On a Bipolar Model of Hyperbolic Geometry and its Relation to Hyperbolic Friedmann-Robertson-Walker Space

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Abstract: - Negatively curved regions of space in a Friedmann-Robertson-Walker (FRW) universe are a realistic possibility. These regions would occur in voids in the large-scale structure of the universe where there is no dark matter with only dark energy present. Hyperbolic space is strange from a physical point of view and various models of hyperbolic space have been introduced, each offering a clarifying view. In the present work we develop a new bipolar model of hyperbolic geometry and show that it provides new insights toward an understanding of hyperbolic as well as elliptic FRW space. In particular, using the bipolar model, we show that the circular geodesics of an FRW space can be referenced to two real centers – a Euclidean center and a hyperbolic center. Considering the physics of elliptic FRW space is so well confirmed in the ΛCDM model describing the expansion of the universe with respect to a Euclidean center, it is possible that the hyperbolic center also plays a physical role in regions of hyperbolic space.

Key-Words: - Hyperbolic geometry, Bipolar coordinates, Robertson-Walker space
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1 Introduction

Bipolar coordinates are an unusual system, typically not even considered in most applications. However, in physical applications, this system is sometimes advantageous, for example as a natural coordinate choice for a variety of electromagnetic solutions of Laplace's equation [1], and, as we shall show below, for negatively curved FRW space in Cosmology. Negatively curved FRW space will arise in Cosmology in regions of voids in the large-scale structure of the universe where dark energy but no dark matter resides [2]. At a sufficient distance away, one sees a uniform outward expansion of galaxies, the Hubble flow, described by comoving or so-called "frozen" coordinates so that only the expansion of 3-space is affected. Locally, however, one must "unfreeze" the coordinates. One will then need to address particle dynamics in such regions. Particle geodesics behave oddly in hyperbolic space and various "models" of hyperbolic space have been introduced, each offering some enlightened view. Below we first describe relevant features of three existing models; the Poincare` half-plane, Poincare` disk, and the more recent Hubbard band model. All of these 2-dimensional models utilize Cartesian coordinates in the hyperbolic plane. We then develop a new model of hyperbolic geometry, closely related to the band model, but using bipolar coordinates rather than Cartesian and show that it provides new insights toward an understanding of hyperbolic FRW space.

2 Conformal Models of Hyperbolic Geometry

There are three often cited conformal models of hyperbolic geometry [3]: the Poincare` half-plane, the Poincare` disk, and the Minkowski model. The three models are isomorphic and each displays different insights of hyperbolic space under varying boundary constraints. For example, consider the Poincare` half-plane. This hyperbolic 2-space has the metric,

$$ds^2 = \lambda^2 \left(\frac{dx^2 + dy^2}{y^2} \right), \qquad (1)$$

and represents a plane of constant negative curvature, $-1/\lambda^2$, “the hyperbolic plane”, described by Cartesian coordinates (x,y), excluding the x-axis. Geodesics are semi-circles centered on the x-axis and perpendicular to it, and lines perpendicular to it (Fig. 1).

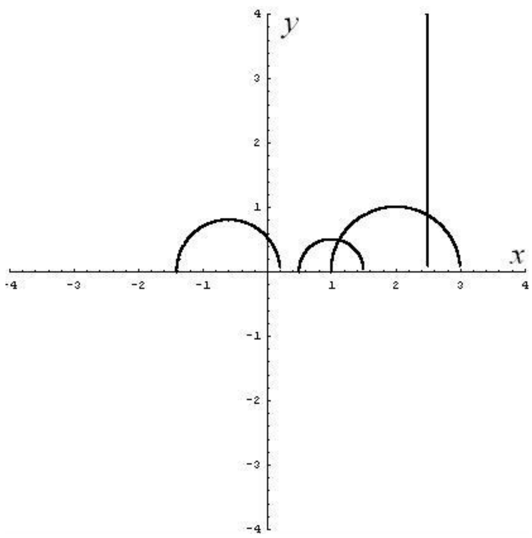

Fig.1. Geodesics of the Poincare` Half-Plane Model of the hyperbolic plane

Recent work by Hubbard [4], describes a 4th conformal model of hyperbolic geometry, the band model.

The metric for this model is:

$$ds^2 = \frac{d\eta^2 + d\psi^2}{(\cos\psi)^2} \qquad (2)$$

where (η, ψ) are Cartesian coordinates. The geodesic equations are:

$$\begin{aligned} &\eta'' + 2\eta'\psi' \tan\psi = 0 \\ &\psi'' - (\eta'^2 - \psi'^2)\tan\psi = 0 \end{aligned} \qquad (3)$$

These equations yield two geodesics:

$\eta = constant$ (vertical lines in Fig.2 – not plotted)

$$\eta = \eta_0 \pm \log\left(K \sin\psi + \sqrt{1 - K^2 (\cos\psi)^2} \right) \qquad (4)$$

K and η_0 are constants. The Cartesian representation, (η, ψ), of the geodesics, Eq. (5), is plotted in Fig. 2. These are the geodesics described in Hubbard’s work. These are not the usual semi-circles. It is called the band model because the space of its complex representation is the band, $\pi/2 > \psi > -\pi/2$.

