Maxwell’s Transport Equations and Its Solutions at Pre- and Superluminal Speeds

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Abstract: Transport solutions of Maxwell’s equations at pre- and superluminal speeds are considered. The Green tensor and fundamental solutions are constructed in analytical form from sublight to superluminal speeds. Formulas for calculating the electromagnetic fields of mobile radiators of various types, useful for radio engineering applications, are given.

Key-Words: Maxwell's transport equations, Green tensor, Kronecker symbol, Fourier transform of generalized functions, shock electromagnetic waves.

Received: April 29, 2024. Revised: December 14, 2024. Accepted: January 19, 2025. Published: March 5, 2025.

1. Introduction

The basis of modern electrodynamics are Maxwell's equations, which make it possible to determine the electromagnetic field with knowledge of the active charges and currents (radiation sources) and vice versa. Among the existing sources, the most common are mobile ones located on platforms of various vehicles. Research in this area is not so numerous and is associated with a certain type of radiation source.

In this paper, we construct fundamental and generalized solutions of Maxwell's equations for a radiation source moving at an arbitrary constant speed V. It can be less than or greater than the speed of light $c=\sqrt{1/\mu\varepsilon}$, which affects the type of Maxwell's transport equations in a moving coordinate system, from elliptical to hyperbolic, depending on the ratio $V/c=M$ - the Mach number.

Therefore, the construction of solutions to these equations, as well as their properties, requires separate study for each speed range: $M<11$ and $M\ge 1$.

1.1. Maxwell’s transport equations. Statement of the task

Mobile transport sources of electromagnetic waves are considered, which do not change their form, but move with a constant velocity V along axis X_3. They can be described by currents of the form $\mathbf{J}(x,z)$, where $x_3+VT=z$. In a moving coordinate (x,z), $\mathbf{x}=(x_1,x_2)$, the system of Maxwell equations for currents has the form [1]:

$$\mathbf{M}(\partial_1,\partial_2,\partial_z)\mathbf{u}(\mathbf{x},z)=\mathbf{J}(\mathbf{x},z), \qquad (1)$$

where the following designations are introduced:

$$\mathbf{u}(\mathbf{x},z)=\begin{pmatrix}\mathbf{E}(\mathbf{x},z)\\ \mathbf{H}(\mathbf{x},z)\end{pmatrix},\quad \mathbf{J}(\mathbf{x},z)=\begin{pmatrix}\mathbf{j}^{\mathrm{m}}(\mathbf{x},z)\\ \mathbf{j}^{\mathrm{e}}(\mathbf{x},z)\end{pmatrix}$$

the differential matrix operator $\mathbf{M}(\partial_1,\partial_2,\partial_z)$ is equal to

$$\bar{\mathbf{M}}(\partial_1,\partial_2,\partial_z)=$$

$$\begin{pmatrix}0 & -\partial_z & \partial_2 & -\mu V\partial_z & 0 & 0\\ \partial_z & 0 & -\partial_1 & 0 & -V\mu\partial_z & 0\\ -\partial_2 & \partial_1 & 0 & 0 & 0 & -V\mu\partial_z\\ V\varepsilon\partial_z & 0 & 0 & 0 & -\partial_z & \partial_2\\ 0 & V\varepsilon\partial_z & 0 & \partial_z & 0 & -\partial_1\\ 0 & 0 & V\varepsilon\partial_z & -\partial_2 & \partial_1 & 0\end{pmatrix} \qquad (2)$$

Here $\mathbf{E}(\mathbf{x},z)$, $\mathbf{H}(\mathbf{x},z)$ are the vectors of electric and magnetic strengths of EM field; $\mathbf{j}^{\mathrm{m}}(\mathbf{x},z)$, $\mathbf{j}^{\mathrm{e}}(\mathbf{x},z)$ are the magnetic and electric current densities,

$$\partial_z=\frac{\partial}{\partial z},\quad \partial_j=\frac{\partial}{\partial x_j}.$$

It is required to construct solutions of equations (1) for any Mach numbers.

