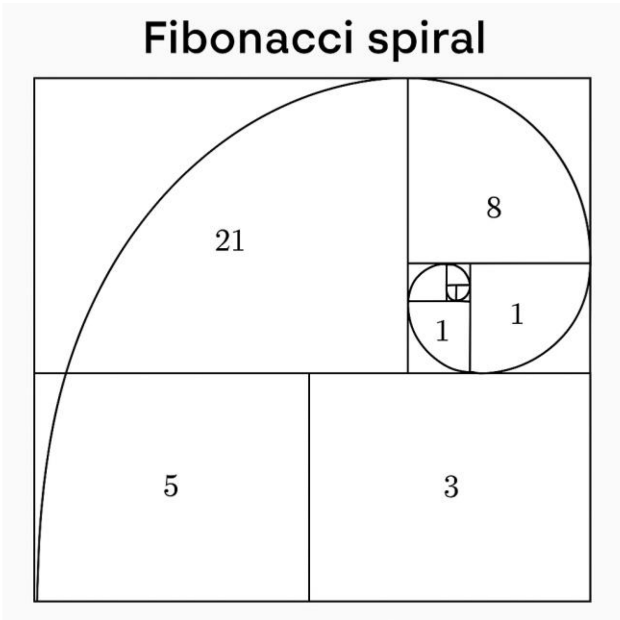
From Eq. (4), the sequence of the function $f(n) = (\phi^2 - \phi)^{\frac{1}{n}}, n \in \mathbb{N}$ can be calculated as followed:

$$(\phi^2 - \phi)^1 = 1$$
$$(\phi^2 - \phi)^{\frac{1}{2}} = \pm 1$$
$$(\phi^2 - \phi)^{\frac{1}{3}} = 1$$
$$(\phi^2 - \phi)^{\frac{1}{4}} = i \text{ or } \pm 1$$
$$(\phi^2 - \phi)^{\frac{1}{5}} = 1$$
$$(\phi^2 - \phi)^{\frac{1}{6}} = \pm 1$$
$$(\phi^2 - \phi)^{\frac{1}{7}} = 1$$
$$(\phi^2 - \phi)^{\frac{1}{8}} = i^{\frac{1}{4}} \text{ or } i^{\frac{1}{2}} \text{ or } i \text{ or } \pm 1$$
$$\ldots$$
$$(\phi^2 - \phi)^{\frac{1}{16}} = i^{\frac{1}{8}} \text{ or } i^{\frac{1}{4}} \text{ or } i^{\frac{1}{2}} \text{ or } i \text{ or } \pm 1$$
$$\ldots$$
$$(\phi^2 - \phi)^{\frac{1}{2^n}} = i^{\frac{1}{2^{n-1}}} \text{ or } \ldots \text{ or } i^{\frac{1}{2}} \text{ or } i \text{ or } \pm 1$$
$$\ldots$$
$$(\phi^2 - \phi)^{\frac{1}{4n}} = i \text{ or } \pm 1, 4n \notin 2^n$$
$$\ldots$$
$$(\phi^2 - \phi)^{\frac{1}{2n}} = \pm 1, 2n \notin (4n \cup 2^n). \quad (5)$$

Similarly noted by García-Caballero, Moreno [3] with Viéte's formula, intriguing results arise by adding up the continued fraction form of φ with infinite sum and infinite products, revolved around the number 2 and correlated to π. By the Fibonacci spiral's geometric similarities to the polar graph of the positive nontrivial positive solutions of RZF (with *prima facie* visualization in Fig. 1), I speculate that the negative value of **Phi** is correlated to my conjecture on the negative critical line of the RZF supplementary to the Riemann Hypothesis (RH) [2, 4]. From Eq. (3) & (4), *lemma 1* is obtained

$$1 - \varphi = \frac{e^{i\pi}}{\varphi}. \quad (6)$$

The approximation and oscillation patterns for $(\varphi - \varphi^2)^{\frac{1}{\pi}}$ can be seen in Eq. (5), and a metric space is conceived to be needed in the form of $\frac{d}{dx}e^{ix} = ie^{ix}$, whereas $\frac{d}{dx}e^{\pi x} = 0$. With reference to Eq. (1), it is observed that $(\frac{1}{b^{n-2}})(\varphi^2 - \frac{\varphi}{b} - 1)$ has an oscillation between the value of $\frac{1}{b}$ and ∓ 1.



