Results Concerning Paracompact and Strongly Paracompact Spaces

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Abstract: In the present study, we introduce the notion of properly closed hereditary qualities in topological spaces. We discuss whether precompact, metacompact, and k-spaces display the correctly closed hereditary character. By providing definitive answers to these queries and introducing characterizations of these spaces, a contribution to a deeper comprehension of the topological structures of paracompact and metacompact spaces is offered. Notably, we showed that the space E is strongly paracompact if and only if it is paracompact and extending a well-known statement concerning locally compact spaces. In closing, some new findings on paracompact spaces are presented, which contribute to the current discussion in this area and shed light on the complex interrelationships among topological features—using strict proofs and comprehensive. We show that the space E is strongly paracompact if and only if it is paracompact, notably extending a well-known result concerning locally compact spaces. Additional discoveries about paracompact spaces are presented after the work, contributing to the current discussion on this topic and providing insightful information on the complex interrelationships among topological aspects. Our work advances the investigation of topological spaces and their inherent qualities by providing robust proof and in-depth analysis.

Key-words: α −paracompact space, α −Metacompact space, strongly paracompact space.

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1. Introduction and Preliminaries

Over the years, there have been major additions to the literature on correctly closed hereditary qualities in topological spaces, with a focus on paracompactness and metacompactness. Closure-preserving closed covers have been widely used in general topology since E. Michael employed them to provide a crucial description of paracompactness. The highly paracompact characteristic has shown to be an intriguing covering property. It makes sense to generalize compact spaces in this way. It is sufficiently generic to encompass a far wider class of spaces while retaining enough structure to benefit from many of the characteristics of compact spaces. If each countable open cover has a locally finite (or point-finite) open refinement, then the space is countably paracompact (or countably metacompact). In terms of the interests of topologists today, these classes of spaces are significantly different, despite the apparent similarities in their names and some of their respective equivalents. Generally speaking, countably paracompactness and normalcy go hand in hand. In fact, normal but non-countably paracompact areas are referred to as "Dowker spaces," while countably paracompact but non-normal environments are known as "anti-Dowker spaces." However, only a small number of regular spaces were known to be countably metacompact 25 years ago. The strongly paracompact characteristic is unique on the one hand because it differs from other covering qualities in numerous ways. However, as every regular Lindelof space is ¨strongly paracompact, the property is generic. Deeper investigations are made possible by the fundamental idea of properly closed hereditary properties, which is the idea that a property of a topological space E extends to every proper closed subspace of E. Regarding paracompact areas, Csaszar's 2007 work is particularly noteworthy as a seminal contribution. Csaszar explores the field of hereditary qualities and offers insightful information about the extension of certain topological characteristics to closed subspaces. Adding to this, the research conducted in 2007 by Choban, Mihaylova, and Nedev enhances the body of literature by examining certain aspects of paracompactness. Their investigation broadens our knowledge of paracompact spaces by clarifying the complex connection between topological characteristics and genetic traits. We have also studied in detail the notion of metacompactness, which is defined by point finite open refinements for countable open covers. Understanding the wider ramifications of metacompactness in different topological spaces becomes possible when metacompactness is considered as a properly closed hereditary characteristic. Furthermore, the notion of α-metacompactness, as presented in [4], offers a more sophisticated perspective to the research, emphasizing subsets of spaces and their open cover characteristics.

This survey of the literature shows how properly closed hereditary features in topological spaces have been understood via research, with a focus on paracompactness and metacompactness. The knowledge obtained from these investigations not only broadens our theoretical grasp but also has applications in a variety of scientific fields and mathematical specializations.

