Gyrogroup

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Abstract: - Gyrogroups are generalized groups, which are best motivated by the algebra of Möbius transformations of the complex open unit disc. Groups are classified into commutative and non-commutative groups and, in full analogy, gyrogroups are classified into gyrocommutative and non-gyrocommutative gyrogroups. Some commutative groups admit scalar multiplication, giving rise to vector spaces. In full analogy, some gyrocommutative gyrogroups admit scalar multiplication, giving rise to gyrovector spaces. Furthermore, vector spaces form the algebraic setting for the standard model of Euclidean geometry and, in full analogy, gyrovector spaces form the algebraic setting for various models of the hyperbolic geometry of Bolyai and Lobachevsky.The special grouplike loops, known as gyrocommutative gyrogroups, have thrust the Einstein velocity addition law, which previously has operated mostly in the shadows, into the spotlight. We will find that Einstein (Möbius) addition is a gyrocommutative gyrogroup operation that forms the setting for the Beltrami-Klein (Poincaré) ball model of hyperbolic geometry just as the common vector addition is a commutative group operation that forms the setting for the standard model of Euclidean geometry. The resulting analogies to which the grouplike loops give rise lead us to new results in (i) hyperbolic geometry; (ii) relativistic physics; and (iii) quantum information and computation.Time Tensors functions have been used to describe the flows of time. The magnitude of the value of time tensor function means the temporal coordinates in a flow of time. We also use a function to describe the motion of particles in quantum mechanics but it has different meanings. The function is called time tensor function. Time tensor imposes space and time measurements and space and time probing. Although using optimised space and time probe fields will allow to deep probing in a position and time measurement beyond the space and time measurements of the probe field stil result in a time tensor.

Key-Words: - Grouplike Loops, Gyrogroups, Gyrovector Spaces, Hyperbolic Geometry, Special Relativity,Time Tensors

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1 Introduction

The evolution from Möbius to gyrogroups began soon after the discovery in 1988 [1] that Einstein velocity addition law of relativistically admissible velocities,which is seemingly structureless, is in fact rich of structure. Later, the rich structure that Einstein velocity addition law encodes turned out to be a gyrocommutative gyrogroup and a gyrovector space. The resulting notions of gyrogroups and gyrovector spaces preserve the flavor of their classical counterparts, groups and vector spaces. They are useful and fascinating enough to be made part of the lore learned by all undergraduate and graduate mathematics and physics students. Being a natural generalization of groups and vector spaces, gyrogroups and gyrovector spaces lay a fruitful bridge between nonassociative algebra and hyperbolic geometry, just as groups and vector spaces lay a fruitful bridge between associative algebra and Euclidean geometry.More than 150 years have passed since the German mathematician August Ferdinand Möbius (1790 – 1868) first studied the transformations that now bear his name [33, p. 71]. Yet, the rich structure he thereby exposed is still far from being exhausted, as the evolution from Möbius to gyrogroups demonstrates in its.

Time Tensors concern the whole of physical reality, considered in usefully physical fields. The physical world appears to have temporal aspects, so the existence and nature of time are general fields. We analyze time tensors and space and time curvatures, using the framework of fluctuation and dissipation mechanisms arising when time tensors and spacetime metric are combined.

Mathematical foundations and applications" [2], and "Beyond the Einstein addition law and its gyroscopic Thomas precession: The theory of gyrogroups and gyrovector spaces" [4, 12], raise expectations for novel applications of special grouplike loops in hyperbolic geometry and in relativistic physics. Indeed, these fields lead their readers to see what some special grouplike loops have to offer, and thereby give them a taste of loops in the service of the hyperbolic geometry of Bolyai and Lobachevsky and the special relativity theory of Einstein.

Seemingly structureless, Einstein's relativistic velocity addition is neither commutative nor associative. Einstein's failure to recognize and advance the rich, grouplike loop structure [5] that regulates his relativistic velocity addition law contributed to the eclipse of his velocity addition law of relativistically admissible 3-velocities, creating a void that could be filled only with the Lorentz transformation of 4-velocities, along with its Minkowski's geometry.

Minkowski characterized his spacetime geometry as evidence that preestablished

harmony between pure mathematics and applied physics does exist [6]. Subsequently, the study of special relativity followed the lines

laid down by Minkowski, in which the role of Einstein velocity addition and its interpretation in the hyperbolic geometry of Bolyai and Lobachevsky are ignored [5]. The tension created by the mathematician Minkowski into the specialized realm of theoretical physics, as well as Minkowski's strategy to overcome disciplinary obstacles to the acceptance of his reformulation of Einstein's special relativity is skillfully described by Scott Walter in [16].