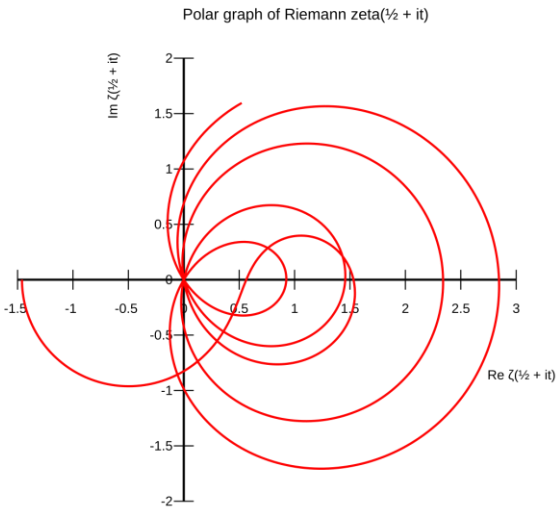


Fig. 1 Graphical comparison of the Fibonacci spiral generated by ChatGTP and the path of RZF along the conjectured critical line in RH created by Vepstas [5].

Even though by a slight distortion with Wallis product applied on the Fibonacci relationship, it can also be defined that [2, 3]

$$\frac{1}{\varphi} = \prod_{n=1}^{\infty}(1 - \frac{1}{F_{2n}F_{2n+1}}), \tag{7}$$

a more fundamental approach is needed to understand the factor $\lfloor b \rfloor$'s constraint to the value of $\lceil a \rceil$, in order to better understand the meaning of the definition of zero with the GR, namely, the vectorization of lengths by the extension from positive spaces.

3 Problem Solution

With the theoretical framework provided above, I formulate my solutions currently in the Harmonic Series (HS).

3.1 Breakdown in the Harmonic Series

From Eq. (5), it can be summarized

I. $\forall p$ as odd numbers $\in n \in \mathbb{N}$, $(\phi^2 - \phi)^{\frac{1}{p}} = 1$;
II. $\forall q$ as even numbers $\in m \in n \in \mathbb{N}$, in which $\sqrt{m} \notin \mathbb{N}$, $(\phi^2 - \phi)^{\frac{1}{q}} = \pm 1$;
III. $\forall l \in 2^n \in n \in \mathbb{N}$,

$$(\phi^2 - \phi)^{\frac{1}{l}} = \{i^{\frac{1}{2^{l-1}}}, i^{\frac{1}{2^{l-2}}}, \ldots, i^{\frac{1}{2}}, i\} \cup \pm 1 \tag{8}$$

By the axiom of choice in case III of Eq. (8),

$$(\phi^2 - \phi)^{\frac{1}{l}} \in f(k) = i^{\frac{4}{2^k}}, l \in 2^n, k \geq 0, k \in W, \tag{9}$$

whereas {case II}∪{case I}∈ $\{-1,\ 1\} \in f(k)$. Therefore, the injective, if not surjective, function

$$(\phi^2 - \phi)^{\frac{1}{n}} \in f'(n) = i^{\frac{1}{2^{n-2}}}, n \in \mathbb{W}, n \geq 0 \tag{10}$$

is descriptive of the value outputs of Eq. (5). The summation of the sequence $f(n)$ in Eq. (5) can oscillate amongst divergence in positive number values, partial or complete symmetry with the critical point $\in [-1, 1]$, and convergence in negative values; only the divergence, partial symmetry with the critical point $\in [-1, 0) \cup (0, 1]$, and exoccipital cases in positive value divergence may contain an imaginary part. Even though oscillatory integral seems to serve better for analyzing the value distribution of $f(n)$, the significant proportions do not fit into the definitive framework. Therefore, the Leibniz notation is preferred to serve the purpose.

