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Cooperative effects in risk models with discrete time

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Abstract: In this paper risk models with small initial capital, insurance percent and ruin probability are constructed. These models may be used in different modern applications among which an insurance of a franchisee is one the most important. The models are based on a principle of a mutual insurance that is a considered system is an aggregation of a large number of identical insurance systems. We assume that these identical systems may be as independent so weak dependent. In such risk models phase transition phenomena are detected also. Main method to obtain these results is an estimate of rate convergence in limit theorems from probability theory.

Key–Words: Risk model, initial capital, insurance percent, mutual insurance, phase transition, a franchisee.

Introduction

Known mathematical models of an insurance are characterized by the following parameters: the initial capital x, the ruin probability $p(x)$ and the insurance percent b and the distribution of the risk (the loss). Usually a behavior of the function $p(x)$ is investigated in a case of a fixed but not small b and large x.

There is a lot of articles and monographs devoted to an analysis of ruin probability behavior for large initial capitals: $x \to \infty$. An asymptotic of the function $p(x)$ depends significantly on an asymptotic of tails of insurance loss distribution. These considerations are made when the distribution of insurance loss have light [1], [2], [3], [4] or heavy tails [5], [6], [7], [8]. Another topic of this investigation is asymptotic analysis of risk models under constant [9], [10] [11] or stochastic [12], [13], [14] interest forces. There are papers devoted not to asymptotic analysis but to a construction of upper and low bounds of ruin probabilities [15], [16].

But modern applied insurance systems of natural catastrophes: floods, draughts, forest fires, earthquakes, tsunami, etc. demand to introduce changes to such a formulation of a problem. A necessity of a construction and an investigation of insurance systems with a small insurance percent b, a small initial capital x and a small ruin probability $p(x)$ appears in a case of large risks. Existing risk models do not possess these properties. Now new applications of such risk models appear in the insurance of a franchisee [17]. These applications are widely used in manifold spheres of modern business.

In this paper a risk model, satisfying these properties is constructed. It is based on a principle of a mutual insurance that a considered system is an aggregation of n independent and identical insurance systems. It is possible not only to recognize the cooperative effects in the aggregated system but to extract in the parameter set the regions, where for $n \to \infty$ these effects are significant, and the regions, where the effects are small. A specifics of a suggested model of a mutual insurance with independent and weak dependent risks is an existence of clear boundary between these regions. Such boundaries may be interpreted as phase transition phenomena.

1 Model of mutual insurance with independent risks

Consider n independent and identical insurance companies. Suppose that the annual risk of the j-th company in the k-th year is $x(k,j)$ and the random variables (r.v.'s) $x(k,j),\ j = 1,\dots,n,\ k = 1,2,\dots,$ are independent and identically distributed,

$$M\,x(k,j) = 1,\ p(x(k,j) < t) = G(t).$$

Suppose that the annual prizes of the single company equal to $1+b$, where $b = n^{-\gamma},\ \gamma > 0$. As the common prize of n companies aggregation is $(1+b)n$ and the common risk is $\sum_{j=1}^{n} x(k,j)$ so the ruin probability

$$p_n = p_n(x) = P\left(\sup_{m>0} \sum_{k=1}^{m} \sum_{j=1}^{n} (x(k,j) - 1 - b) > x\right).$$

Suppose that x is a fixed and sufficiently small quantity, for example $x = 0$.

Consider a case of large risks. The risk is large if [18] its distribution function (d.f.) ($G(t) = 0,\ t \le 0$) for some $\alpha,\ C,\ 1 < \alpha < 2,\ C > 0$, satisfies the condition

$$1 - G(t) \sim \frac{C(2-\alpha)}{\alpha}\, t^{-\alpha}, \quad t \to +\infty. \tag{1}$$

Theorem 1. *Suppose that for some $\alpha,\ 1 < \alpha < 2$, there exists $C > 0$ so that d.f. G satisfies the condition (1). If the inequality*

$$\gamma < 1 - 1/\alpha \tag{2}$$

is true then for any $\tau,\ 1/(1-\gamma) < \tau < \alpha$, there exists a positive number $C_1 = C_1(\tau)$ so that

$$p_n \le C_1 n^{1-\tau(1-\gamma)}, \quad n = 1, 2, \dots \tag{3}$$

If the inequality (2) is not true then

$$\lim_{n\to\infty} \inf(p_n,\ n \ge 1) > 0. \tag{4}$$

Corollary 1. *If the inequality (2) is true then*

$$\lim_{n\to\infty} p_n = 0. \tag{5}$$

Theorem 2. *Suppose that d.f. G satisfies the theorem 1 conditions. If the inequality*

$$\gamma > 1 - 1/\alpha \tag{6}$$

is true then

$$\lim_{n\to\infty} p_n = 1. \tag{7}$$

For a comparison consider a case when the risks are not large (d.f. G has a finite variation and does not satisfies the condition (1)):

$$D\, x(k,j) = \sigma^2,\ 0 < \sigma^2 < \infty. \tag{8}$$

Theorem 3. *Suppose that d.f. G satisfies the condition (8). If the inequality*

$$\gamma < 1/2 \tag{9}$$

is true then there exists a positive number C_2 so that

$$p_n \le C_2 n^{2\gamma-1}, \quad n = 1, 2, \dots \tag{10}$$

If the condition (9) is not true then the formula (4) takes place.

Corollary 2. *If the inequality (9) is true then the formula (5) takes place.*

Theorem 4. *Suppose that d.f. G satisfies the conditions of the theorem 3. If the inequality*

$$\gamma > 1/2 \tag{11}$$

is true then (7) takes place.

In more strong conditions on d.f. G the theorem 3 has the following modification.

Theorem 5. *Suppose that the condition (9) is true then the following statements take place.*
1. If d.f. G has a density and there exists $\nu > 0$ so that

$$M\ \exp(\nu x(k,j)) < \infty \tag{12}$$

then

$$\ln p_n \sim -\frac{n^{1-2\gamma}}{2\sigma^2}, \quad n \to \infty. \tag{13}$$

2. If there exists $\mu > 2$ so that

$$M\, x^{\mu}(k,j) < \infty \tag{14}$$

then there is a positive number q_μ so that

$$p_n \le \frac{q_\mu}{n^{\mu-1-\mu\gamma}},\ n = 1, 2, \dots \tag{15}$$

2 Mutual insurance models with weak dependent risks

In the previous subsection the ruin probability

$$\Phi = \lim_{n\to\infty} p_n$$

of the aggregated insurance system, consisting of n subsystems with independent and identically distributed annual risks, was considered. In different conditions the parameter $\gamma^* > 0$ satisfying

$$\Phi = \begin{cases} 0, & \gamma < \gamma^*, \\ 1, & \gamma > \gamma^*. \end{cases} \tag{16}$$

was found.

P. Embrechts suggested to consider the phase transition (16) in the case when the risks of different united subsystems are weak dependent. In this subsection there is an exhaustive solution of this question, based on a special stochastic model of a weak dependence between annual risks $x(k,j)$ of aggregated subsystems. This dependence supposed that the fluctuation of the risks $x(k,j)$ is divided into a common part with the small order $n^{-\delta},\ \delta > 0$, and an individual part with the finite order 1 as follows

$$x(k,j) - 1 = n^{-\delta}\Delta x(k) + \Delta x(k,j). \tag{17}$$

Here $\Delta x(k),\ \ \Delta x(k,j)$ are independent and identically distributed r.v.'s with common d.f. $U(t)$, $E\Delta x(k) = E\Delta x(k,j) = 0,\ k \geq 1,\ 1 \leq j \leq n$. As in the case of the independent risks the phase transition phenomenon is recognized. This phenomenon is showed by the following formula:

$$\Phi = \begin{cases} 0,\ 0 < \gamma < \gamma^* \text{ and } \delta > \gamma, \\ 1,\ \gamma > \gamma^* \text{ or } 0 < \delta < \gamma. \end{cases} \tag{18}$$

Theorem 6. *Suppose that d.f. $U(t),\ U(-1/2) = 0,$ has bounded density, $D\Delta x(k) = D\Delta x(k,j) = \sigma^2$, $0 < \sigma^2 < \infty$, $k \geq 1$, $1 \leq j \leq n$. If $0 < \gamma < 1/2$ and $\delta > \gamma$ then*

$$\lim_{n\to\infty} p_n = 0. \tag{19}$$

If $\gamma > 1/2$ or $0 < \delta < \gamma$ then

$$\lim_{n\to\infty} p_n = 1. \tag{20}$$

Theorem 7. *Suppose that d.f. $U(t),\ U(-1/2) = 0,$ has bounded density and for some $1 < \alpha < 2,\ C > 0$*

$$1 - U(t) \sim \frac{C(2-\alpha)}{\alpha} t^{-\alpha},\ t \to \infty. \tag{21}$$

If $0 < \gamma < 1 - 1/\alpha$ and $\delta > \gamma$ then (19) is true.
If $\gamma > 1 - 1/\alpha$ or $0 < \delta < \gamma$ then (20) is true.