According to Leo Corry [11], Einstein considered Minkowski's reformulation of his theory in terms of four-dimensional spacetime to be no more than "superfluous erudition". Admitting that, unlike his seemingly structureless relativistic velocity addition law, the Lorentz transformation is an elegant group operation.

Therefore, suppose that there is a price to pay in mathematical elegance and regularity when replacing ordinary vector addition approach to Euclidean geometry with Einstein vector addition approach to hyperbolic geometry. But, this is not the case since grouplike loops, called gyrocommutative gyrogroups, come to the rescue. It turns out that Einstein addition of vectors with magnitudes $v < c$ is a gyrocommutative gyrogroup operation and, as such, it possesses a rich nonassociative algebraic and geometric structure. The best way to introduce the gyrocommutative gyrogroup notion that regulates the algebra of Einstein's relativistic velocity addition law is offered by Möbius transformations of the disc [2]. The subsequent transition from Möbius addition, which regulates the Poincaré ball model of hyperbolic geometry, Fig. 1, to Einstein addition, which regulates the Beltrami-Klein ball model of hyperbolic geometry, Fig. 6, expressed in gyrolanguage, will then turn out to be remarkably simple and elegant [5, 7].

Evidently, the grouplike loops that we naturally call gyrocommutative gyrogroups, along with their extension to gyrovector spaces, form a new tool for the twenty-first century exploration of classical hyperbolic geometry and its use in physics.

2 Möbius transformations of the disc

Möbius transformations of the disc D,

$$D = \{z \in C : |z| < 1\} \qquad (1)$$

of the complex plane C offer an elegant way to introduce the grouplike loops that we call gyrogroups. More than 150 years have passed since August Ferdinand Möbius first studied the transformations that now bear his name [35]. Yet, the rich structure he thereby exposed is still far from being exhausted.

Ahlfors' book [1], Conformal Invariants: Topics in Geometric Function Theory, begins with a presentation of the Möbius self-transformation of the complex open unit disc $D = \{z \in C : |z| < 1\}$

$$z \mapsto e^{i\theta} \frac{a+z}{1+\bar{a}z} = e^{i\theta}(a \oplus_M z) \qquad (2)$$

$$a, z \in D, \theta \in R$$

where $\bar{a}$ is the complex conjugate.

Suggestively, the polar decomposition (2) of Möbius transformation of the disc gives rise to

Möbius addition, $\oplus_M$,

$$a \oplus_M z = \frac{a+z}{1+\bar{a}z} \quad (3)$$

Naturally, Möbius subtraction, ∇_M is given by

$$a \nabla_M z = a \oplus_M (-z)$$

so that

$$z \nabla_M z = 0$$

$$\nabla_M z = 0 \nabla_M z = 0 \oplus_M (-z) = -z$$

Remarkably, Möbius addition possesses the automorphic inverse property

$$\nabla_M (a \oplus_M b) = \nabla_M a \nabla_M b \quad (4)$$

and the left cancellation law

$$\nabla_M a \oplus_M (a \oplus_M z) = z \quad (5)$$

for all $a, b, z \in D$

Möbius addition gives rise to the Möbius disc groupoid $(D, \oplus_M)$, recalling that a groupoid $(G, \oplus)$ is a nonempty set, G, with a binary operation, $\oplus$ and that an automorphism of a groupoid $(G, \oplus)$ is a bijective self map f of G that respects its binary operation $\oplus$ that is,

$$f(a \oplus b) = f(a) \oplus f(b).$$

The set of all automorphisms of a groupoid $(G, \oplus)$ forms a group, denoted $Aut(G, \oplus)$.

Möbius addition $\oplus_M$ in the disc is neither commutative nor associative.

To measure the extent to which Möbius addition deviates from associativity we define the gyrator

$$gyr : D \times D \rightarrow Aut(D, \oplus_M) \quad (6)$$

by the equation

$$gyr[a,b]z = \nabla_M (a \oplus_M b) \oplus_M \{a \oplus_M (b \oplus_M)\} (7)$$

for all $a, b, z \in D$.

The automorphisms

$$gyr[a,b] \in Aut(D, \oplus_M) \quad (8)$$

of D, $a, b \in D$ called gyrations of D, have an important hyperbolic geometric interpretation [6].

