Properties of Abiyev's Triangle

POLADOVA VUSALA NADIR

Ministry of Science and Education Republic of Azerbaijan Institute of Radiation Problems
Address: 9 B.Vahabzade str., Baku, AZERBAIJAN, AZ1143

Abstract: The procedure for writing Abiyev's triangle using symmetric graphs has been explained. The Fibonacci and Lucas sequences are formed, respectively, by adding the numbers from the lines of the triangles on the left and right sides of this triangle. Symmetrical diagonal elements of these triangles generate Pascal's and analogous triangles, and vice versa. The diagonal numbers of these triangles make up the elements of the left and right triangles of Abiyev's triangle. The numbers in the lines of these ultimate triangles represent the coefficients of the polynomial terms corresponding to the binomial $a^n \pm b^n = f_n[(a+b), ab]$. Mathematical conversions are considerably simpler when polynomials are used in the calculations. Examples of this have been provided. Using this method, we obtain a simple solution to one of the Ramanujan's problems.

1. Introduction

It is known that Fibonacci numbers arise as sums of diagonals in Pascal's triangle.

$$F_n = \sum_{k=0}^{l\frac{n}{2}J} \binom{n-1-k}{k} = \frac{(n-1-k)!}{k!(n-1-2k)!} \quad (1)$$

for all $n \geq 0$.

$$F_{7+1} = F_8 = \sum_{k=0}^{3} \binom{n-k}{k} =$$
$$= \binom{7}{0} + \binom{6}{1} + \binom{5}{2} + \binom{4}{3} = 1+6+10+4 = 21$$

This is the sum of the numbers in the 8th line on the left side of Abiyev's triangle [1, 2]. This leads us to the conclusion that the sum of the diagonal numbers in Pascal's triangle forms the spectrum of Fibonacci numbers. The objective of this paper is to calculate the spectrum of Fibonacci and Lucas numbers using the $a^n \pm b^n = f_n[(a+b), ab]$ equation and to demonstrate the application of the Abiyev Triangle in solving algebraic equations [4, 5, 6].

2.1. Writing a triangle

First, let's form the middle column using natural numbers, starting from 0 (Table 1). Then, we write 0 in front of even numbers and 1 in front of odd numbers on the left side of the middle column. On the right side of this middle column, we write 2 in front of the even numbers, and in front of the odd numbers, we write odd numbers themselves to form the 1st column on the left and right. Next, we write 1;1 in front of the numbers 0;1 on the left side of these columns and 1;1 in front of the numbers 2;3 on the right side. We use mutually symmetrical columns to generate the second column on the left and right sides of this table. By repeating this rule, we also make the 3rd, 4th,... columns. As you can see, the number of columns is equal to $\frac{n+1}{2}$ for odd n's and $\frac{n+2}{2}$ for even n's (Table 1).

If we add the numbers in the lines of this table, we get the Fibonacci sequence on the left, and the Lucas sequence on the right:

$$F_n = \sum_{k=0}^{l\frac{n}{2}J} \binom{n-1-k}{k}; \ L_n = \sum_{k=0}^{l\frac{n}{2}J} \frac{n}{k}\binom{n-1-k}{k-1}$$

Examples:

$$F_7 = \sum_{k=0}^{3} \binom{n-k}{k} = \binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3} =$$
$$= 1+5+6+1 = 13$$

(2)

$$L_6 = \sum_{k=0}^{3} \frac{6}{k}\binom{5-k}{k-1} = \frac{6}{0}\binom{5}{-1} + \frac{6}{1}\binom{4}{0} +$$

$$+\frac{6}{2}\binom{3}{1}+\frac{6}{3}\binom{2}{2}=1+6+9+2=18 \qquad (3)$$

2.2. Spectrum of Fibonacci and Lucas sequences

Note that Abiyev proposed to write the function $f_n\big[(a+b),ab\big]$ in the form of a binomial $a^n \pm b^n$ [3]:

$$f_n\big[(a+b),ab\big]=$$
$$=c_0(a+b)^n-c_1(a+b)^{n-2}ab+$$
$$+c_2(a+b)^{n-4}(ab)^2+...+c_n(ab)^n$$

Given the roots of the golden ratio equation $\big[a=\tfrac{1}{2}(1+\sqrt{5});\ b=\tfrac{1}{2}(1-\sqrt{5})\big]$, substituting the values of their sum $(a+b=1)$ and product $(ab=-1)$ into the above function we obtain:

$$f_n=c_0+c_1+c_2+...+c_n$$

The Fibonacci number spectrum corresponds to the coefficients of a^n-b^n, the numbers (in orange) on the left side of Abiyev's triangle, while the Lucas numbers spectrum corresponds to coefficients of a^n+b^n, the numbers (in green) on the right side of the triangle (Table 1).