2. Green's tensor of Maxwell's transport equations

The Green's tensor of Maxwell's transport equations is the matrix $\mathbf{U}(x_1, x_2, z)$ of fundamental solutions of Eqs (1) by

$$\mathbf{J} = \delta(x_1), \delta(x_2), \delta(z)\left\{\delta_{ij}\right\}_{6\times 6},$$

which satisfies to the radiation conditions and describe waves diverging from a moving source and decaying at infinity:

$$\begin{aligned}&\mathbf{M}(\partial_1, \partial_2, \partial_z)\mathbf{U}(x_1, x_2, z) = \\ &= \delta(x_1)\delta(x_2)\delta(z)\left\{\delta_{ij}\right\}_{6\times 6},\end{aligned} \qquad (3)$$

To construct it, we use the Fourier transform in the space of generalized functions of slow growth. In the space of Fourier transforms $x_1, x_2, z \leftrightarrow k_1, k_2, k_3$, the relationship with the original of regular functions has the form:

$$\begin{aligned}&\overline{f}(k_1, k_2, k_3) = \\ &= \int_{R^3} f(x_1, x_2, z) e^{i(x_1 k_1 + x_2 k_2 + z k_3)} dx_1 dx_2 dz \\ &f(x_1, x_2, z) = \\ &= \frac{1}{(2\pi)^3} \int_{R^3} \overline{f}(k_1, k_2, k_3) e^{-i(x_1 k_1 + x_2 k_2 + z k_3)} dk_1 dk_2 dk_3\end{aligned} \qquad (4)$$

Using the Fourier transform property of the derivative: $\partial_j \leftrightarrow -ik_j$, from Eqs (2) we obtain a system of linear algebraic equations of the form

$$\mathbf{M}(-ik_1, -ik_2, -ik_z)\overline{\mathbf{U}}(k_1, k_2, k_3) = \left\{\delta_{ij}\right\}_{6\times 6}, \qquad (5)$$

Here δ_{ij} is the Kronecker symbol, $\delta(z)$ is singular delta function.

$$\mathbf{M}(-ik_1, -ik_2, -ik_z) =$$

$$= \begin{pmatrix} 0 & ik_3 & -ik_2 & -ik_3 V\mu & 0 & 0 \\ -ik_3 & 0 & ik_1 & 0 & -ik_3 V\mu & 0 \\ ik_2 & -ik_1 & 0 & 0 & 0 & -ik_3 V\mu \\ ik_3 V\varepsilon & 0 & 0 & 0 & ik_3 & -ik_2 \\ 0 & ik_3 V\varepsilon & 0 & -ik_3 & 0 & ik_1 \\ 0 & 0 & ik_3 V\varepsilon & ik_2 & -ik_1 & 0 \end{pmatrix} \qquad (6)$$

It follows from this

$$\overline{\mathbf{U}}(k_1, k_2, k_3) = \left(\mathbf{M}(-ik_1, -ik_2, -ik_z)\right)^{-1} \qquad (7)$$

The components of the inverse matrix are obtained by solving symbolic equations in MatCad-15. They have the following form for the columns of the matrix

$$\left\{\overline{\mathrm{U}}_{m1}\right\} = \begin{bmatrix} 0 \\ ik_3 \overline{f}_0(k_1, k_2, k_3) \\ -ik_2 \overline{f}_0(k_1, k_2, k_3) \\ \dfrac{k_1^2 - k_3^2 M^2}{\varepsilon V} \overline{f}_1(k_1, k_2, k_3) \\ \dfrac{ik_1\, ik_2}{\varepsilon V} \overline{f}_1(k_1, k_2, k_3) \\ -\dfrac{ik_1}{\varepsilon V} \overline{f}_0(k_1, k_2, k_3) \end{bmatrix}$$

$$\left\{\overline{\mathrm{U}}_{m2}\right\} = \begin{bmatrix} -ik_3 \overline{f}_0(k_1, k_2, k_3) \\ 0 \\ ik_1 \overline{f}_0(k_1, k_2, k_3) \\ \dfrac{ik_1\, ik_2}{\varepsilon V} \overline{f}_1(k_1, k_2, k_3) \\ -\dfrac{k_2^2 - k_3^2 M^2}{\varepsilon V} \overline{f}_1(k_1, k_2, k_3) \\ -\dfrac{ik_2}{\varepsilon V} \overline{f}_0(k_1, k_2, k_3) \end{bmatrix}$$

$$\left\{\overline{\mathrm{U}}_{m3}\right\} = \begin{bmatrix} ik_2 \overline{f}_0(k_1, k_2, k_3) \\ -ik_1 \overline{f}_0(k_1, k_2, k_3) \\ 0 \\ -\dfrac{ik_1}{\varepsilon V} \overline{f}_0(k_1, k_2, k_3) \\ -\dfrac{ik_2}{\varepsilon V} \overline{f}_0(k_1, k_2, k_3) \\ -\dfrac{ik_3 - ik_3 M^2}{\varepsilon V} \overline{f}_0(k_1, k_2, k_3) \end{bmatrix}$$