3 Proofs of main results

Theorems 1, 3 proofs. Suppose that $X_k,\ k = 1, 2, \ldots,$ is the sequence of independent and identically distributed r.v.'s (the sequence of i.i.d.r.v.'s), $MX_k = 0$, $p(X_k < x) = F(x)$, and for some $1 < \alpha < 2$ there is $C > 0$ and $p,\ q \geq 0,\ p+q = 1$, so that for $x \to +\infty$ the following formulas are true:

$$1 - F(x) + F(-x) \sim \frac{C(2-\alpha)}{\alpha} x^{-\alpha}, \tag{22}$$

$$\frac{1-F(x)}{1-F(x)+F(-x)} \to p, \frac{F(-x)}{1-F(x)+F(-x)} \to q. \tag{23}$$

Then according to [22, chapter 8, §9, theorem 15, remark 13], [23, chapter XVII, §5, theorem 3] for any u, $-\infty < u < \infty$:

$$\lim_{n\to\infty} p \left(\frac{\sum_{i=1}^{n} X_i}{n^{1/\alpha}} \geq u \right) = 1 - P(u; \alpha, C, p, q). \tag{24}$$

D.f. $P(u; \alpha, C, p, q)$ is stable and has the characteristic function $\varphi(t) = e^{\psi(t)}$ where

$$\psi(t) = \frac{|t|^\alpha C\ \Gamma(3-\alpha)}{\alpha(\alpha-1)} \left[\cos \frac{\pi\alpha}{2} \pm \right. \tag{25}$$

$$\left. \pm i(p-q) \sin \frac{\pi\alpha}{2} \right]$$

and for $p > 0$ there exists $C'(\alpha, C, p, q) > 0$ so that for $u \to +\infty$

$$1 - P(u; \alpha, C, p, q) \sim C'(\alpha, C, p, q) u^{-\alpha}. \tag{26}$$

In the formula (25) for $t > 0$ the upper sign is "+" and for $t < 0$ the low sign is "−". More detailed information about the function $P(u; \alpha, C, p, q)$ is in [24, chapter. 2, §7, the figure 4]. If the conditions (22), (23) are true then the formulas (24), (25), (26) lead to

$$\lim_{n\to\infty} P\left(\sum_{i=1}^{n} X_i \geq 0 \right) = 1 - P(0; \alpha, C, p, q) > 0. \tag{27}$$

Lemma 1. *Suppose that $F(x) = G(x+1)$ and for some $1 < \alpha < 2$ d.f. G satisfies the theorem 1 conditions then*

$$\lim_{n\to\infty} P\left(\sum_{i=1}^{n} X_i \geq 0 \right) = 1 - P(0; \alpha, C, p, 0) > 0. \tag{28}$$

Proof. The equality $F(x) = G(x+1)$ and the condition (1) lead to the formulas (22), (23) for $p = 1$, $q = 0$. Then the formula (26) is true and so the formulas (27), (28) are true. The lemma is proved.

Denote

$$S_k = X_1 + \ldots + X_k,\ M_k = \max(S_1, \ldots, S_k),$$

$$B = \{S_{2n} > b\},\ A = \{M_n > b\},$$

$$A_k = \{S_i \leq b,\ 1 \leq i \leq k-1,\ S_k > b\},$$

$$a(\alpha) = 1 - P(0; \alpha, C, 1, 0) > 0.$$

Using the formula (28) choose $N(\alpha) > 0$ so that for $n \geq N(\alpha)$

$$P(X_1 + \ldots + X_n \geq 0) \geq \frac{a(\alpha)}{2}. \tag{29}$$

Lemma 2. *If the lemma 1 conditions are true for then*

$$P(M_n > b) \leq \frac{2}{a(\alpha)} P(S_{2n} > b),\ n \geq N(\alpha). \tag{30}$$

Proof. Using the construction of the monograph [25] (see the proofs of the lemmas 1, §4, chapter 4) obtain for $k = 1, \ldots, n$

$$P(B \cap A_k) \geq P((S_{2n} \geq S_k) \cap A_k) =$$

$$= P(A_k) P(X_{k+1} + \ldots + X_{2n} \geq 0) =$$

$$= P(A_k) P(S_{2n-k} \geq 0). \tag{31}$$

As the condition (29) is true then for $k=1,\dots,n$

$$P(S_{2n-k}\geq 0)\geq \frac{a(\alpha)}{2}. \qquad (32)$$

The events $A_k,\ k=1,\dots,n$, are mutually nonintersecting and so from (31), (32)

$$P(B)\geq \sum_{k=1}^{n} P(B\cap A_k)\geq$$

$$\geq \frac{a(\alpha)}{2}\sum_{k=1}^{n} P(A_k)=\frac{a(\alpha)}{2}P(A). \qquad (33)$$

Put the events A, B into the formula (33) which is true for $n\geq N(\alpha)$ and obtain (30). The lemma is proved.

Estimate now the probability

$$\mathcal{A}(n)=p\left(\sup_{m\geq 1}(S_{mn}-mnb)>0\right),$$

denoting

$$C_k=\left\{\max\left(\frac{S_j}{j},\ \ n2^{k-1}\leq j<n2^k\right)>b\right\}.$$

Lemma 3. *If the lemma 1 conditions and the formulas (29) for $n\geq N(\alpha)$ are true then*

$$\mathcal{A}(n)\leq \sum_{k=1}^{\infty}\frac{2}{a(\alpha)}P(S_{n2^{k+1}}>nb2^{k-1}). \qquad (34)$$

Proof. It is clear that

$$\mathcal{A}(n)=P\left(\sup_{m\geq 1}\left(\frac{S_{mn}}{m}-nb\right)>0\right)=$$

$$=P\left(\sup_{m\geq 1}\frac{S_{mn}}{m}>nb\right). \qquad (35)$$

Using the formula (35) and the construction of the monograph [26, chapter 8, §4, theorem 5] obtain

$$\mathcal{A}(n)\leq P\left(\sup_{m\geq 1}\frac{S_{mn}}{m}>nb\right)\leq \sum_{k=1}^{\infty}P(C_k)\leq$$

$$\leq \sum_{k=1}^{\infty}P(M_{n2^k}>nb2^{k-1}). \qquad (36)$$

Using the inequality (30) from the formula (36) obtain the formula (34). The lemma is proved.

Denote

$$\overline{F}(t)=1-F(t),\quad \overline{F}_n(t)=p(S_n\geq t),$$

$$\mu_1(y)=\int_{-y}^{y} sdF(s),\ \gamma_t(y)=\int_{-y}^{y}|s|^t dF(s),$$

$$C_t=\int_{-\infty}^{\infty}|s|^t dF(s).$$

Lemma 4. *If the conditions of the lemma 1 are true for any $y,\ y>0,\ t,\ 1<t<\alpha$, then*

$$C_t<\infty,\ \ |\mu_1(y)|\leq \frac{C_t}{y^{t-1}},\ \overline{F}(y)\leq \frac{C_t}{y^t}. \qquad (37)$$

Proof. As the lemma 1 conditions are true then $C_t<\infty$ for $1<t<\alpha$. It is clear that

$$\mu_1(y)=-\int_{|s|\geq y} sdF(s),\ y>0,$$

and consequently for $y>0$

$$|\mu_1(y)|\leq \int_{|s|\geq y}|s|dF(s)=\int_{|s|\geq y}\frac{|s|^t}{|s|^{t-1}}dF(s)\leq$$

$$\leq \int_{|s|\geq y}\frac{|s|^t}{y^{t-1}}dF(s)\leq \frac{\int_{-\infty}^{\infty}|s|^t dF(s)}{y^{t-1}}=\frac{C_t}{y^{t-1}}. \qquad (38)$$

Analogously the inequality $\overline{F}(y)\leq C_t/y^t$ is proved. The lemma is proved.

Lemma 5. *If the conditions of the lemma 1 are true then for any $\tau,\ c,\ 1<\tau<\alpha<2,\ c>0$, there exist $N(\tau,c),\ Q(\tau,c)$ so that for all $n>N(\tau,c)$ the inequality*

$$\overline{F}_n(x)\leq \frac{nQ(\tau,c)}{x^\tau},\ x\geq cn^{1/\tau}\ln^2 n \qquad (39)$$

is true.