From Eq. (6) (*lemma 1*), we get $\ln f(0) = \ln f(1) = i\pi$ (*hypothesis 0*). Let

$$\exists g(n)\ \forall n \in \mathbb{W} \cap n > 0, = \ln f(n) = (i\pi)^{\frac{1}{n}}, \tag{11}$$

and the integration $\int_{n=1}^{\infty} g(n)\, dn$ is conventionally solved by de'Moivre's theorem with $\arg\theta$. Herein in the context, I propose *lemma 2* according to ZFC

$$f(n) \in f'(n) = i^{\frac{1}{2^{n-2}}} = \frac{i^{\frac{1}{2^n}}}{\sqrt{i}}, \tag{12}$$

where both the function set $f(n)$ is bounded by $i^{-\frac{1}{2}}$, proving *hypothesis 0* as *lemma 0*; its closed form by the bound can be derived

$$\prod_{n=0}^{\infty} f(n) \in \prod_{n=0}^{\infty} f'^{(n)} = \frac{i^{\sum_{n=0}^{\infty}\frac{1}{2^n}}}{\sqrt{i}} = i^{\frac{3}{2}}, n \in \mathbb{W}, n \geq 0. \tag{13}$$

The comparison with the approach with de'Moivre's theorem can be seen in Fig. 2 and Fig. 3 with ChatGTP's standard solution's result plots on Eq. (11).

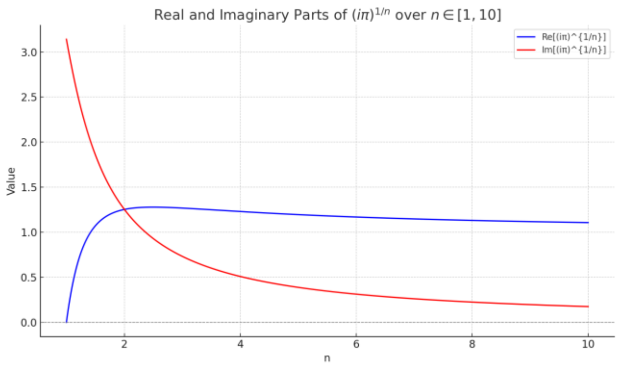


Fig. 2 Numerical stimulation with de'Moivre's theorem from n = 1 to 10.

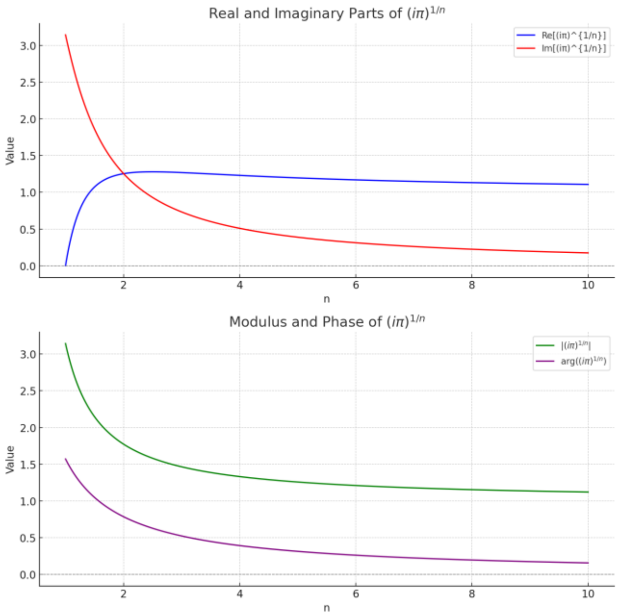


Fig. 3 Plot of the function $g(n) = (i\pi)^{\frac{1}{n}}$ from Eq. (11) with modulus and phase by the de'Moivre's theorem approach.

Albeit the de'Moivre's theorem approach do not include $n = 1$, it can be seen that the convergence exists with at least one bound value. So, the solution for Eq. (11) is necessary but not sufficient, unless the premise is set to ∃!.