$$\{\bar{\mathrm{U}}_{m4}\}=\begin{bmatrix} -\dfrac{k_3^2M^2-k_1^2}{\mu V}\bar{f}_1(k_1,k_2,k_3) \\ -\dfrac{ik_1\,ik_2}{\mu V}\bar{f}_1(k_1,k_2,k_3) \\ \dfrac{ik_1}{\mu V}\bar{f}_0(k_1,k_2,k_3) \\ 0 \\ ik_3\,\bar{f}_0(k_1,k_2,k_3) \\ -ik_2\,\bar{f}_0(k_1,k_2,k_3) \end{bmatrix}$$

$$\{\bar{\mathrm{U}}_{m5}\}=\begin{bmatrix} \dfrac{ik_1\,ik_2}{\mu V}\bar{f}_1(k_1,k_2,k_3) \\ \dfrac{k_3^2M^2-k_2^2}{\mu V}\bar{f}_1(k_1,k_2,k_3) \\ -\dfrac{ik_2}{\mu V}\bar{f}_0(k_1,k_2,k_3) \\ -ik_3\,\bar{f}_0(k_1,k_2,k_3) \\ 0 \\ ik_1\,\bar{f}_0(k_1,k_2,k_3) \end{bmatrix}$$

$$\{\bar{\mathrm{U}}_{m6}\}=\begin{bmatrix} -\dfrac{ik_1}{\mu V}\bar{f}_0(k_1,k_2,k_3) \\ -\dfrac{ik_2}{\mu V}\bar{f}_0(k_1,k_2,k_3) \\ \dfrac{ik_3M^2-ik_3}{\mu V}\bar{f}_0(k_1,k_2,k_3) \\ ik_2\,\bar{f}_0(k_1,k_2,k_3) \\ -ik_1\,\bar{f}_0(k_1,k_2,k_3) \\ 0 \end{bmatrix} \qquad (7)$$

Здесь мы ввели базисные функции, которые имеют следующий вид:

$$\bar{f}_0(k_1,k_2,k_3)=\frac{1}{k_1^2+k_2^2+m^2k_3^2} \qquad (8)$$

$$\bar{f}_1(k_1,k_2,k_3)=-\frac{1}{k_3^2}\bar{f}_0(k_1,k_2,k_3)$$

The original of basic functions $f_0(\mathbf{x},z)$, $f_1(\mathbf{x},z)$ have different values depending on the Mach number [2].

3. The Green tensor of Maxwell's transport equations

Using the Fourier transform of generalized functions, the Green tensor is constructed $\mathbf{U}(\mathbf{x},z)$, which satisfies (1) and the radiation conditions at . $\mathbf{J}(\mathbf{x},z)=\delta(\mathbf{x})\delta(z)\delta_{ij}$, $i,j=1,..6.$ It has the following form [1]:

$$\mathbf{U}(\mathbf{x},z)=$$

$$\begin{pmatrix} 0 & -\partial_3 f_0 & \partial_2 f_0 & b & \dfrac{\partial_1\partial_2 f_0}{\mu V} & \dfrac{\partial_1 f_0}{\mu V} \\ \partial_3 f_0 & 0 & -\partial_1 f_0 & \dfrac{\partial_1\partial_2 f_1}{\mu\cdot V} & b & \dfrac{\partial_2 f_0}{\mu V} \\ -\partial_2 f_0 & \partial_1 f_0 & 0 & \dfrac{-\partial_1 f_0}{\mu\cdot V} & -\dfrac{\partial_2 f_0}{\mu V} & \dfrac{m^2\partial_3 f_0}{\mu V} \\ a & \dfrac{-\partial_1\partial_2 f_1}{\varepsilon V} & \dfrac{\partial_1 f_0}{\varepsilon V} & 0 & -\partial_3 f_0 & \partial_2 f_0 \\ \dfrac{\partial_1\partial_2 f_1}{\varepsilon\cdot V} & -a & \dfrac{\partial_2 f_0}{\varepsilon V} & \partial_3 f_0 & 0 & -\partial_1 f_0 \\ -\dfrac{\partial_1 f_0}{\varepsilon V} & -\dfrac{\partial_2 f_0}{\varepsilon V} & \dfrac{-m^2\partial_3 f_0}{\varepsilon V} & -\partial_2 f_0 & \partial_1 f_0 & 0 \end{pmatrix}$$

where

$$\frac{(\partial_1^2-M^2\partial_3^2)\,f_1}{\varepsilon V}=a\,,\quad \frac{(\partial_2^2-M^2\partial_3^2)\,f_1}{\mu V}=b\,.$$

At $M<1$ (*sublight speed*):

$$f_0(\mathbf{x},z)=\frac{1}{4\pi\sqrt{m^2r^2+z^2}}\,,$$

$$f_1(\mathbf{x},z)=\frac{\operatorname{sgn}(z)}{4\pi}\ln\left(\frac{|z|+\sqrt{m^2r^2+z^2}}{mr}\right) \qquad (9)$$

where $m=\sqrt{1-M^2}$, $r=\sqrt{x_1^2+x_2^2}$.