Proof. Fix $c,\ c>0$, and $\tau,\ 1<\tau<\alpha$. Choose t, satisfying the inequality $1<\tau<t<\alpha$, and use the theorem 2 from [27]. Then in conditions of the lemma 5 obtain

$$\overline{F}_n(x)\leq n\overline{F}(y)+\exp(Z),\ x>0,\ y>0,\ n=1,2,...,$$

$$Z=\frac{x}{y}-\left(\frac{x-n\mu_1(y)}{y}+\frac{n\gamma_t(y)}{y^t}\right)\ln\left(\frac{xy^{t-1}}{n\gamma_t(y)}+1\right). \qquad (40)$$

Denote $R_n(x)=nC_t\ln^t x/x^t$. The function $R_n(x)$ and the function $R_n(x)/\ln x$ monotonically decrease for $x>e$. Analogously to [28] define $y=x/\ln x$. Then according to the inequality (37) the formula (40) leads to

$$\overline{F}_n(x)\leq R_n(x)+e^{\ln x-(\ln x-R_n(x))\ln(1+\ln x/R_n(x))}\leq$$

$$\leq R_n(x)+e^{\ln x\left(1-\left(1-\frac{R_n(x)}{\ln x}\right)\ln\left(1+\frac{\ln x}{R_n(x)}\right)\right)}. \qquad (41)$$

Choose $N_0 > N(\alpha)$ from the condition: $cN_0^{1/t}\ln^2 N_0 > e$, where e is the base of the natural logarithm. As the function $R_n(x),\ x > e$, monotonically decreases by x for $n \geq N_0$ then

$$\sup\Big(R_n(x),\ x \geq cn^{1/t}\ln^2 n\Big) = R_n(cn^{1/t}\ln^2 n) =$$

$$= \frac{C_t(\ln c + \ln n^{1/t} + 2\ln\ln n)^t}{c^t \ln^{2t} n}. \qquad (42)$$

It is possible to choose $Q_1,\ Q_1 > 0$, and $N_1,\ N_1 > N_0$, so that for $n \geq N_1$

$$R_n(x) \leq \frac{Q_1}{\ln^t n},\ x \geq cn^{1/t}\ln^2 n. \qquad (43)$$

According to (43)

$$\inf\left(\frac{\ln x}{R_n(x)}\ , x \geq cn^{1/t}\ln^2 n\right) = \frac{\ln(cn^{1/t}\ln^2 n)}{R_n(cn^{1/t}\ln^2 n)} \geq$$

$$\geq \frac{\ln^t n\ \ln(cn^{1/t}\ln^2 n)}{Q_1}.$$

So it is possible to choose $N_2 > N_1,\ Q_2 > 0$ so that for $n \geq N_2$

$$\frac{\ln x}{R_n(x)} \geq Q_2 \ln^{t+1} n = e_n,\ x \geq cn^{1/t}\ln^2 n. \qquad (44)$$

Combine the formulas (41), (44) and find for $n \geq N_2$

$$\overline{F}_n(x) \leq R_n(x) + \exp\{[1-(1-e_n^{-1})\ln(1+e_n)]\ln x\} =$$

$$= \frac{nC_t \ln^t x}{x^t} + \exp\{[1-(1-e_n^{-1})\ln(1+e_n)]\ln x\},$$

$$x \geq cn^{1/t}\ln^2 n.$$

The inequality may be rewritten in the form

$$\overline{F}_n(x) \leq \frac{nC_t \ln^t x}{x^t} + x^{-b_n} \qquad (45)$$

where

$$b_n = -1+(1-e_n^{-1})\ln(1+e_n) \to \infty,\ n \to \infty. \qquad (46)$$

Using the formula (46) choose $N_3 > N_2$ so that for $n \geq N_3$

$$b_n > 2t. \qquad (47)$$

Then for $n \geq N_3,\ x \geq cn^{1/t}\ln^2 n$ it is possible to rewrite the inequality (45) with the help of the formula (47) as follows

$$\overline{F}_n(x) \leq \frac{nC_t \ln^t x}{x^t} + \frac{1}{x^{2t}} \leq \qquad (48)$$

$$\leq \frac{nC_t \ln^t x}{x^t}\left(1 + \frac{1}{C_t x^t}\right) \leq \frac{nC_t \ln^t x}{x^t}\left(1 + \frac{1}{C_t c^t n}\right).$$

Denote $Q_3 = C_t\left(1 + 1/C_t c^t\right)$ and obtain from the formula (48) for $n \geq N_3$ that

$$\overline{F}_n(x) \leq \frac{nQ_3 \ln^t x}{x^t},\ x \geq cn^{1/t}\ln^2 n. \qquad (49)$$

Choose Q_4 from the condition

$$\frac{ln^t x}{x^t} < \frac{Q_4}{x^\tau},\ x \geq e.$$

Put $N(\tau,\ c) = N_3,\ Q(\tau,\ c) = Q_3\ Q_4$. With the help of the inequality (49) it is possible to prove that for $n > N(\tau,\ c)$

$$\overline{F}_n(x) \leq \frac{nQ(\tau,\ c)}{x^\tau},\ x \geq cn^{1/t}\ln^2 n.$$

So for $n > N(\tau,\ c)$

$$\overline{F}_n(x) \leq \frac{nQ(\tau,\ c)}{x^\tau},\ x \geq cn^{1/\tau}\ln^2 n.$$

Lemma 6. *If $0 < \gamma < 1 - 1/\alpha$ then for each τ, $1/(1-\gamma) < \tau < \alpha$, it is possible to choose N'_τ, c_τ so that for $n \geq N'_\tau,\ k = 1, 2, ...$*

$$n^{1-\gamma}\ 2^{k-1} \geq c_\tau\ (n2^{k+1})^{1/\tau}\ \ln^2(n2^{k+1}). \qquad (50)$$

Proof. Choose $N'_\tau,\ \ c_\tau$ from the conditions

$$\frac{n^{1-\gamma-1/\tau}}{\ln^2 n} \geq 1,\ n \geq N'_\tau > e, \qquad (51)$$

$$c_\tau = \min\left\{\frac{2^{k(1-1/\tau)}2^{-2-1/\tau}}{1+(k+1)^2\ln^2 2}, k = 1, 2, ...\right\}. \qquad (52)$$

If the formula (51) and the lemma 6 conditions are true then there exists the finite number N'_τ. The formula (52) leads to the inequality $c_\tau > 0$. Estimate the right side of the formula (50) denoting it by J. For $n \geq N'_\tau,\ k = 1, 2, \ldots :$

$$J = c_\tau\ (n2^{k+1})^{1/\tau}\ln^2(n2^{k+1}) =$$

$$= c_\tau(n2^{k+1})^{1/\tau}\left(\ln n + \ln 2^{k+1}\right)^2 \leq$$

$$\leq c_\tau n^{1/\tau}2^{(k+1)/\tau}2\left(\ln^2 n + (k+1)^2\ln^2 2\right) =$$

$$= c_\tau n^{1/\tau}\ln^2 n\ 2^{1+(k+1)/\tau}\left(1 + \frac{(k+1)^2\ln^2 2}{\ln^2 n}\right).$$

As $n \geq N'_\tau > e$ then

$$J \leq c_\tau n^{1/\tau}\ln^2 n\ 2^{1+(k+1)/\tau}\left(1 + (k+1)^2\ln^2 2\right).$$

According to (51) obtain

$$J \le c_\tau n^{1-\gamma} 2^{k-1} 2^{1+(k+1)/\tau} 2^{1-k} \left(1+(k+1)^2 \ln^2 2\right) =$$

$$= n^{1-\gamma} 2^{k-1} c_\tau \left\{ 2^{k(1/\tau-1)} 2^{2+1/\tau} \left(1+(k+1)^2 \ln^2 2\right)\right\} \le$$

$$\le n^{1-\gamma} 2^{k-1}.$$

The last inequality is the corollary of the formula (52). The lemma is proved.

Lemma 7. *Suppose that the conditions of the lemma 1 are true. If* $0 < \gamma < 1 - 1/\alpha$ *then for each* τ, $1/(1-\gamma) < \tau < \alpha$, *for* $n \ge N_\tau$ *and* $N_\tau = \max(N'_\tau,\ N(\tau, c_\tau))$, $Q_\tau = Q(\tau,\ c_\tau)$ *obtain*

$$\mathcal{A}(n) \le \frac{8Q_\tau}{a(\alpha)(1-2^{1-\tau})n^{(1-\gamma)\tau-1}}. \qquad (53)$$

Proof. Fix τ satisfying the inequality $1/(1-\gamma) < \tau <$ $< \alpha$. As for all $n \ge N_\tau \ge N'_\tau$, $k = 1, 2, ...$, the lemma 6 leads to the formula (50). So the lemma 5 with $c = c_\tau$ and n replaced by $n2^{k+1}$ may be applied to the inequality $P\left(S_{n2^{k+1}} > nb2^{k-1}\right)$:

$$P\left(S_{n2^{k+1}} \ge nb2^{k-1}\right) = \overline{F}_{n2^{k+1}}(n^{1-\gamma}2^{k-1}) \le \qquad (54)$$

$$\le \frac{n2^{k+1}Q_\tau}{(n^{1-\gamma}2^{k-1})^\tau},\ n \ge N_\tau.$$

Put the inequality (54) into (34) and obtain for $n \ge N_\tau$:

$$\mathcal{A}(n) \le \sum_{k=1}^{\infty} \frac{2}{a(\alpha)} P(S_{n2^{k+1}} > nb2^{k-1}) \le$$

$$\le \sum_{k=1}^{\infty} \frac{2}{a(\alpha)} \frac{Q_\tau 2^{k(1-\tau)} 2^{1+\tau}}{n^{(1-\gamma)\tau-1}} <$$

$$< \frac{8Q_\tau}{a(\alpha)(1-2^{1-\tau})n^{(1-\gamma)\tau-1}}. \qquad (55)$$

The formula (53) is proved.