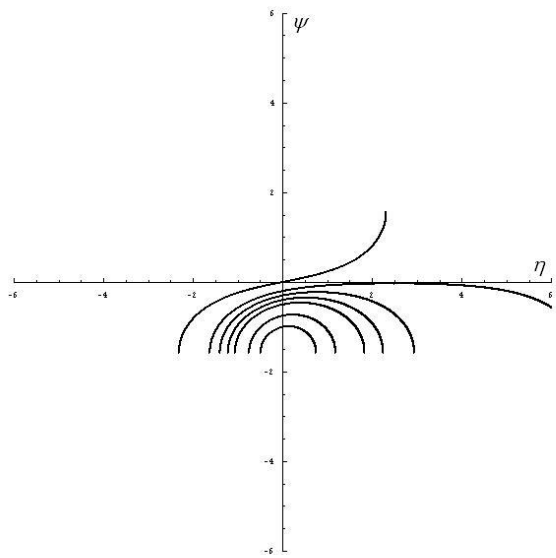

Fig.2. Geodesics of the Hubbard Band-Model of the hyperbolic plane from eqn. (5)

3 The Bipolar Model of Hyperbolic Geometry

We shall now prove the following proposition:

Proposition 1: The band model is the Euclidean bipolar representation of the Cartesian half-plane model.

Proof: Consider the half-plane metric in Cartesian coordinates (x, y) of Eq.(1).
Under the standard Euclidean coordinate transformation to a bipolar system $(x, y) \to (\eta, \psi)$,

$$x = \frac{h \sinh \eta}{\cosh \eta + \sin \psi}, \qquad (5)$$

$$y = \frac{h \cos \psi}{\cosh \eta + \sin \psi}$$

where, $-\infty < \eta < \infty$ $\quad 0 \le \psi \le 2\pi$ and the two poles are at $x = \pm h$.
For $h = 1$, $\lambda \equiv 1$, the metric (1) transforms to:

$$ds^2 = \frac{d\eta^2 + d\psi^2}{(\cos \psi)^2}. \qquad (6)$$

This is the band model metric, Eq.(2), and the proof is complete. We have shown that the band model is simply a different Euclidean view of the half-plane model. Rather than view (η, ψ) as Cartesian, as in Fig. 2, it is natural here to take them as the Euclidean bipolar coordinates defined with respect to Cartesian coordinates from Eq.(5). This is an entirely new representation of the half-plane or band model which we shall call the *bipolar model* of hyperbolic geometry.

For convenience, we rotate the metric, Eq.(6), for a bipolar representation on the x-axis (Fig.3).

$$ds^2 = \frac{d\eta^2 + d\psi^2}{(\sin \psi)^2} \qquad (7)$$

The geodesics of the bipolar model are traditional semi-circles about the poles at $h = \pm 1$ on the x-axis.

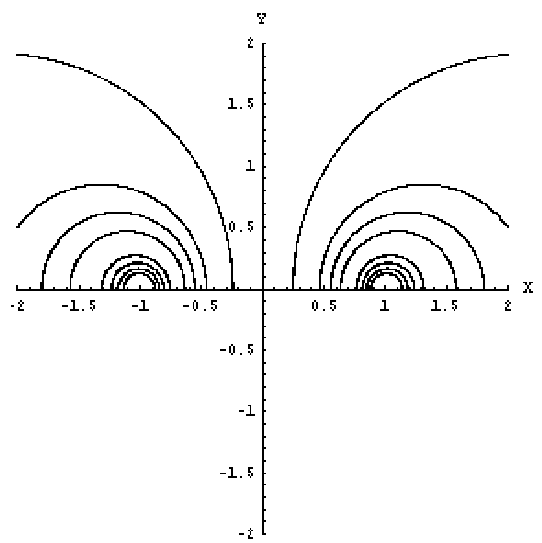

Fig.3. Geodesics of the Bipolar Model of the hyperbolic plane

The polar equations of these circles with respect to $(x, y) = (0, 0)$ are given by:

$$r_x(\phi) = h\left(\cos \phi \coth \eta - \sqrt{\cos^2 \phi \coth^2 \eta - 1}\right) \qquad (8)$$

The Cartesian equation of these "x-circles" is given by:

$$(x - h \coth \eta)^2 + y^2 = h^2 \text{csch}^2 \eta \qquad (9)$$

We demonstrate below that this family of circles has, in fact, *two centers*: a Euclidean center at $x = h \coth \eta$ and a hyperbolic center at the pole $x = h = 1$.