Examples:

$$a^n-b^n=a^8-b^8=(a-b)x(x^6-6x^4y+$$
$$+10x^2y^2-4y^2);$$
$$a=3;\ b=2\ \text{for}\ a+b=x=5;\ ab=y=6.$$

$$3^8-2^8=$$
$$=1\cdot5\big(5^6-6\cdot5^4\cdot6+10\cdot5^2\cdot6^2-$$
$$-4\cdot6^3\big)=5\big(24625-23364\big)=$$
$$=5\cdot1261=6305$$

Note that the binomial $a^n \pm b^n$ is easily calculated for conjugate complex numbers a and b:

$$a=c+id,\ b=c-id,\ a+b=2c,$$
$$ab=c^2+d^2.$$

2.3. Examples of the application of this formula.

Example 1.
$$\begin{cases}x+y=1\\x^2+y^2=2\end{cases}\Longrightarrow xy=\frac{-1}{2}$$

$$x^{11}+y^{11}=?$$

$$\frac{x^{11}+y^{11}}{x+y}=(x+y)^{10}-11(x+y)^8xy+$$
$$+44(x+y)^6(xy)^2-77(x+y)^4(xy)^3+$$
$$+55(x+y)^2(xy)^4-11(xy)^5=$$
$$=1^{10}-11\cdot1^8\cdot\left(\frac{-1}{2}\right)+44\cdot1^6\cdot\left(\frac{-1}{2}\right)^2-$$
$$-77\cdot1^4\cdot\left(\frac{-1}{2}\right)^3+55\cdot1^2\cdot\left(\frac{-1}{2}\right)^4-$$
$$-11\cdot\left(\frac{-1}{2}\right)^5=1+11\cdot\frac{1}{2}+44\cdot\frac{1}{4}+77\cdot\frac{1}{8}+$$
$$+55\cdot\frac{1}{16}+11\cdot\frac{1}{32}=$$
$$=\frac{32+11\cdot16+44\cdot8+77\cdot4+55\cdot2+11}{32}=$$
$$\frac{32+176+352+308+110+11}{32}=\frac{989}{32}=30.90625$$

Example 2.

$$\begin{cases}x+x^{-1}=\sqrt{13}\\x^5-x^{-5}=?\end{cases}$$

$$x^2+x^{-2}=\left(\sqrt{13}\right)^2-2=11$$

$$\big(x-x^{-1}\big)^2=x^2+x^{-2}-2=9;$$
$$x-x^{-1}=3$$

$$\frac{x^5-x^{-5}}{x-x^{-1}}=\big(x+x^{-1}\big)^4-$$
$$-3\big(x+x^{-1}\big)^2\big(x\cdot x^{-1}\big)+\big(x\cdot x^{-1}\big)^2=$$
$$=\left(\sqrt{13}\right)^4-3\left(\sqrt{13}\right)^2+1=169-39+1=131$$

$$x^5-x^{-5}=\big(x-x^{-1}\big)\cdot131=3\cdot131=393$$

Example 3.

$$\begin{cases}x+y=10\\x^5+y^5=20000\end{cases}$$
$$x=?,\ y=?$$

$$\frac{x^5+y^5}{x+y}=(x+y)^4-5(x+y)^2xy+5(xy)^2$$

$$x^5+y^5=10\big(10^4-5\cdot10^2xy+5(xy)^2\big)=$$
$$=10\big[10000-500xy+5(xy)^2\big]=20000$$

$$5(xy)^2 - 500\,xy + 8000 = 0$$
$$(xy)^2 - 100\,xy + 1600 = 0$$

$$(xy) = 50 \pm \sqrt{2500 - 1600} = 50 \pm 30$$

$$(xy)_1 = 20; \qquad\qquad (xy)_2 = 80$$

$$\begin{cases} x + y = 10 \\ xy = 20 \end{cases} \qquad \begin{cases} x + y = 10 \\ xy = 80 \end{cases}$$