Now begin to prove the theorems 1, 3. For this aim choose

$$X_{n(k-1)+j} = x(k, j) - 1, \quad k \ge 1,\ j = 1, \ldots, n.$$

Then according to the theorems 1, 3 conditions obtain

$$p_n = \mathcal{A}(n),\ n = 1, 2, \ldots \qquad (56)$$

Theorem 1 proof. Suppose that $\gamma < 1 - 1/\alpha$. Choose arbitrary τ : $1/(1-\gamma) < \tau < \alpha$. Using the lemma 7 define Q_τ, N_τ so that for $n \ge N_\tau$ the inequality (53) is true. Put

$$C_1(\tau) = \frac{8Q_\tau}{a(\alpha)(1-2^{1-\tau})}.$$

Then from the formulas (53), (56) obtain the inequality (3).

Suppose now that $\gamma \ge 1 - 1/\alpha$ then from the equality (56) obtain

$$p_n \ge P(S_n > n^{1-\gamma}) = P\left(\frac{S_n}{n^{1/\alpha}} > n^{1-\gamma-1/\alpha}\right) \ge \qquad (57)$$

$$\ge P\left(\frac{S_n}{n^{1/\alpha}} \ge 1\right),\ n = 1, 2, ...$$

From the formulas (24), (27) find that

$$\lim_{n\to\infty} P\left(\frac{S_n}{n^{1/\alpha}} \ge 1\right) = 1 - P(1, \alpha, C, 1, 0) > 0. \qquad (58)$$

The formulas (57), (58) lead to

$$\liminf_{n\to\infty}\ p_n \ge 1 - P(1, \alpha, C, 1, 0) > 0$$

so the inequality (4) is true. The theorem 1 is proved.

Theorem 3 proof. According to the formula (35) obtain

$$\mathcal{A}(n) = P\left(\sup_{m\ge 1} \frac{S_{mn}}{m} > nb\right) \le$$

$$\le P\left(\sup_{m\ge 1} \left|\frac{S_{mn}}{m}\right| > nb\right). \qquad (59)$$

Denote

$$\overline{C}_k = \left\{\max\left(\left|\frac{S_j}{j}\right|, \quad n2^{k-1} \le j < n2^k\right) > b\right\},$$

$$\overline{M}_n = \max(|S_1|, \ldots, |S_n|) \qquad (60)$$

and from the inequality (59) analogously to the formula (36) obtain

$$\mathcal{A}(n) \le \sum_{k=1}^{\infty} P(\overline{M}_{n2^k} > nb2^{k-1}). \qquad (61)$$

Using the Kolmogorov's inequality obtain from (61):

$$\mathcal{A}(n) \le \sum_{k=1}^{\infty} \frac{n2^k\sigma^2}{(nb2^{k-1})^2}. \qquad (62)$$

If the condition $\gamma < 1/2$ is true then from the formula (62) obtain

$$\mathcal{A}(n) \le \frac{4\sigma^2}{n^{1-2\gamma}} \to 0, \quad n \to \infty. \qquad (63)$$

Put $C_2 = 4\sigma^2$ and obtain from the formulas (56), (63) that

$$p_n \leq \frac{C_2}{n^{1-2\gamma}}, \quad n = 1, 2, \dots \tag{64}$$

The formula (64) leads to the inequality (10).

Suppose that $\gamma \geq 1/2$ and analogously to the formula (57) obtain

$$p_n \geq P(S_n > n^{1-\gamma}) =$$

$$= P\left(\frac{S_n}{n^{1/2}} > n^{1-\gamma-1/2}\right) \geq P\left(\frac{S_n}{n^{1/2}} \geq 1\right). \tag{65}$$

According to the theorem 3 conditions and to the Lindeberg theorem corollary [29, chapter 8, §40] obtain the equality

$$\lim_{n\to\infty} P\left(\frac{S_n}{n^{1/2}} \geq 1\right) = \overline{\Phi}_{0,\sigma^2}(1) > 0 \tag{66}$$

where $\overline{\Phi}_{0,\sigma^2}(t)$ is the tail of the Gaussian distribution $\Phi_{0,\sigma^2}(t)$ with the mean 0 and the variance σ^2.

Then from (65) and (66) obtain the formula

$$\lim_{n\to\infty} inf\ p_n \geq \overline{\Phi}_{0,\sigma^2}(1) > 0$$

and so the inequality (4). The theorem 3 is proved.

Theorems 2, 4 proofs

Denote

$$y(k,j) = x(k,j) - 1,$$

$$z_n(k) = \frac{1}{n^{1/\alpha}} \sum_{j=1}^{n} y(k,j),\ k \geq 1,\ n \geq 1.$$

Lemma 8. *Suppose that for some $\alpha,\ 1 < \alpha < 2$, d.f. G satisfies the theorem 1 conditions. Then for $n \to \infty$ the weak convergence of d.f. $P(z_n(k){<}t)$ to d.f. $P(t;\alpha,C,1,0)$ (in all continuous points of $P(t;\alpha,C,1,0)$) is true, where $P(t;\alpha,C,1,0)$ is the stable distribution with the characteristic function (c.f.) $\varphi(t) = e^{\psi(t)}$ and $\psi(t)$ is defined in (25).*

Proof. This statement is the formulas (22) - (25) corollary for $F(x) = G(x+1),\ p = 1,\ q = 0$.

Remark 1. *According to the formula (25) d.f. $P(t;\alpha,C,1,0)$ has a density. The density of d.f. $P(t;\alpha,C,1,0)$ [24, theorem 2.7.5] is bounded $P(t;\alpha,C,1,0)$.*

Lemma 9. *Suppose that d.f. G satisfies the theorem 3 conditions. Then for $n \to \infty$ the weak convergence of d.f. $P(z_n(k) < t)$ to d.f. $\Phi_{0,\sigma^2}(t)$ is true.*

Proof. The lemma 9 statement is the direct Lindeberg theorem [29, chapter 8, §40] corollary.

Lemma 10. *The following equality is true*

$$p_n{=}P\left(\sup\left(\sum_{k=1}^{m}\left(z_n(k){-}\frac{n}{n^{\gamma+\frac{1}{\alpha}}}\right), m \in N\right){>}0\right).$$

Proof.

$$p_n{=}P\left(\sup\left(\sum_{k=1}^{m}\sum_{j=1}^{n}(x(k,j){-}1{-}b)\,, m \in N\right){>}0\right) =$$

$$= P\left(\sup\left(\sum_{k=1}^{m}\sum_{j=1}^{n}(y(k,j){-}n^{-\gamma})\,, m \in N\right){>}0\right) =$$

$$= P\left(\sup\left(\sum_{k=1}^{m}\left(z_n(k) - n^{1-\gamma-1/\alpha}\right), m \in N\right){>}0\right).$$

The lemma is proved.

Suppose that $z(1), z(2), \dots$ are i.i.d.r.v.'s with the following d.f.: if $\alpha,\ 1 < \alpha < 2$, then d.f.

$$P(z_1 < t) = G_\alpha(t) = P(t;\alpha,C,1,0)$$

if $\alpha = 2$ then d.f. $P(z_1 < t) = G_2(t) = \Phi_{0,\sigma^2}(t)$. Put

$$Z(M){=}\sup\left(\sum_{k=1}^{m} z(k), 1{\leq}m{\leq}M\right),\ Z(\infty){=}Z,$$

$$Z_n(M){=}\sup\left(\sum_{k=1}^{m} z_n(k), 1{\leq}m{\leq}M\right),\ Z_n(\infty){=}Z_n.$$

Lemma 11. *For any $\alpha,\ 1 < \alpha \leq 2$, the equality*

$$P\left(Z > 1\right) = 1 \tag{67}$$

is true.