Thus, the gyrator in (6) generates the gyrations in (8).

In order to emphasize that gyrations of D are also automorphisms of $(D, \oplus_M)$, as we will see below, they are also called gyroautomorphisms.

Clearly, in the special case when the binary operation $\oplus_M$ in (7) is associative,

$gyr[a,b]$ reduces to the trivial automorphism,

$gyr[a,b]z = z$ for all $z \in D$.

Hence, indeed, the self map $gyr[a,b]$ of the disc D measures the extent to which Möbius addition $\oplus_M$ in the disc D deviates from associativity.

One can readily simplify (7) in terms of (3), obtaining

$$gyr[a,b]z = \frac{1+a\bar{b}}{1+\bar{a}b} z \quad (9)$$

$a, b, z \in D$, so that the gyrations

$$gyr[a,b] = \frac{1+a\bar{b}}{1+\bar{a}b} = \frac{a \oplus_M b}{b \oplus_M a} \quad (10)$$

are unimodular complex numbers. As such, gyrations represent rotations of the disc D about its center, as shown in (9).

Gyrations are invertible.

The inverse, $gyr^{-1}[a,b] = (gyr[a,b])^{-1}$, of a gyration $gyr[a,b]$ is the gyration $gyr[b,a]$

$$gyr^{-1}[a,b] = gyr[b,a] \quad (11)$$

Moreover, gyrations respect Möbius addition in the disc,

$$gyr[a,b](c \oplus_M d) = gyr[a,b]c \oplus_M gyr[a,b]d$$

(12) for all a, b, c, d∈D, so that gyrations of the disc are automorphisms of the disc, as anticipated in (8).

Identity (10) can be written as

$(a \oplus_M b) = gyr[a,b](b \oplus_M a)$ (13) thus giving rise to the gyrocommutative law of Möbius addition.

Furthermore, Identity (7) can be manipulated, by mean of the left cancellation law (5), into the identity

$$a \oplus_M (b \oplus_M z) = (a \oplus_M b) \oplus_M gyr[a,b]z \quad (14)$$

thus giving rise to the left gyroassociative law of Möbius addition. The gyrocommutative law, (13), and the left gyroassociative law, (14), of Möbius addition in the disc reveal the grouplike structure of Möbius groupoid $(D, \oplus_M)$, that we naturally call a gyrocommutative gyrogroup. Taking the key features of Möbius groupoid $(D, \oplus_M)$ as axioms, and guided by analogies with group theory, we thus obtain the following definitions of gyrogroups and gyrocommutative gyrogroups.

Definition 1. (Gyrogroups).

A groupoid $(G, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms. In G there is at least one element, 0, called a left identity, satisfying

(G1) $0 + \oplus a = a$ for all $a \in G$.

(G2) $\nabla a \oplus a = 0$.

(G3) $a \oplus (b \oplus c) = (a \oplus b) gyr[a,b]c$.

(G4) $gyr[a,b] \in Aut(G, \oplus)$

(G5) $gyr[a,b] = gyr[a \oplus b, b]$

The gyrogroup axioms (G1) – (G5) in Definition 1 are classified into three classes:

(1) The first pair of axioms, (G1) and (G2), is a reminiscent of the group axioms.

(2) The last pair of axioms, (G4) and (G5), presents the gyrator axioms.

(3) The middle axiom, (G3), is a hybrid axiom linking the two pairs of axioms in (1) and (2).

Definition 2. (Gyrocommutative Gyrogroups).

A gyrogroup $(G, \oplus)$ is gyrocommutative if its binary operation obeys the gyrocommutative law

(G6) $a \oplus b = gyr[a,b](b \oplus a)$ for all $a, b \in G$.

Temporal space is a set of temporal elements or points satisfying specified time dimensions. Von Neumann says that "First of all we must admit that this objection points at an essential weakness which is, in fact, the chief weakness of quantum mechanics: its non-relativistic character, which distinguishes the time t from the three space coordinates x, y, z and presupposes an objective simultaneity concept. In fact, while all other quantities especially those x, y, z closely connected with t by the Lorentz transformation are represented by operators, there corresponds to the time an ordinary number-parameter t, just as in classical mechanics."

Reference Frames

A frame of reference or reference frame is a coordinate system or set of axes used by an observer to measure the position, orientation, everything of objects in space..