At $M>1$ (*superlight speed*):

$$f_0(\mathbf{x},z)=\frac{-H(z-mr)}{2\pi\sqrt{z^2-m^2r^2}}\,,$$

$$f_1(\mathbf{x},z)=\frac{-H(z-mr)}{2\pi}\ln\left(\frac{z+\sqrt{z^2-m^2r^2}}{mr}\right) \qquad (10)$$

where $m=\sqrt{M^2-1}$, $H(z)$ is Heaviside step function.

At $M = 1$ (*light speed*):

$$f_0(\mathbf{x}, z) = -\frac{1}{2\pi}\delta(z)\ln r,$$
$$f_1(\mathbf{x}, z) = -\frac{1}{2\pi}H(z)\ln\, r. \qquad (11)$$

4. Construction of solutions to Maxwell's transport equations

The solution of equations (1) satisfying the radiation conditions has the form of a tensor-functional convolution of the Green tensor with the right side of equation (1):

$$\mathbf{u}(\mathbf{x}, z) = \mathbf{U}(\mathbf{x}, z) * \mathbf{J}(\mathbf{x}, z), \qquad (12)$$

From here follow

$$\mathrm{E}_j = \sum_{k=1}^{3} \mathrm{U}_{jk} * \mathrm{j}_k^m(\mathbf{x}, z) + \sum_{k=4}^{6} \mathrm{U}_{jk} * \mathrm{j}_{k-3}^e(\mathbf{x}, z),$$
$$\mathrm{H}_j = \sum_{k=1}^{3} \mathrm{U}_{(j+3)k} * \mathrm{j}_k^m(\mathbf{x}, z) + \sum_{k=4}^{6} \mathrm{U}_{(j+3)k} * \mathrm{j}_{k-3}^e(\mathbf{x}, z).$$
$$j = 1, 2, 3.$$

The integral representation of this solution depends on the type $\mathbf{J}(\mathbf{x}, z)$. Integral representations for radiation sources are obtained, which are described by regular functions, as well as by singular simple layers on the surfaces of *S*: $\mathbf{J} = \mathbf{J}(\mathbf{x}, z)\delta_S(\mathbf{x}, z)$ and curves L: $\mathbf{J} = \mathbf{J}(\mathbf{x}, z)\delta_L(\mathbf{x}, z)$ (see [1]).

This makes it possible to study the EM fields of moving emitters focused on surfaces and curves.

5. Electromagnetic shock waves. Conditions on the fronts

At light and superluminal speeds, the system of Maxwell's transport equations (1) is strictly hyperbolic. The radiation source generates shock electromagnetic waves, at the fronts of which the following conditions are met for jumps in the intensity of the electromagnetic field:

$$\mu V[\mathbf{H}(\mathbf{x}, z)]_F = [\mathbf{E}]_F \times \mathbf{n}(\mathbf{x}, z),$$

$$\varepsilon V[\mathbf{E}(\mathbf{x}, z)]_F = [\mathbf{H}]_F \times \mathbf{n}(\mathbf{x}, z),$$

where on the right in the equations are the vector products of the jumps of the intensity vectors at the front *F* of the EM wave to the $\mathbf{n}(\mathbf{x}, z)$ - normal to the front (wave vector).

It follows that the electric field and magnetic field strength jumps are orthogonal to each other and orthogonal to the normal to the wave front. If in front of the wave front $\mathbf{E}(\mathbf{x}, z) = 0$, $\mathbf{H}(\mathbf{x}, z) = 0$ then (6), (7) follows:

$$\mathbf{H}(\mathbf{x}, z)\big|_F = \frac{1}{\mu V}[\mathbf{E} \times \mathbf{n}(\mathbf{x}, z)]\big|_F,$$
$$\mathbf{E}(\mathbf{x}, z)\big|_F = \frac{1}{\varepsilon V}[\mathbf{H} \times \mathbf{n}(\mathbf{x}, z)]\big|_F,$$

That is, the shock electromagnetic waves are transverse and the vectors of electric and magnetic intensity at the front of the shock electromagnetic wave are orthogonal to each other and lie in a tangent bundle to it.

6. Conclusion

The results obtained can be used to study the electromagnetic fields of various light emitters and radio wave emitters located on mobile objects (trains, cars, ships, etc.), as well as to solve transport boundary value problems of electrodynamics in limited areas based on the method of boundary integral equations, which the authors plan to do in the near future using the method of generalized functions, similar to [3,4].

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