Proof. Fix $\alpha,\ 1{<}\alpha{\leq}2$. By the definition $Mz(k) = 0$ then for any bounded interval Δ on the straight line

$$\limsup_{m\to\infty} P\left(\sum_{k=1}^{m} z(k) \in \Delta\right) =$$

$$= \limsup_{m\to\infty} P\left(\frac{1}{m^{1/\alpha}}\sum_{k=1}^{m} z(k) \in \frac{1}{m^{1/\alpha}}\Delta\right). \tag{68}$$

If Δ is fixed then there exists M so that for $m \geq M$

$$\frac{1}{m^{1/\alpha}}\Delta \subset (-1, 1)\,. \tag{69}$$

Then according to the formulas (68), (69)

$$\limsup_{m\to\infty} P\left(\sum_{k=1}^{m} z(k) \in \Delta\right) \leq$$

$$\leq \limsup_{m\to\infty} P\left(\frac{1}{m^{1/\alpha}}\sum_{k=1}^{m} z(k) \in (-1,1)\right). \quad (70)$$

As r.v.'s $z(k)$ for some $\alpha,\ 1<\alpha\leq 2$, have d.f. G_α then

$$P\left(\frac{1}{m^{1/\alpha}}\sum_{k=1}^{m} z(k) \in (-1,1)\right) \equiv P\left(z(1)\in(-1,1)\right).$$

So

$$\limsup_{m\to\infty} P\left(\frac{1}{m^{1/\alpha}}\sum_{k=1}^{m} z(k) \in (-1,1)\right) =$$

$$= P\left(z(1)\in(-1,1)\right) = 1-\epsilon_1, \quad \epsilon_1>0. \quad (71)$$

The formulas (70), (71) allow to define $\epsilon_1>0$, satisfying for any bounded interval Δ the inequality

$$\limsup_{m\to\infty} P\left(\sum_{k=1}^{m} z(k)\in\Delta\right) \leq 1-\epsilon_1.$$

The conditions of [19, chapter 1, §3, theorem 7] are true and so the equality (67) takes place. The lemma is proved.

Lemma 12. *For any $\alpha,\ 1<\alpha\leq 2,\ \epsilon,\ \epsilon>0$, there exists $M'(\alpha,\epsilon)$ so that for all $M\geq M'(\alpha,\epsilon)$*

$$P\left(Z(M)>1\right) > 1-\epsilon.$$

Proof. Using the lemma 11 statement (see the formula (67)) find

$$1 = P\left(Z = Z(\infty) > 1\right) = \lim_{M\to\infty} P\left(Z(M)>1\right).$$

The lemma is proved.

Introduce the Markov chains $w_n(m),\ \ w(m)$, $m\geq 1$, by the formulas $w_n(0)=0,\ w(0)=0$,

$$w_n(m+1) = \max(w_n(m)+z_n(m+1),0), \quad (72)$$

$$w(m+1) = \max(w(m)+z(m+1),0). \quad (73)$$

From [19, chapter 1, §3, theorem 2] obtain the coincidence of r.v.'s $Z_n(m),\ w_n(m),\ m=1,2,...$, by the distribution. Analogously r.v.'s $Z(m),\ w(m),\ m=1,2,...$, coincide by the distribution too.

Lemma 13. *For any $\alpha,\ 1<\alpha\leq 2$, and for any $m>0$ if $n\to\infty$ then there is the weak convergence of r.v.'s $w_n(m)$ distributions to r.v. $w(m)$ distribution (and so for r.v.'s $Z_n(m)$ distributions to r.v. $Z(m)$ distribution).*

Proof. From the lemmas 8, 9 obtain that if $n\to\infty$ so r.v.'s $z_n(k)$ distributions converge weakly to r.v. $z(k)$ distribution for $\alpha,\ 1<\alpha\leq 2$. As the result r.v.'s $w_n(1)$ distributions converge weakly to r.v. $w(1)$ distribution for $n\to\infty$.

Suppose that r.v.'s $w_n(m)$ distributions converge weakly to r.v. $w(m)$ distribution for $n\to\infty$. Then from [30, theorem of §7] obtain that c.f.'s of r.v.'s $w_n(m)$ converge uniformly to c.f. of r.v. $w(m)$ on each finite interval for $n\to\infty$.

As for $n\to\infty$ r.v.'s $z_n(k)$ distributions converge weakly to r.v. $z(k)$ distribution so c.f. of r.v.'s $z_n(m+1)$ converge to c.f. of r.v. $z(m+1)$ for $n\to\infty$ uniformly on at any finite interval. Consequently for $n\to\infty$ c.f.'s of r.v.'s $w_n(m)+z_n(m+1)$ converge to c.f. of r.v. $w(m)+z(m+1)$ uniformly on any finite interval. So if $n\to\infty$ then there is weak convergence of r.v.'s $w_n(m)+z_n(m+1)$ distributions to r.v. $w(m)+z(m+1)$ distribution. As the result there is the weak convergence of r.v.'s $w_n(m+1)$ distributions to r.v. $w(m+1)$ distribution for $n\to\infty$. The induction statement is proved.

Lemma 14. *For any $\alpha,\ 1<\alpha\leq 2$, and any $m>0$ there exist the nonnegative numbers $p(m),\ q(m)$: $p(m)+q(m)=1$ and d.f. $F_m(t)$ with bounded density $f_m(t)$, $-\infty<t<\infty$, $f_m(t)=0,\ t\leq 0$, so that for $-\infty<t<\infty$*

$$P\left(w(m)<t\right) = p(m)\theta(t)+q(m)F_m(t). \quad (74)$$

Here $\theta(t)=0,\ t\leq 0,\ \theta(t)=1,\ t>0$.

Proof. Denote

$$g_\alpha(t) = \frac{d}{dt}P\left(z(1)<t\right) = \frac{d}{dt}G_\alpha(t).$$

As the lemmas 8 and 9 are true so for $1<\alpha\leq 2$ the function $g_\alpha(t)$ is unimodal (that is this function possesses single local and so global extremum – maximum). Consequently $g_\alpha(t)$ has a finite upper bound on $(-\infty,\ \infty)$. Choose

$$p(1) = \int_{-\infty}^{0} g_\alpha(\tau)d\tau, q(1)=1-p(1),$$

$$f_1(t)=0,\ t\leq 0,\ f_1(t)=\frac{g_\alpha(t)}{q(1)},\ t>0, \quad (75)$$

and denote by $F_1(t)$ d.f. with the density $f_1(t)$. Then

$$P\left(w(1)<t\right) = P\left(\max(0,z(1))<t\right) =$$

$$= \theta(t)G_\alpha(t) = p(1)\theta(t)+q(1)F_1(t) \quad (76)$$

and the density $f_1(t)$ is bounded.

Suppose that the representation (74) is true for $m=k$ and for d.f. $F_k(t)$ with bounded density $f_k(t)$.

Prove this representation for $m = k + 1$. Denote by "$*$" the operation of d.f. conjuncture. From the equality (74), which is true for $m = k$, obtain

$$P\left(w(k) + z(k+1) < t\right) =$$

$$= \left(p(k)\theta(t) + q(k)F_k(t)\right) * G_\alpha(t) =$$

$$= p(k)G_\alpha(t) + q(k)F_k(t) * G_\alpha(t) =$$

$$= p(k)G_\alpha(t) + q(k)R_k(t), \tag{77}$$

$$R_k(t) = F_k(t) * G_\alpha(t),$$

where d.f. $R_k(t)$ has bounded density

$$r_k(t) = \int_{-\infty}^{\infty} f_k(t-\tau)g_\alpha(\tau)d\tau.$$

It is clear that the density $\psi_k(t)$ of the distribution

$$P\left(w(k) + z(k+1) < t\right) = \Psi_k(t)$$

is the bounded function and

$$\psi_k(t) = p(k)g_\alpha(t) + q(k)r_k(t).$$

Analogously to (75) choose

$$p(k+1) = \int_{-\infty}^{0} \psi_k(\tau)d\tau,\ q(k+1) = 1 - p(k+1),$$

$$f_{k+1}(t)=0,\ t\le 0,\ f_{k+1}(t)=\frac{\psi_k(t)}{q(k+1)},\ t>0, \tag{78}$$

and denote by $F_{k+1}(t)$ d.f. with the bounded density $f_{k+1}(t)$. Then

$$P\left(w(k+1) < t\right) = \theta(t)\Psi_k(t) =$$

$$= p_{k+1}\theta(t) + q_{k+1}F_{k+1}(t). \tag{79}$$

Consequently for $m = k + 1$ the representation (74) is true too. The theorem is proved.

Lemma 15. *For any* $\alpha,\ 1 < \alpha \le 2$, *and for any* $m > 0$

$$\lim_{n\to\infty} P\left(Z_n(m) > 1\right) = P\left(Z(m) > 1\right).$$

Proof. The equalities (74) and the lemma 13 it follows that in each continuity point $t = T$ of d.f. $P\left(w(m) < t\right)$

$$\lim_{n\to\infty} P\left(Z_n(m) > T\right) = P\left(Z(m) > T\right).$$

As the lemma 14 is true so the point $T = 1$ is continuity point of d.f. $P\left(w(m) < t\right)$.