Lorentz Tensor

Lorentz tensor is, by definition, an object whose indices transform like a tensor under Lorentz transformations; what we mean by this precisely will be explained below.

4-vector is a tensor with a first rank tensor.

We write a 4-vector in components as

$$G^{\mu} = \begin{pmatrix} G^0 \\ G^1 \\ G^2 \\ G^3 \end{pmatrix}$$

where we use Greek indices to run over all the spacetime indices, $\mu \in [0,3]$.

The Lorentz transformation

We write the components of the Lorentz transformation matrix in index notation.

We transform the components of a 4-vector from an unprimed frame to a frame which is moving at speed v in the x direction relative to F .

We use the Lorentz transformation

$$\begin{pmatrix} \Delta x'^0 \\ \Delta x'^1 \\ \Delta x'^2 \\ \Delta x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta x^0 \\ \Delta x^1 \\ \Delta x^2 \\ \Delta x^3 \end{pmatrix}$$

where $\gamma = \dfrac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}$ and $\beta = \dfrac{v}{c}$.

Now we write the components of the Lorentz transformation matrix as S_{ν}^{μ} where μ is a row index and ν is a column index, such that

$$S = \begin{pmatrix} S_0^0 & S_1^0 & S_2^0 & S_3^0 \\ S_0^1 & S_1^1 & S_2^1 & S_3^1 \\ S_0^2 & S_1^2 & S_2^2 & S_3^2 \\ S_0^3 & S_1^3 & S_2^3 & S_3^3 \end{pmatrix}$$

Then, the Lorentz transformation for Δx^{μ} can be written in the compact notation

$$(\Delta x')^{\mu} = \sum_{\nu=0}^{3} S_{\nu}^{\mu} \Delta x^{\nu} = S_{\nu}^{\mu} \Delta x^{\nu}$$

$$(\Delta x')^{\mu} = S_0^{\mu} \Delta x^0 + S_1^{\mu} \Delta x^1 + S_2^{\mu} \Delta x^2 + S_3^{\mu} \Delta x^3$$

$$(\Delta x')^{\mu} = S_0^{\mu} c\Delta t + S_1^{\mu} \Delta x + S_2^{\mu} \Delta y + S_3^{\mu} \Delta z$$

$$(\Delta x')^0 = (c\Delta t') = \gamma(c\Delta t) - \gamma\beta\Delta x$$

$$(\Delta x')^1 = (\Delta x') = -\gamma\beta(c\Delta t) + \gamma\Delta x$$

$$(\Delta x')^2 = (\Delta y') = \Delta y$$

$$(\Delta x')^3 = (\Delta z') = \Delta z$$

is the usual Lorentz transformation to a frame moving in the x direction.

$$\begin{pmatrix} \Delta x'^0 \\ \Delta x'^1 \\ \Delta x'^2 \\ \Delta x'^3 \end{pmatrix} = \begin{pmatrix} (c\Delta t') \\ (\Delta x') \\ (\Delta y') \\ (\Delta z') \end{pmatrix} = \begin{pmatrix} \gamma(c\Delta t) - \gamma\beta\Delta x \\ -\gamma\beta(c\Delta t) + \gamma\Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

The inverse Lorentz transformation should satisfy

$$(S^{-1})_{\nu}^{\xi} S_{\nu}^{\xi} = \delta_{\nu}^{\xi}$$

where $\delta_{\nu}^{\beta} \equiv diag(1,1,1,1)$ is the Kronecker delta.

$$(S^{-1})_{\nu}^{\xi} (\Delta x')^{\mu} = \delta_{\nu}^{\xi} \Delta x^{\nu} = \Delta x^{\xi}$$

The inverse $(S^{-1})_{\nu}^{\xi}$ is also written as S_{ν}^{ξ} .

The left index denotes a row while the right index denotes a column, while the top index denotes the frame we're transforming to and the bottom index denotes the frame we're transforming from.