Hypothesis 1:

$$\exists!\, g(n)\ \forall n \in \mathbb{W} \cap n > 0, = \ln f(n) = (i\pi)^{\frac{1}{n}}. \quad (14)$$

3.2 Complex Logarithm

It is defined from *lemma 0* that $\ln(-1) = i\pi$. Now we define $\varphi - \varphi^2 = e^{i\pi}$ into a complex number

$$z = \log e = \frac{\log(\varphi - \varphi^2)}{i\pi}, \quad \text{(proof trivial)} \quad (15)$$

and then it can be derived that

$$\varphi - \varphi^2 = \log i\pi z = \log(i\pi \log e). \quad \text{(proof trivial)} \quad (16)$$

It is arguable that if $\log(i\pi \log e)$ can be put into $\log e \mp \log i\pi$, but the approximation on the real part is satisfactory

$$Re(\log i\pi z) \doteq \log \pi + e. \quad \text{(proof trivial)} \quad (17)$$

Therefore,

$$Re[-f(1)] \doteq \log \pi + e, \quad \text{(proof trivial)} \quad (18)$$

implying the potential of a complex value expression of $f(1)$ with the form

$$Re[f(1)] \doteq -(\log \pi + e). \quad (19)$$

Thereby, the proof by contradiction exists, with regard to Eq. (5), to justify Eq. (14), and *hypothesis 1* is regarded for *corollary 1* in the following texts.

3.3 In Relation to RZF

Let $g(n) = \ln f(n) = (i\pi)^{\frac{1}{n}}, n \in W, n \geq 0$, the integration

$$\int g(n)\, dn = i\pi + \int_{n=1}^{\infty} g(n)\, dn \quad (20)$$

is evaluated at the value point $\int g(z)\, dn|z = i\pi + (i\pi)^{\frac{1}{z}}$ by the axiom of choice in Eq. (10) with presumed continuity in $\sum_{n=1}^{\infty} f(n), n \in \mathbb{N}$, and with the value of $z = \log e$ from Eq. (15)

$$\int g(z)\, dn|z = i\pi + (i\pi)^{\ln 10}. \quad (21)$$

Therefore, the condition is met for applying de'Moivre's theorem approach with the element $(i\pi)^{\ln 10} = e^{\ln 10 \times \ln(i\pi)}$ with the polar form $\ln(i\pi) = \ln \pi + i\frac{\pi}{2}$. The expression from Eq. (21) then becomes

$$\int g(z)\, dn|z = i\pi + e^{\ln(10 \ln \pi)} \times e^{i\frac{\pi}{2}\ln 10}. \quad (22)$$

From Eq. (13), it is seen that the imaginary part of the function set $f(n)$ is not only bounded by $\frac{1}{\sqrt{i}}$, but also by an exponential function that is involved in Eq. (17) to (19). To further explore the potential correlations, the function $f'(n)$ is also evaluated at the value of z, with the presupposition on the imaginary part being capable of having a continuous pattern, which is proofed in the expression

$$f'(z) = \frac{i^{\frac{1}{2^{\log e}}}}{\sqrt{i}} = i^{e^{\frac{\ln 2}{\ln 10}} - \frac{1}{2}} = i^{e^{\log 2} - \frac{1}{2}}. \quad (23)$$

Let's take $\partial = \ln n + \gamma_n$, as in the Euler–Mascheroni constant [6], to denote the HS, then $\prod f(n) = (\phi^2 - \phi)^{\partial} = \left(-e^{i\pi}\right)^{\partial}$. By Eq. (16), it is derived that that $z = \frac{\log e^{i\pi}}{i\pi}$, and from Eq. (13) with *hypothesis 0*, it is inferred

$$\prod f(n) \in \left(\prod_{n=0}^{\infty} f'^{(n)}\right) - i\pi$$
$$= i^{\frac{3}{2}} - i\pi = i\sqrt{i} - i\pi = i\left(\sqrt{i} - \pi\right)$$
(proof trivial) (24)

$$\prod f(n) = (\phi^2 - \phi)^{\partial} = \left(-e^{i\pi}\right)^{\partial} \equiv i\left(\sqrt{i} - \pi\right)$$

can be expressed as

$$[-(\phi - \phi^2)]^{\partial} = (-\log i\pi z)^{\partial}$$
$$= [-\log(i\pi \log e)]^{\partial} = \left(-e^{i\pi}\right)^{\partial}. \quad (24)$$