Lemma 16. *For any* $\alpha,\ 1 < \alpha \le 2$, *and for any* $\gamma > 1 - 1/\alpha$

$$\lim_{n\to\infty} p_n = 1.$$

Proof. Suppose that $1 < \alpha \le 2,\ \epsilon > 0$. Define by the lemma 12 $M' = M'(\alpha, \epsilon)$ so that

$$P\left(Z(M') > 1\right) > 1 - \epsilon. \tag{80}$$

Using the lemma 15 for fixed $M' = M'(\alpha, \epsilon),\ \alpha,\ \epsilon$, find $N_1 = N_1(\alpha, \epsilon)$ so that for any $n \ge N_1$

$$\mid P\left(Z_n(M') > 1\right) - P\left(Z(M') > 1\right) \mid < \epsilon. \tag{81}$$

From the inequalities (80), (81) find that for $n \ge N_1$

$$P\left(Z_n(M') > 1\right) > 1 - 2\epsilon. \tag{82}$$

The lemma 10 leads to

$$1 \ge p_n \ge P\left(\sup\left(K(m), 1 \le m \le M'\right) > 0\right) \ge$$

$$\ge P\left(Z_n(M') > M' n^{1-\gamma-1/\alpha}\right),$$

$$K(m) = \sum_{k=1}^{m}\left(z_n(k) - \frac{n}{n^{\gamma+1/\alpha}}\right)$$

Choose $N_2 = N_2(\alpha, \epsilon)$ so that $M' N_2^{1-\gamma-1/\alpha} < 1$. Then for $n \ge \max(N_1(\alpha, \epsilon),\ N_2(\alpha, \epsilon))$ from the formula (82)obtain

$$1 \ge p_n \ge P\left(Z_n(M') > 1\right) > 1 - 2\epsilon.$$

The lemma is proved.

The lemma 16 leads to the theorems 2, 4 statements.

Theorem 5 proof

From the definition of p_n obtain

$$P\left(\sum_{j=1}^{n}(x(1,j) - 1) > nb\right) \le p_n \le$$

$$\le \sum_{m=1}^{\infty} P\left(\sum_{k=1}^{m}\sum_{j=1}^{n}(x(k,j) - 1) > mnb\right). \tag{83}$$

Denote $y_j = x(1,j) - 1,\ j \ge 1$ and put $S_n = \sum\limits_{j=1}^{n} y_j$, $n = 1, 2, ...$ Rewrite the inequality (83) as follows

$$P(S_n > nb) \le p_n \le \sum_{m=1}^{\infty} P(S_{mn} > mnb). \tag{84}$$

Prove the theorem 5 using the inequality (84) in two steps.

The step 1. To estimate the probability $p(S_k>kb)$ in the conditions (12) use the Cramer theorem [31] with the remained member in the Petrov form [32].
Theorem *. *If $x \geq 0,\ x = o(\sqrt{k})$ and the condition (12) is true then*

$$p(S_k>\sigma x\sqrt{k})=$$
$$=\overline{\Phi}_{0,1}(x)e^{\frac{x^3}{\sqrt{k}}\lambda\left(\frac{x}{\sqrt{k}}\right)}\left[1+O\left(\frac{x+1}{\sqrt{k}}\right)\right], \qquad (85)$$

where
$$\lambda(t)=\sum_{j=0}^{\infty}a_j t^j.$$

Here the row $\lambda(t)$ with the coefficients calculated via

$$\gamma_k=\frac{1}{i^k}\left(\frac{d^k}{dt^k}\ln E\exp(ity_1)\right)_{t=0},$$

where i is the imaginary unit and the symbol $\ln$ *denotes main meaning of the logarithm so that*

$$\ln Ee^{ity_1}\mid_{t=0}=0.$$

The function $\overline{\Phi}_{0,1}(x)$ may be represented in the Feller form [33]:

$$\overline{\Phi}_{0,1}(x)=\frac{e^{-x^2/2}}{x\sqrt{2\pi}}\left(1-\frac{v(x)}{x^2}\right),\ x>1, \qquad (86)$$

where $v(x)$ is some function satisfying the inequality $0 \leq v(x) \leq 1,\ x>1.$

Suppose that the inequality $\gamma < 1/2$ is true. Introduce auxiliary designations

$$V_n=1-\frac{v(n^{-\gamma}\sqrt{n}\sigma^{-1})}{n^{1-2\gamma}\sigma^{-2}},$$

$$W_{n,m}=1+O\left(\frac{n^{-\gamma}\sqrt{nm}\sigma^{-1}+1}{\sqrt{nm}}\right),$$

$$U_{n,m}=\exp\left(\frac{-n^{1-2\gamma}m}{2\sigma^2}+\frac{n^{1-3\gamma}m}{\sigma^3}\lambda\left(\frac{n^{-\gamma}}{\sigma}\right)\right)=$$
$$=\exp\left(\frac{-n^{1-2\gamma}m}{2\sigma^2}\left(1-\frac{2n^{-\gamma}}{\sigma}\lambda\left(\frac{n^{-\gamma}}{\sigma}\right)\right)\right).$$

Then the inequality (84) may be rewritten with the help of the formulas (85), (86) as follows

$$\frac{\sigma}{n^{-\gamma}\sqrt{2\pi n}}V_nW_{n,1}U_{n,1} \leq p_n \leq \qquad (87)$$
$$\leq \sum_{m=1}^{\infty}\frac{\sigma}{n^{-\gamma}\sqrt{2\pi nm}}W_{n,m}U_{n,m}.$$

Choose $c>0$ so that

$$\begin{array}{l} W_{n,m} \leq (1+cn^{-\gamma}),\ m=1,2,\ldots, \\ W_{n,1} \geq (1-cn^{-\gamma}),\ n=1,2,\ldots \end{array} \qquad (88)$$

From the inequalities (87), (88) obtain

$$\frac{\sigma}{n^{-\gamma}\sqrt{2\pi n}}V_n(1-cn^{-\gamma})U_{n,1} \leq p_n \leq \qquad (89)$$
$$\leq \frac{\sigma(1+cn^{-\gamma})U_{n,1}}{(1-U_{n,1})n^{-\gamma}\sqrt{2\pi n}},\ n \geq 1.$$

Then
$$\ln p_n \sim -\frac{n^{1-2\gamma}}{2\sigma^2},\ n\to\infty,$$

that is
$$\lim_{n\to\infty} p_n=0. \qquad (90)$$

Suppose that $\gamma \geq 1/2$ then from the inequality (84) obtain

$$p_n \geq P(S_n \geq n^{1-\gamma}) \geq P(S_n \geq \sqrt{n}). \qquad (91)$$

Using the central limit theorem [29, the Lindeberg theorem corollary] obtain from (91)

$$\liminf_{n\to\infty}\ p_n \geq \overline{\Phi}_{0,1}(1)>0.$$

So we have proved that the equality (90) is true if and only if the inequality $\gamma < 1/2$ takes place.

Step 2. To estimate the probability $p(S_k>kb)$ in the conditions (14) use the Nagaev inequality [34] in the following theorem ** form.
Theorem **. *If the conditions (14) are true then there exist the positive and finite numbers $n_\mu,\ g_\mu$ so that for $x \geq n_\mu\sqrt{k}$*

$$p(S_k>x) \leq \frac{kg_\mu}{x^\mu},\ k=1,2,\ldots \qquad (92)$$

It follows from the theorem ** that for

$$n \geq N_\mu=n_\mu^\beta,\ \beta=\frac{2}{1-2\gamma}$$

the following inequalities for $m=1,2,...$ are true

$$p(S_{mn}>mnb) \leq \frac{mng_\mu}{(mnb)^\mu}=\frac{g_\mu}{b^\mu(mn)^{\mu-1}}. \qquad (93)$$

Using the inequalities (84), (93) obtain for $n \geq N_\mu$

$$p_n \leq \frac{q_\mu}{b^\mu n^{\mu-1}}=\frac{q_\mu}{n^{\mu-1-\mu\gamma}},\ n\geq 1,$$

$$q_\mu=g_\mu\sum_{m=1}^{\infty}m^{1-\mu}<\infty.$$

So in this case the equality (90) is true if and only if $\gamma<1/2$. The theorem 5 is proved.