$$t^2 - 10t + 20 = 0; \qquad u^2 - 10u + 80 = 0$$
$$t = 5 \pm \sqrt{25 - 20} = 5 \pm \sqrt{5}; \quad u = 5 \pm \sqrt{-55}$$
$$(x_1; y_1) = (5 - \sqrt{5}; 5 + \sqrt{5})$$

$$(x_2; y_2) = (5 + \sqrt{5}; 5 - \sqrt{5})$$
$$(x_3; y_3) = (5 - i\sqrt{5}; 5 + i\sqrt{5})$$
$$(x_4; y_4) = (5 + i\sqrt{5}; 5 - i\sqrt{5})$$

Conclusion

The diagonal elements in the left side of the Abiyev Triangle (red cells) constitute the Pascal Triangle and vice versa. On the other hand, the diagonal elements in the right side of the Abiyev Triangle (blue cells) constitute the Pascal Triangle analogue and vice versa (Table 2, 3).

Table 1.

Abiyev's triangle

												n												
0											0	0	2											2
1											1	1	1											1
1										1	0	2	2	1										3
2										1	1	3	3	1										4
3									1	2	0	4	2	4	1									7
5									1	3	1	5	5	5	1									11
8								1	4	3	0	6	2	9	6	1								18
13								1	5	6	1	7	7	14	7	1								29
21							1	6	10	4	0	8	2	16	20	8	1							47
34							1	7	15	10	1	9	9	30	27	9	1							76
55						1	8	21	20	5	0	10	2	25	50	35	10	1						123
89						1	9	28	35	15	1	11	11	55	77	44	11	1						199
144					1	10	36	56	35	6	0	12	2	36	105	112	54	12	1					322
233					1	11	45	84	70	21	1	13	13	91	182	156	65	13	1					521
377				1	12	55	120	126	56	7	0	14	2	49	196	294	210	77	14	1				843
610				1	13	66	165	210	126	28	1	15	15	140	378	450	275	90	15	1				1364
987			1	14	78	220	330	252	84	8	0	16	2	64	336	672	660	352	104	16	1			2207
1597			1	15	91	286	495	462	210	36	1	17	17	204	714	1122	935	442	119	17	1			3571
2584		1	16	105	364	715	792	462	120	9	0	18	2	81	540	1386	1782	1287	546	135	18	1		5778
4181		1	17	120	455	1001	1287	924	330	45	1	19	19	285	1254	2508	2717	1729	665	152	19	1		9349
6765	1	18	136	560	1365	2002	1716	792	165	10	0	20	2	100	825	2640	4290	4004	2275	800	170	20	1	15127

Table 2.

Pascal's triangle

														Power
						1								2^0
					1		1							2^1
				1		2		1						2^2
			1		3		3		1					2^3
		1		4		6		4		1				2^4
	1		5		10		10		5		1			2^5
1		6		15		20		15		6		**1**		2^6
1	7		21		35		35		**21**		7		1	2^7
1	8	28		56		**70**		56		28		8	1	2^8
1	9	36	**84**		126		126		84		36	9	1	2^9
1	10	**45**	120	210	252	210	120	45	10	1				2^{10}
1	**11**	55	165	330	462	462	330	165	55	11	1			2^{11}
1	12	66	220	495	792	924	792	495	220	66	12	1		2^{12}
1	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1	2^{13}

Table 3.

Analogous Pascal's triangle

																Sum
						1		2								3
					1		3		2							6
				1		4		5		2						12
			1		5		9		7		2					24
		1		6		14		16		9		2				48
	1		7		20		30		25		11		2			96
1		8		27		50		55		36		**13**		2		192
1	9		35		77		105		**91**		49		15		2	384
1	10	44		112		**182**		196		140		64		17	2	768
1	11	54	**156**		294		378		336		204		81	19	2	1536
1	12	**65**	210	450	672	714	540	285	100	21	2					3072
1	**13**	77	275	660	1122	1386	1254	825	385	121	23	2				6144
1	14	90	352	935	1782	2508	2640	2079	1210	506	144	25	2			12288
1	15	104	442	1287	2717	4290	5148	4719	3289	1716	650	169	27	2		24576
1	16	119	546	1729	4004	7007	9438	9867	8008	5005	2366	819	196	29	2	49152

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