In this study, Section 2, we address answering issue (A) in the affirmative and give characterizations of paracompact and metacompact spaces. We explore the subtleties of these characteristics, illuminating how they interact in the context of topological spaces. The response in the positive to query (A) advances the current investigation of central issues in topological theory. We also present characterizations that improve our knowledge of paracompact and metacompact regions and provide important new perspectives on their structural characteristics. By deriving a generalization of a well-known result mentioned as [6], we go one step further in Section 3. We broaden our knowledge of the connection between locally compact spaces and the interaction between paracompactness and strong paracompactness. The finding proves that the requirements of being strongly paracompact and paracompact are similar for a locally compact space E. This generalization extends the current body of knowledge and offers a more comprehensive understanding of the relationship between these two important topological features. Furthermore, Section 3 delves into additional results concerning paracompact spaces, contributing to the broader discourse on their properties. Using carefully established definitions and facts, we navigate the intricacies of topological spaces, laying the groundwork for a comprehensive understanding of their characteristics. We use a few key definitions and data to help us with our investigation, which lays a strong basis for the developments that follow in Sections 2 and 3. A thorough and perceptive analysis of the connections and characteristics present in paracompact, metacompact, and strongly paracompact regions is made possible by these fundamental components.

Definition 1.1: [7] A subset A of a space E is called α-paracompact if every open cover of A in E has a refinement which is open and locally finite and covers A.

Definition 1.2: [8] If $M \subseteq E$, then M distinguished with respect to the cover U of E if $\forall a, b \in M : a \neq b, a \in U$ implies $b \notin U$.

We say that the set M is maximal distinguished with respect to a cover $\mathcal{U}$ of E if it is not a proper subset of any distinguished subset of E with respect to $\mathfrak{U}$.

Definition 1.3: [9] A family $\mathcal{U}$ of subsets of a space E is called locally finite outside closed sets if $\forall F \subseteq E$ a closed subset and $\forall n \in E - F$, $\exists V$ an open neighborhood of n which intersects finitely many members of: $V(F) = (V \in \mathcal{U} | V \cap F \neq \phi)$

Definition 1.4: [10] A space E is C-scattered if every non-empty closed subset F of E has a point with a compact neighborhood in F.

Definition 1.5: [11] A collection U of subsets of a space E is called directed if $U_1, U_2 \in \mathcal{U}$ implies that $\exists V \in \mathcal{U}$ such that $U_1 \cup U_2 \subseteq W$.

Theorem 1.6: [11] In the space E, the followings are equivalent:
(a) E is metacompact
(b) Each directed open cover has a closure preserving closed refinement.

Theorem 1.7: [6] If the space E has a locally finite closed cover consisting of paracompact subspaces, then E is paracompact.

Theorem 1.8: [6] If the space E has a locally finite closed cover consisting of locally compact subspaces, then E is locally compact.

Theorem 1.9: [6] If the space E has an open cover consisting of locally compact subspaces, then E is locally compact.

Theorem 1.10:[6] A locally compact space E *is* strongly paracompact *iff it* is paracompact.

Theorem 1.11: [6] If $\mathcal{U}$ be a locally finite open cover of a regular space E and each $U \in \mathcal{U}$ is paracompact and the boundary $Bd(U)$ is lindelof, then E is paracompact.

Theorem 1.12: [8] If $\mathcal{U}$ is an open cover of the space E, then the following hold:
(a) If $K \subseteq E$ is a distinguished subset with respect to $\mathcal{U}$, then it is discrete and closed.
(b) There exists a maximal distinguished subset of E with respect to $\mathcal{U}$.
(c) If M is a maximal distinguished subset of E with respect to $\mathcal{U}$, then $W = (U | U \in \mathcal{U}, U \cap M \neq \phi)$ covers E. The concepts and terminologies are not defined here, see Engelking [6].

2. Structural characteristics of paracompactness and metacompactness

Theorem 2.1. Let E be a T_2 -space. If each proper closed subspace of E is paracompact, then E is para- compact.
Proof: Suppose that U is a non-empty open subset of E whose closure is not equal E. Then
$\{\bar{U}, E - U\}$ is a locally finite closed cover of E consisting of paracompact subsets. Hence by theorem 1.7, E is paracompact.

Theorem 2.2. Let E be a T_2 -space. Then the followings are equivalent:
(a) E is paracompact.
(b) Every closed subspace of E is α -paracompact.
(c) Each closed subspace of E is paracompact.
(d) The set of non-isolated points of E is α-paracompact.