Remark 2. *The analyzed risk model may be improved by a consideration of finite horizon ruin probabilities. On the one hand it allows to investigate both a possibility and a necessity of current time coalitions caused by some short time factors. From an another side this suggestion allows to simplify the model analysis.*

Theorem 6 proof

The following formula is true

$$p_n=P\left(\sup_{m>0}\left(\sum_{k=1}^m\sum_{j=1}^n(x(k,j)-1-n^{-\gamma})\right)>0\right)=$$

$$=P\left(\sup_{m>0}\frac{\sum_{k=1}^m\frac{\Delta x(k)}{n^{\delta-1}}+\sum_{k=1}^m\sum_{j=1}^n\Delta x(k,j)}{m}>n^{1-\gamma}\right)\leq$$

$$\leq P\left(\sup_{m>0}\frac{n^{1-\delta}}{m}\sum_{k=1}^m\Delta x(k)+\right.$$

$$\left.+\sup_{m>0}\frac{1}{m}\sum_{k=1}^m\sum_{j=1}^n\Delta x(k,j)>n^{1-\gamma}\right). \qquad (94)$$

Then from (94) obtain the inequality

$$p_n\leq P_1(n)+P_2(n), \qquad (95)$$

in which

$$P_1(n)=P\left(\sup_{m>0}\frac{n^{1-\delta}}{m}\sum_{k=1}^m\Delta x(k)>\frac{n^{1-\gamma}}{2}\right),$$

$$P_2(n)=P\left(\sup_{m>0}\frac{1}{m}\sum_{k=1}^m\sum_{j=1}^n\Delta x(k,j)>\frac{n^{1-\gamma}}{2}\right).$$

Denote

$$M_t=\max(|\Delta x(1)|,|\Delta x(1)+\Delta x(2)|,\ldots,$$

$$|\Delta x(1)+\Delta x(2)+\ldots+\Delta x(t)|). \qquad (96)$$

Suppose now that $0<\gamma<1/2$ and $\delta>\gamma$. Then, using the algorithm of the theorem 1 proof obtain

$$P_1(n)=P\left(\sup_{m>0}\frac{1}{m}\sum_{k=1}^m\Delta x(k)>\frac{n^{\delta-\gamma}}{2}\right)\leq$$

$$\leq\sum_{k=1}^\infty P\left(\max_{j:\,2^{k-1}\leq j<2^k}\frac{|\sum_{i=1}^j\Delta x(i)|}{j}>\frac{n^{\delta-\gamma}}{2}\right)\leq$$

$$\leq\sum_{k=1}^\infty P\left(M_{2^k}>n^{\delta-\gamma}2^{k-2}\right)\leq$$

$$\leq\sum_{k=1}^\infty\frac{2^kD\Delta x(k)}{(n^{\delta-\gamma}2^{k-2})^2}=\frac{16\sigma^2}{n^{2\delta-2\gamma}}, \qquad (97)$$

$$P_2(n)=P\left(\sup_{m>0}\frac{\sum_{k=1}^m\sum_{j=1}^n\Delta x(k,j)}{mn}>\frac{n^{-\gamma}}{2}\right)\leq$$

$$\leq\frac{16\sigma^2}{n^{1-2\gamma}}. \qquad (98)$$

The formulas (95), (97), (98) lead to the equality (19).

Prove now the equality (20). For this aim divide the set $G=\{(\gamma,\delta):0<\delta<\gamma\ or\ \gamma>1/2\}$ into the nonintersecting subsets

$$G_1=\{(\gamma,\delta):\delta\geq 1/2,\ \gamma>1/2\},$$

$$G_2=\{(\gamma,\delta):0<\delta<\gamma,\ \delta<1/2\},\ G_1\cup G_2=G.$$

Consider the case when $(\gamma,\delta)\in G_1$. From (94) obtain that $\forall M,\ M\in N=\{1,2,...\}$,

$$p_n=P\left(\sup_{m>0}\frac{\sum_{k=1}^m\frac{\Delta x(k)}{n^{\delta-1}}+\sum_{k=1}^m\sum_{j=1}^n\Delta x(k,j)}{\sqrt{n}m}>n^{\frac{1}{2}-\gamma}\right)\geq$$

$$\geq P\left(\max_{0<m\leq M}\sum_{k=1}^m\left(\frac{\Delta x(k)}{n^{\delta-\frac{1}{2}}}+\sum_{j=1}^n\frac{\Delta x(k,j)}{\sqrt{n}}\right)>\frac{M}{n^{\gamma-\frac{1}{2}}}\right).$$

As

$$\forall\epsilon>0\ \exists N_1:\forall n\geq N_1\ n^{1/2-\gamma}M<1$$

so $\forall n\geq N_1\quad p_n\geq$

$$\geq P\left(\max_{0<m\leq M}\sum_{k=1}^m\left(\frac{\Delta x(k)}{n^{\delta-\frac{1}{2}}}+\sum_{j=1}^n\frac{\Delta x(k,j)}{\sqrt{n}}\right)>1\right) \qquad (99)$$

Suppose that $\lambda(s),\ s\geq 1$, is s.i.i.d.r.v.'s with the common gaussian d.f. $\Phi_{0,\sigma^2}(t)$, which has the mean 0 and the variance $\sigma^2(t)$. Denote

$$\lambda_{n,b}(s)=\sum_{j=1}^n\frac{\Delta x(s,j)}{n^{1/b}},\ u_{n,b}(s)=n^{1-1/b-\delta}\Delta x(s),$$

$$z_{n,b}(s)=u_{n,b}(s)+\lambda_{n,b}(s),\ z_b(s)=u_{n,b}(s)+\lambda(s),$$

$$Z_{n,b}(s)=\max\left(0,\max_{0<m\leq s}\sum_{k=1}^m z_{n,b}(k)\right),$$

$$Z_b(s)=\max\left(0,\max_{0<m\leq s}\sum_{k=1}^m z_b(k)\right),$$

$$w_{n,b}(s) = \max\left(0, w_{n,b}(s-1) + z_{n,b}(s)\right),$$
$$w_b(s) = \max\left(0, w_b(s-1) + z_b(s)\right),\ s \geq 1,$$
$$w_{n,b}(0) = w_b(0) = 0. \quad (100)$$

In our case $b = 2$ so from the lemma 9 obtain

$$F_{\lambda_{n,2(s)}}(t) = P(\lambda_{n,2}(s) < t) \Rightarrow \Phi_{0,\sigma^2}(t),\ n \to \infty, \quad (101)$$

where " $\Rightarrow$ " means the weak convergence of d.f. As the formulas

$$P(u_{n,2}(s) < t) = U(t) \text{ if } \delta = 1/2,$$
$$P(u_{n,2}(s) < t) \Rightarrow I(t)\ n \to \infty, \text{ if } \delta > 1/2$$

with

$$I(t) = \begin{cases} 1, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

are true so from the continuity theorem [26, chapter 7, § 3] obtain for $n \to \infty$

$$P(z_{n,2}(s) < t) \Rightarrow \begin{cases} (U * \Phi_{0,\sigma^2})(t),\ \delta = \frac{1}{2}, \\ F_{\lambda_{n,2(s)}}(t),\ \delta > 1/2. \end{cases} \quad (102)$$

Then according to [23, tom 2, chapter 6, § 9] the following equalities

$$P(w_{n,2}(s) < t) = P(Z_{n,2}(s) < t),$$
$$P(w_2(s) < t) = P(Z_2(s) < t),\ s \geq 1, \quad (103)$$

are true and from the formulas (101), (102)

$$P(z_{n,2} < t) \Rightarrow \Phi_{0,\sigma^2}.$$

So the lemma 13 leads to

$$P(Z_{n,2}(s) < t) \Rightarrow P(Z_2(s) < t),\ \ n \to \infty. \quad (104)$$

The condition, that d.f. $U(t)$ density is bounded and so d.f. $(U * \Phi_{0,\sigma^2})(t)$ is bounded too, is necessary to obtain from the lemma 14 the following corollary. D.f. $F_{w_2(s)}(t) = P(w_2(s) < t)$ is continuous at the point $t = 1$. So from the formula (104) and from the lemma 15 obtain that for $\forall \epsilon > 0\ \exists N_2 \in N :\ \forall n \geq N_2$ the inequality

$$P(Z_{n,2}(s) > 1) > P(Z_2(s) > 1) - \epsilon$$

is true. Then for $\forall n \geq \max(N_1, N_2)$ obtain

$$p_n \geq P\left(\max_{0<m\leq M} \sum_{k=1}^{m} z_{n,2}(k) > 1\right) =$$
$$= P(Z_{n,2}(M) > 1) > P(Z_2(M) > 1) - \epsilon. \quad (105)$$

The lemma 12 leads to

$$P(Z_2(s) > 1) \to 1,\ \ M \to \infty.$$

Consequently $\exists M^* \in N :\ \forall M \geq M^*$ so that

$$p_n > 1 - 2\epsilon,$$

and then the equality (20) is true too.

Consider now the case $(\gamma, \delta) \in G_2$. In this case analogously to the formula (99) obtain that for $\forall \epsilon > 0$ $\exists N :\ \forall n \geq N$

$$p_n \geq P\left(\max_{0<m\leq M} \sum_{k=1}^{m} (\Delta x(k) + n^{\delta-1/2}\lambda_{n,2}(k)) > 1\right).$$

It is clear that

$$P(n^{\delta-1/2}\lambda_{n,2}(s) < t) \Rightarrow I(t),\ n \to \infty,$$

and consequently

$$P(\Delta x(k) + n^{\delta-1/2}\lambda_{n,2}(s) < t) \Rightarrow U(t),\ n \to \infty.$$

Then introducing Markov chains

$$w'_{n,2}(s) = \max\left(0, w'_{n,2}(s-1) + \Delta x(s) + \frac{\lambda_{n,2}(s)}{n^{1/2-\delta}}\right),$$
$$w'_2(s) = \max\left(0, w'_2(s-1) + \Delta x(s)\right),\ s \geq 1,$$
$$w'_{n,2}(0) = w'_2(0) = 0$$

and repeating the word by the word the proof in previous case obtain the equality (20).