It is then derived that $z = \frac{10^{e^{i\pi}}}{i\pi} = \frac{1}{10 i\pi}$, and according to Eq. (6) & (16), it is justified that

$$\begin{cases} (\phi - \phi^2)^{\partial} = (\log 10^{-1})^{\partial} \\ \log 10^{-1} = -1 = \log(i\pi \log e) \end{cases}. \quad (25)$$

And an alternative expression for the value of *e* is obtained and seen in Fig. 4

$$e = 10^{\frac{1}{10 i\pi}}. \quad (26)$$

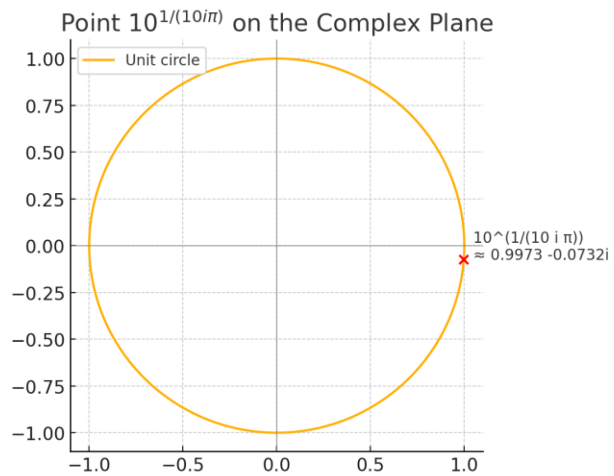


Fig. 3 The visual presentation of Eq. (26) rendered by ChatGTP.

The criteria for necessity in Eq. (11) and *corollary 1* (Eq. (14)) are thus met

$$g(n) = \ln f(n) = (10 \log e)^{-\frac{1}{n}}, n \in \mathbb{N}. \quad (27)$$

Thereby, it can also be expressed that

$$\sum_{n=1}^{\infty} g(n) = \ln \sum_{n=1}^{\infty} f(n) = \ln(\phi^2 - \phi)^{\partial}. \quad (28)$$

From Eq. (16), it is derived $\varphi^2 - \varphi = -\log i\pi z = -e^{i\pi}$, therefore,

$$\sum_{n=1}^{\infty} g(n) = \ln \sum_{n=1}^{\infty} f(n) = \ln(e^{2i\pi})^{\partial} = 2i\pi\partial. \quad (29)$$

And Eq. (24) provides the sufficiency criteria in proving *corollary 1*, justifying the *theorem 1* in Eq. (14) and *theorem 2* in Eq. (29).

Theorem 3 is further derived in relation to the Euler–Mascheroni constant [6]

$$\sum_{n=1}^{\infty} g(n) = \ln(e^{2i\pi})^{\ln n + \gamma_n} = 2i\pi(\ln n + \gamma_n), \quad (30)$$

and I further hypothesize $n \in \mathbb{W}$ holds for the theorem due to the symmetry of *lemma 0*, and it is plotted with ChatGTP, albeit the magnitude and phase in my proposed negative critical strip $-\frac{1}{2}$ was not visualized [4]. The proof for *hypothesis 2* is significant for the extension of Gamma Function, hence the proof for the correlations between *theorem 2* and *theorem 3*.