Under the transformation $(\eta,\psi)\to(\psi,\eta)$, retaining the definition of the bipolar coordinates in Eq. (5), metric (7) becomes:

$$ds^2 = \frac{d\eta^2 + d\psi^2}{(\sin\eta)^2} \tag{10}$$

The geodesics of the "complimentary" metric space (10) yield circles centered on the y-axis. Their polar representation is given by:

$$r_y(\phi) = h\left(\sin\phi\cot\psi - \sqrt{\sin^2\phi\cot^2\psi + 1}\right) \tag{11}$$

These "y-circle" geodesics pass through both poles and are shown in Fig. 4.

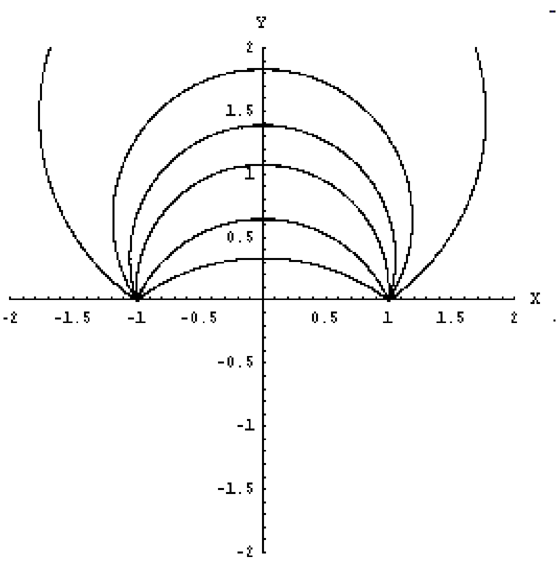

Fig.4. Geodesics of the complimentary bipolar hyperbolic metric, Eq. (10).

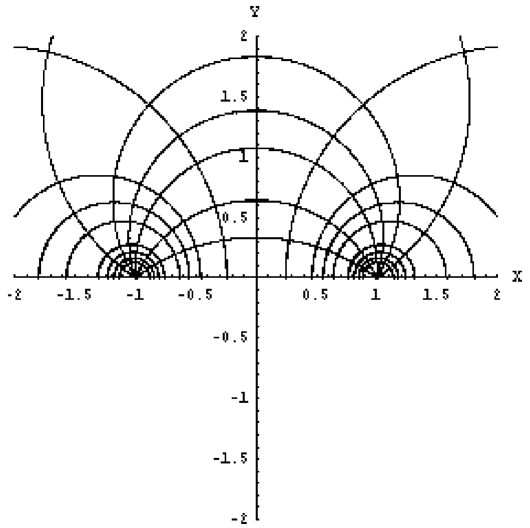

Fig.5. Geodesics of Fig.3 and Fig. 4 overlaid.

These circles are orthogonal to the circles (9) and are given in Cartesian coordinates by:

$$(y - h\cot\psi)^2 + x^2 = h^2\csc^2\psi \tag{12}$$

With respect to the metric space (10), the geodesics of its complimentary space (7) are simply Euclidean circles and therefore, by a fundamental theorem of hyperbolic geometry [5], are also hyperbolic circles about their hyperbolic center. Overlaying the geodesics of the two complimentary spaces in the same Cartesian system yields Fig. 5.

One must, of course, make a choice as to which metric, (7) or (10), will govern the space of the full set of orthogonal curves which coincide with the natural orthogonal bipolar coordinate Euclidean space. For present purposes, we choose metric (10). Then the y-circles are geodesics while the x-circles are Euclidean circles described by Eq. (9) as illustrated in Fig. 6. This graph then becomes, in essence, a demonstration of the "standard construction" [6] required in the well-known proof in hyperbolic geometry that Euclidean circles in the hyperbolic plane are also hyperbolic circles . To this end, it can be shown with respect to metric (10) that the "curved" radii from the hyperbolic center at the pole to a given circle are constant and are given by:

$$R = \frac{\lambda}{2} log\left(\frac{\lambda+\rho}{\lambda-\rho}\right) \qquad (13)$$

where ρ is the Euclidean radius of that circle found from Eq.(9). This is shown in Fig. 6.