Theorem 7 proof

Consider the case $\gamma < 1 - 1/\alpha,\ \delta > \gamma$. From the theorem 1 obtain that for $\forall \tau,\ 1/(1-\gamma) < \tau < \alpha$,

$$\exists\ C_1(\tau) > 0\ :\ P_2(n) \leq C_1(\tau) n^{1-\tau(1-\gamma)} 2^{\tau}.$$

The multiplier 2^{τ} occurs because here we consider the probability of the inequality

$$\sup_{m>0} \frac{1}{mn} \sum_{k=1}^{m} \sum_{j=1}^{n} \Delta x(k,j) > n^{-\gamma}/2,$$

but not the probability of the inequality

$$\sup_{m>0} \frac{1}{mn} \sum_{k=1}^{m} \sum_{j=1}^{n} \Delta x(k,j) > n^{-\gamma},$$

as it was made in the proof of the theorem 1.

Consequently the equality

$$\lim_{n\to\infty} P_2(n) = 0$$

is true. As the equality (95) takes place so we are to show only that

$$\lim_{n\to\infty} P_1(n) = 0. \tag{106}$$

In the proof of previous theorem it was shown that in the case $0 < \gamma < 1/2$ and $\delta > \gamma$

$$P_1(n) \le \sum_{k=1}^{\infty} P(M_{2^k} > n^{\delta-\gamma}2^{k-2}) = A. \tag{107}$$

Choose

$$K^* > 0 : \; K^* \ge \max(\log_2 N(\tau,1) - 1, \log_2 N(\alpha))$$

where $N(\alpha),\; N(\alpha) > 0$, is so that $\forall t \ge N(\alpha)$ (see the formula (29))

$$P(\Delta x(1) + \Delta x(2) + \ldots + \Delta x(t) > 0) \ge a(\alpha)/2$$

where $a(\alpha){=}1{-}P(0;\alpha,C,1,0),\; C > 0, 1 < \alpha < 2$, and the constant $N(\tau,1)$ is defined by the lemma 5. Then from the formula (107) obtain

$$A = \sum_{k=1}^{K^*} P(M_{2^k} > n^{\delta-\gamma}2^{k-2}) +$$

$$+ \sum_{k=K^*+1}^{\infty} P(M_{2^k} > n^{\delta-\gamma}2^{k-2}). \tag{108}$$

As

$$\sum_{k=1}^{K^*} P(M_{2^k} > n^{\delta-\gamma}2^{k-2}) \le$$

$$\sum_{k=1}^{K^*} P(M_{2^k} > \frac{2^{k-2}}{n^{-\delta+\gamma}}) \le K^* P(M_{2^{K^*}} > \frac{n^{\delta-\gamma}}{2}), \tag{109}$$

so the formulas (108), (109) and the lemma 2 lead to

$$A \le \frac{2K^*}{a(\alpha)} P\left(\sum_{k=1}^{2^{K^*+1}} \Delta x(k) > n^{\delta-\gamma}/2\right) +$$

$$+ \frac{2}{a(\alpha)} \sum_{k=K^*+1}^{\infty} P\left(\sum_{j=1}^{2^{k+1}} \Delta x(j) > n^{\delta-\gamma}2^{k-2}\right) =$$

$$= A_1 + A_2. \tag{110}$$

From the lemma 5 obtain that for $\forall\tau,\; 1{<}\tau{<}\alpha{<}2$, $\exists N(\tau,1) \in N = \{1,2,...\},\; Q(\tau,1)$ so that for $\forall k,\; k \ge \log_2 N(\tau,1) - 1$,

$$P\left(\sum_{j=1}^{2^{k+1}} \Delta x(j) > s\right) \le \frac{2^{k+1}Q(\tau,1)}{s^{\tau}} \tag{111}$$

for $s \ge 2^{(k+1)/\tau}((k+1)ln2)^2$.

In the case A_1 $\; s = n^{\delta-\gamma}/2$. As $\delta > \gamma$ so $\exists N_1 :$ $\forall n \ge N_1$

$$n^{\delta-\gamma}/2 \ge 2^{(K^*+1)/\tau}((K^*+1)ln2)^2.$$

In the case A_2 $\; s = n^{\delta-\gamma}2^{k-2}$. So for $\delta > \gamma$ obtain that $\exists N_2 : \forall n \ge N_2$ and for $\forall k \ge K^*$

$$n^{\delta-\gamma}2^{k-2} \ge 2^{(k+1)/\tau}((k+1)ln2)^2.$$

Then for $\forall n \ge \max(N_1, N_2)$ from (110), (111) obtain that

$$A_1 + A_2 \le \frac{2}{a(\alpha)} \left(\frac{4K^*2^{K^*+1}Q(\tau,1)}{(n^{\delta-\gamma})^{\tau}} + \right.$$

$$\left. + \sum_{k=K^*+1}^{\infty} \frac{2^{k+1}Q(\tau,1)}{(n^{\delta-\gamma}2^{k-2})^{\tau}} \right) =$$

$$= \frac{2Q(\tau,1)\left(4K^*2^{K^*+1} + \sum_{k=K^*+1}^{\infty} 2^{(1-\tau)k+2\tau+1}\right)}{a(\alpha)(n^{\delta-\gamma})^{\tau}} \tag{112}$$

Denoting

$$L(\alpha,\tau,1,K^*) = \frac{2Q(\tau,1)}{a(\alpha)} \left(4K^*2^{K^*+1} + \right.$$

$$\left. + 2^{2\tau+1} \sum_{k=K^*+1}^{\infty} 2^{(1-\tau)k} \right),$$

from the formulas (107), (110), (112) obtain that

$$P_1(n) \le \frac{L(\alpha,\tau,1,K^*)}{(n^{\delta-\gamma})^{\tau}}.$$

So the equality (106) is true.

Suppose now that $\gamma > 1 - 1/\alpha$ or $\delta < \gamma$. Divide the set $G = \{(\gamma,\delta) : 0 < \delta < \gamma \; or \; \gamma > 1 - 1/\alpha\}$ into the nonintersecting subsets

$$G_{1,\alpha} = \{(\gamma,\delta) : \delta \ge 1 - 1/\alpha,\; \gamma > 1 - 1/\alpha\},$$

$$G_{2,\alpha} = \{(\gamma,\delta) : 0 < \delta < \gamma,\; \delta < 1 - 1/\alpha\},$$

$$G_{1,\alpha} \cup G_{2,\alpha} = G.$$

Consider the case when $(\gamma,\delta) \in G_{1,\alpha}$. From the formula (94) obtain that for $\forall M,\; M \in N,\;\; p_n =$

$$= P\left(\sup_{m>0} \sum_{k=1}^{m} \left(\frac{\Delta x(k)}{n^{\frac{1}{\alpha}-1+\delta}} + \sum_{j=1}^{n} \frac{\Delta x(k,j)}{n^{1/\alpha}} - \frac{n}{n^{\frac{1}{\alpha}+\gamma}} \right) > 0 \right) \ge$$

$$\geq P\left(\max_{0<m\leq M}\sum_{k=1}^{m}\left(\frac{\Delta x(k)}{n^{\frac{1}{\alpha}-1+\delta}}+\sum_{j=1}^{n}\frac{\Delta x(k,j)}{n^{1/\alpha}}\right)>\frac{nM}{n^{\frac{1}{\alpha}+\gamma}}\right)$$

Analogously to the proof of the previous theorem in the case $(\gamma,\delta)\in G_1$ (with the single correction that $b{=}\alpha,\ 1{<}\alpha{<}2$, and $\lambda(k),\ k\geq 1$, are i.i.d.r.v.'s with the common stable d.f. $P(u;\alpha,C,1,0))$ obtain that for $\forall\epsilon{>}\ 0\ \exists N^*$: $\forall n\geq N^*$

$$p_n\geq 1-2\epsilon. \tag{113}$$

Suppose that $(\gamma,\delta)\in G_{2,\alpha}$. In this case for $\forall M,\ M\in N$, the following inequality $p_n=$

$$=P\left(\sup_{m>0}\sum_{k=1}^{m}\left(\Delta x(k)+\sum_{j=1}^{n}\frac{\Delta x(k,j)}{n^{1-\delta+\frac{2}{\alpha}}}-\frac{n}{n^{\frac{1}{\alpha}+\gamma}}\right)>0\right)\geq$$

$$\geq P\left(\max_{0<m\leq M}\sum_{k=1}^{m}\left(\Delta x(k)+\sum_{j=1}^{n}\frac{\Delta x(k,j)}{n^{1-\delta+\frac{2}{\alpha}}}\right)>\frac{nM}{n^{\frac{1}{\alpha}+\gamma}}\right).$$

is true. Analogously to the proof of the previous theorem in the case $(\gamma,\delta)\in G_2$ (with the corrections for b and $\lambda(k)$, $k\geq 1$) obtain the inequality (113).

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