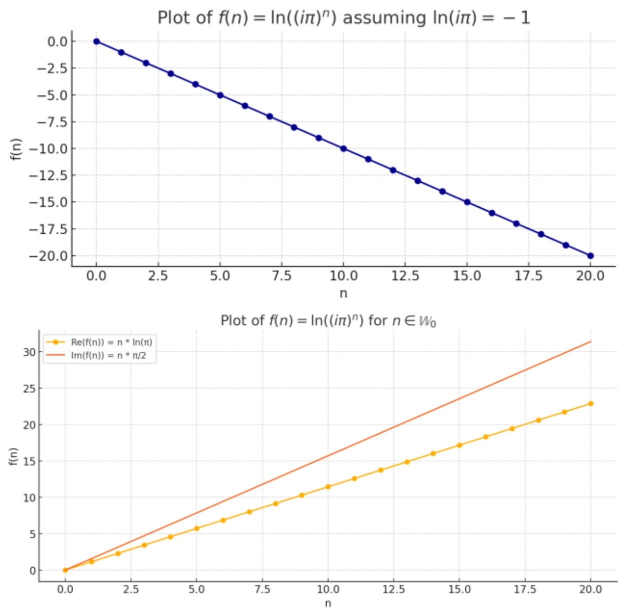


Fig. 4 The visual presentation of *lemma 0*'s applicability in the whole number line with zero included.

3.3 Back to the Harmonic Series

By theorem 2 and $f(n)$ in Eq. (5), it is derived from Eq. (15) & (27) that

$$\sum_{n=1}^{\infty}\frac{1}{n}=\frac{\sum_{n=1}^{\infty}g(n)}{2i\pi}=\frac{\sum_{n=1}^{\infty}g(n)}{2i\pi}$$
$$=\frac{\sum_{n=1}^{\infty}(10\log e)^{-\frac{1}{n}}}{2i\pi}=\frac{\left(\log e^{10}\right)^{-\partial}}{2i\pi}. \quad (31)$$

It is further derived

$$2i\pi\partial=(\log e^{10})^{-\partial}$$
$$2i\pi\partial=(\log e^{10})^{-\partial}$$
$$\log 2i\pi\partial=\log(10\log e)^{-\partial}$$
$$\log 2i\pi\partial\times 10^{-\partial}=\log(-\partial\log e), \quad (32)$$

And from Eq. (26), it is obtained

$$2\mathrm{i}\pi\times 10^{-\partial}=-\log e=-\frac{1}{10i\pi}$$
$$\sum_{n=1}^{\infty}\frac{1}{n}=-\log\frac{1}{20\pi^2}. \quad (33)$$

Alternatively, it is derived

$$\ln 2i\pi\partial=\ln(10\log e)^{-\partial}, \quad (34)$$

and

$$\ln 2i\pi\partial=\ln(10)^{-\partial}+\ln(\log e)^{-\partial}$$
$$=-\partial\ln 10+\ln[\partial(\log e)]+i\pi$$
$$=\ln(-\partial\log e)=\ln(\partial\log e)+i\pi, \quad (35)$$

which means

$$\ln 2i\pi\partial^2\log e=i\pi$$
$$e^{\partial^2}=1. \quad (36)$$

Therefore, a relatively exact value of the HS is derived in terms of the complex domain, given $n\in\mathbb{N}$

$$\partial^2=i\pi$$
$$\sum_{n=1}^{\infty}\frac{1}{n}=\sqrt{i\pi}=\sqrt{\ln -1}. \quad (37)$$

Alternatively, with *lemma 0*,

$$\partial^2=-\frac{1}{2i\pi}=\frac{i}{2\pi}$$
$$\sum_{n=1}^{\infty}\frac{1}{n}=\sqrt{\frac{i}{2\pi}} \quad (38)$$

The dual results suggest that the sequence formalism of the Euler–Mascheroni constant not only exists in form of limits, but may also have a pair product in the negative whole numbers for $n\neq 0$, corroborating with the previous hypothesis.

4 Conclusion

From the research I find a complex number expression for the transcendental number $e=10^{\frac{1}{10i\pi}}$ with relation to a metric space that can be possibly correlated with the Euler–Mascheroni constant. An exact solution in terms of complex number is derived for HS $\sum_{n=1}^{\infty}\frac{1}{n}=\sqrt{i\pi}=\sqrt{\ln -1}$. Further proofs and analysis, including further derivatives that can serve for proofs of my previous works, will be given in the next papers to come.

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