Proof: (a) $\Rightarrow$ (b), (a) $\Rightarrow$ (c) and (d) are proved by standard techniques.
(d) $\Rightarrow$ (a) and (b) $\Rightarrow$ (a) follows from theorem 2.1.
(d) $\Rightarrow$ (a); let $\mathcal{U}$ be any open cover of E. Let A be the set of non-isolated points and A be a α-paracompact.
Then $\mathcal{U}$ has an open (in E) locally finite (in E) refinement V which covers A.
Let $\mathcal{M} = V \cup \{\{n\} | n \in E - A\}$ then $\mathcal{M}$ is an open locally finite refinement of $\mathcal{U}$.
Thus, E is paracompact.

Theorem 2.3. If E is a T_2 –space and each proper closed subset of E is metacompact, then E is a metacompact.

Proof. Let $W \subseteq E$ be a directed open cover of E. Let A be where B = E − W.
Let $\mathcal{U}$ (A)= { $W \cap A$| $W \in \mathcal{U}$ } and $\mathcal{U}$ (B) = { W then $\mathcal{U}$ (A), $\mathcal{U}(B)$ are directed open covers of A and B respectively.

$\mathcal{U}$ (B) have closure preserving closed refinements say $\mathcal{V}(A)$ and $\mathcal{V}(B)$ respectively.
Now $\mathcal{V} = \mathcal{V}$ (A) $\cup \mathcal{V}$ (B) is a closure preserving closed refinement of $\mathcal{U}$. Hence, by Theorem 1.6, E is metacompact.

Recall that a $A \subseteq E$ is called α -metacompact [4] if each open cover of A in E has a point finite open (in E) refinement which covers A.

Following the method used in the proof of 2.2, we obtain the following:

Theorem 2.4. Let E be a T_2 -space. Then the followings are equivalent:

(a) E is metacompact.
(b) Every closed subspace of E is α -metacompact.
(c) Every closed subspace of E is a metacompact.
(d) The set of non-isolated points of E is α-metacompact.

Theorem 2.5. Let E be a T_2 -space. If every proper closed subset of E is a k -space, then E is a
k -space.

Proof. Let $W \neq \phi$ be an open subset of E: $\overline{W} \neq E$.
Let $A_1 = \overline{W}$, $A_2 = E - W$. Then $E = A_1 \cup A_2$. Let $F \subseteq E$ such that $F \cap K$ is closed in K for any compact subset K of E.
To show that E is a k -space we show that F is closed.
Let K_1 be any compact subset of A_1. Then $F \cap K_1$ is closed in K_1.
Consequently, $F \cap K_1$ is closed in E. Therefore, $(F \cap A_1) \cap K_1$ is closed in A_1.
Since A_1 is a k-space we obtain that $(F \cap A_1)$ is closed in A_1.
Thus $(F \cap A_1)$ is closed in E.

Then A, B are closed subsets W an open subset of E such that E = A ∪ B $\cap B$| $W \in \mathcal{U}$ },
Since A and B are metacompact, $\mathcal{U}$ (A) and
Similarly, we showed that $F \cap A_2$ is closed in E.
Consequently, $F = F \cap A_1 \cup F \cap A_2$ is closed in E. Hence E is k -space.

3. Some Results Concerning Paracompact and Strongly Paracompact Spaces

In Section 3, we mainly concentrate on obtaining important results concerning paracompact spaces and present a generalization of Theorem 1.10. The next part of the section delves into a thorough investigation of the aspects that are intrinsic to paracompact spaces, offering significant new perspectives on the topological features of these spaces. We negotiate complex relationships inside these places by drawing on well-established terminology and basic ideas, illuminating their structural subtleties. The presentation of a generalization of Theorem 1.10, a finding with ramifications for the larger subject of topological theory, is a noteworthy highlight of Section 3. Our understanding of the relationships between paracompactness and other topological features is improved by the theorem's extension or generalization. Our goal is to offer a thorough and sophisticated understanding of the linkages found in paracompact and strongly paracompact spaces through meticulous study and logical reasoning.

Theorem 3.1. Let $\mathcal{U}$ be an open cover of regular space E which is locally finite outside closed sets. If each $U \in \mathcal{U}$ is paracompact and boundary $Bd(U)$ is Lindelof, then E is paracompact.

Proof. Let M be a maximal distinguished subset of E with respect to $\mathcal{U}$.