4 Relation to FRW Metrics

The space inside the Euclidean circle of Fig.6, when referenced to its Euclidean center, can be covered by the Poincare` disk model [6]. The Poincare` disk is the FRW space for curvature $k<0$. It's metric with respect to the Euclidean center is

$$ds^2 = \frac{d\rho^2 + \rho^2 d\theta^2}{\left(1+\frac{k\rho^2}{4}\right)^2} \ , \qquad (14)$$

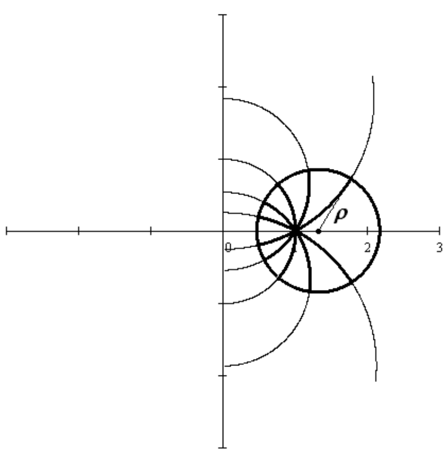

Fig.6. Constant radii y-circle geodesic segments of metric (10) from a pole at $x = 1$ to an x-circle intercept are emphasized. These segments are also hyperbolic radii of the circle from its hyperbolic center. One Euclidean radius, ρ, of the circle from its Euclidean center (dot) is shown.

where ρ and θ are the Euclidean radius and angle from the Euclidean center in Fig. 6. The physical "proper length" of the radius for $\theta = constant$ (a geodesic) is, from (14),

$$s = \frac{\lambda}{2} Ln\left(\frac{\lambda+\rho}{\lambda-\rho}\right), \qquad (15)$$

where $\lambda^2 \equiv -4/k$, and is identical to (13). Thus we have shown for a negatively curved RW space that the "proper", or hyperbolic, radius measured from the Euclidean center of the circle is identical to the hyperbolic length of the curved radii measured from its hyperbolic center. This demonstrates the dual-center aspect of orbits in negatively curved RW spaces.

One more useful isometry regarding the curved radii (13) can be described. We shall use relation (13), which is the curved radius emanating from the hyperbolic center, to transform (14) to a metric with respect to the hyperbolic center. Equation (13) can also be written:

$$R = \frac{\lambda}{2} log\left(\frac{\lambda+\rho}{\lambda-\rho}\right) = \lambda \tanh^{-1}\frac{\rho}{\lambda} \qquad (16)$$

Solving for ρ:

$$\rho = \lambda \tanh\left(\frac{R}{\lambda}\right) \qquad (17)$$

Inserting this into (14) yields

$$ds^2 = dR^2 + \lambda^2 \sinh^2\left(\frac{R}{\lambda}\right) d\phi^2 . \qquad (18)$$

Defining $\chi \equiv \frac{R}{\lambda}$ and simplifying

(18) yields:

$$ds^2 = \lambda^2\left(d\chi^2 + \sinh^2\chi \, d\phi^2\right) \qquad (19)$$

This is the familiar 2-dimensional hyperbolic form of the spatial FRW metric. However, we now understand the meaning of "χ". It is simply the curved hyperbolic radius of Fig. 6 emanating from the hyperbolic center. For positively curved RW spaces, there are still two centers, a Euclidean center and an elliptic center which is now imaginary. Also, it can be shown that the distance between the hyperbolic center and the Euclidean center is given by:

$$\varepsilon = \lambda\left(1-\operatorname{sech}\chi\right) \tag{20}$$

The bipolar model has served to link several isometries of hyperbolic space depending on a judicious choice of origin for measurement: the half-plane model with respect to $(x,y)=(0,0)$; the bipolar model with respect to $(x,y)=(h,0)$; and the disk model (RW space) with respect to $(x,y)=(h\coth\eta,0)$.

5 Conclusions

Negatively curved spaces must eventually play a role in Cosmology – if for no other reason than void regions in the currently accepted large-scale structure of the universe must be negatively curved FRW space due to the absence of matter. There are a number of models describing various aspects of hyperbolic space. In this work, we have created a new model, which we term the "bipolar model". We have shown that the bipolar model is intimately related to negatively curved FRW space and clearly serves to demonstrate an unusual aspect of such spaces – namely that the circular geodesics of such spaces have two real centers, a hyperbolic center as well as a Euclidean center. These are not merely the Euclidean center and poles of the bipolar coordinate system but rather refer to two distinct centers for circular orbits of particles in hyperbolic systems, as shown in Fig. 6. In two dimensions that property is the direct result of the merger of Euclidean plane coordinates and hyperbolic metrics describing the geometry of the "hyperbolic plane". The above work is easily extended to three dimensional space and FRW 4-space. Considering the physics of elliptic FRW space is so well confirmed in the ΛCDM model describing the expansion of the universe with respect to a Euclidean center (Euclidean coordinates with non-Euclidean metric), it is possible that the hyperbolic center also plays a physical role in regions of hyperbolic space where coordinates are no longer frozen. However that issue is beyond the scope of this paper.

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