So, M is a closed discrete.

By 1.12, $\mathcal{U}(M)=\{U \in U | U \cap M \neq \phi\}$ covers E.

$\forall n \in M$, let $U_n \in U(M)$ and $n \in U_n$, put

$U_n = \{U_n\} \cup \{U \in \mathcal{U}(M) | n \in U, U \nsubseteq U_n\}$.

Then $U' = \cup\{ \mathcal{U}_n | n \in M\}$ is a subfamily of $\mathcal{U}$ which covers E and U' is locally finite outsides closed sets. If $n \in E$, then $n \in M$ or $n \notin M$.

If $n \notin M$ then there is an open neighborhood of n which intersects finitely many members of U'. If $n \in M$ then there is an open neighborhood G of E which intersects finitely many members of $U'(E - U_n)$. Clearly, $U' = U'(E - U_n) \cup \{U_n\}$. Hence, G intersects finitely many members of U'. Thus, U' is locally finite. Hence, by theorem 1.11, E is paracompact.

Theorem 3.2. If E is paracompact, C-scattered, then E is locally compact.

Proof. Let E be a paracompact, C-scattered.

Denote by $E^{(1)} = \{m \in E | m\}$ do not have a compact neighborhood in E.

For some ordinal $\alpha > 1$, $E^{(\alpha)}$ is defined [19].

If $\beta = \alpha + 1$ is define, then $E^{(\beta)} = (E^{(\alpha)})^{(1)}$ and $E^{(\beta)} = \cap_{\alpha<\beta} E(\alpha)$.

Otherwise, if E is C-cattered, $E^{(\alpha)} = \phi$ for some ordinal α.

For $\alpha = 1$, if $E^{(\alpha)} = \phi$, then E is locally compact.

Hence the result follows.

If the result holds $\forall \beta < \alpha$ i,e, if $E^{(\beta)} = \phi$ for $\beta < \alpha$ then E is locally compact.

Now we want to prove that if $E^{(\alpha)} = \phi$, then E is locally compact.

Case 1. $\exists \alpha : \alpha = \beta + 1$ for some $\beta < \alpha$.

So, $E^{(\alpha)} = \phi$ implies $E^{(\beta)}$ is locally compact.

For each $m \in E^{(\beta)}$, $\exists U_m$ containing m such that $\bar{U} \cap E^{(\beta)}$ is compact.

For each $n \in E - E^{(\beta)}$ $\exists U_n$ containing n : $n \in \bar{U}_n \subseteq E - E^{(\beta)}$.

Hence, for each $n \in E - E(\beta)$, $(\bar{U}_n)^{\beta} = \phi$ [20].

By the induction assumption for each $n \in E - E^{(\beta)}$, $\bar{U}_n$ is locally compact.

Now $\mathcal{U} = \{U_n | n \in E - E^{(\beta)}\} \cup \{U_m | m \in E^{(\beta)}\}$ consists an open cover of E.

Since E is paracompact, U has a locally finite closed refinement, say $\mathcal{V}$.

Let $\mathcal{V}' = \{\bar{V} | V \in \mathcal{V}, V \subseteq U_n\}$ for some $n \in E - E^{(\beta)}$.

Now $E = E^{(\beta)} \cup \mathcal{V}'$ and $E^{(\beta)} \cup \mathcal{V}'$ is a closed locally finite collection, hence by theorem 1.8 we obtain that E is locally compact.

Case 2. $E^{(\alpha)} = \cap_{\beta<\alpha} E^{(\beta)} = \phi$.

Consider the open cover $\mathcal{U} = \{E - E^{(\beta)} | \beta < \alpha\}$ of E.

By the inductive assumption, $E - E^{(\beta)}$ is a locally compact subspace of E.

Thus, by theorem 1.9, E is locally compact.

Corollary 3.3. If E is paracompact C-scattered space , D is paracompact, then $E \times D$ is paracompact.

The following theorem is a generalization of theorem 1.10.

Theorem 3.4. For C-scattered space E, the following are equivalent:

(a) E is paracompact.

(b) E is strongly paracompact.

The proof follows by theorem 3.2 and theorem 1.10.

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