We present the components of S and S^{-1} in their transformations.

$$S=\begin{pmatrix} S_0^0 & S_1^0 & S_2^0 & S_3^0 \\ S_0^1 & S_1^1 & S_2^1 & S_3^1 \\ S_0^2 & S_1^2 & S_2^2 & S_3^2 \\ S_0^3 & S_1^3 & S_2^3 & S_3^3 \end{pmatrix}$$

$$S^{-1}=\begin{pmatrix} S_0^0 & -S_0^1 & -S_0^2 & -S_0^3 \\ -S_1^0 & S_1^1 & S_1^2 & S_1^3 \\ -S_2^0 & S_2^1 & S_2^2 & S_2^3 \\ -S_3^0 & S_3^1 & S_3^2 & S_3^3 \end{pmatrix}$$

The inverse to the transformation

$$\begin{pmatrix} \Delta x'^0 \\ \Delta x'^1 \\ \Delta x'^2 \\ \Delta x'^3 \end{pmatrix}=\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \Delta x^0 \\ \Delta x^1 \\ \Delta x^2 \\ \Delta x^3 \end{pmatrix}$$

$$\begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \Delta x'^0 \\ \Delta x'^1 \\ \Delta x'^2 \\ \Delta x'^3 \end{pmatrix}=\begin{pmatrix} \Delta x^0 \\ \Delta x^1 \\ \Delta x^2 \\ \Delta x^3 \end{pmatrix}$$

The metric

The metric $L_{\mu\nu}$ is a special Lorentz tensor, which for Minkowski spacetime in our convention is given by

$$L=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}=diag(1,-1,-1,-1)$$

The other convention is to use $L_{\mu\nu}=diag(1,-1,-1,-1)$, which will change around minus signs in various places.

We use the metric to raise and lower Lorentz indices.

By de_nition $G_\mu = L_{\mu\nu}G^\nu$ given a 4-vector G^ν with an upstairs index.

We think that G^ν as a column vector, and G_μ as a row vector.

The inverse metric $L^{\mu\psi}$ with upstairs indices satisfies $L^{\mu\psi}L_{\psi\nu}=\delta_\nu^\mu$ then, we can show that $L^{\mu\psi}=diag(1,-1,-1,-1)$.

In other words, the Minkowski metric is its own inverse. We can then use the

inverse metric to raise indices, as in $G^\mu = L^{\mu\psi}G_\psi$ given a 4-vector with a lower index.

The Lorentz group

We can write down the condition for an object S to be a Lorentz transformation.

$$L_{\mu\nu}=S_\mu^\psi S_\nu^\xi L_{\psi\xi}$$

It translates to $S^T LS = L$ for S^T the matrix transpose of S.

$$S^T LS=\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}=L$$

This condition is both necessary and sufficient for a 4×4 matrix S to leave the inner product of any two 4-vectors invariant.

Any group is a set of elements with an operation that combines any two elements to form a third, which satisfies certain properties are closure, associativity, identity, and inverse.

Here, the elements are the S and the group operation is matrix multiplication.

Closure

The product of any 2 Lorentz transformations is another Lorentz transformation.

Associativity

Associativity of Lorentz transformations which follows from the properties of matrix multiplication.

Identity

The identity is $S_\nu^\mu = \delta_\nu^\mu$

Inverse

The inverse of S_ν^μ is $(S^{-1})_\nu^\mu = S_\nu^\mu$.

3. Möbius Addition in the Ball

If we identify complex numbers of the complex plane C with vectors of the Euclidean plane R^2 in the usual way,

$$\mathbb{C}\ni u=u_1+iu_2=(u_1,u_2)=\mathbf{u}\in\mathbb{R}^2$$

$$\bar{u}v+u\bar{v}=2\mathbf{u}\cdot\mathbf{v}$$

$$|u|=\|\mathbf{u}\|$$

$$\mathbb{R}^2_{s=1}=\{\mathbf{v}\in\mathbb{R}^2:\|\mathbf{v}\|<s=1\}$$

$$\begin{aligned}\mathbb{D}\ni u\oplus_M v&=\frac{u+v}{1+\bar{u}v}\\ &=\frac{(1+u\bar{v})(u+v)}{(1+\bar{u}v)(1+u\bar{v})}\\ &=\frac{(1+\bar{u}v+u\bar{v}+|v|^2)u+(1-|u|^2)v}{1+\bar{u}v+u\bar{v}+|u|^2|v|^2}\\ &=\frac{(1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2)\mathbf{u}+(1-\|\mathbf{u}\|^2)\mathbf{v}}{1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{u}\|^2\|\mathbf{v}\|^2}\\ &=\mathbf{u}\oplus_M\mathbf{v}\in\mathbb{R}^2_{s=1}\end{aligned}$$

As such, it survives unimpaired in higher dimensions, suggesting the following definition of

Möbius addition in the ball of any real inner product space.

Definition 3. (Möbius Addition in the Ball).

Let V be a real inner product space , and let V_s be the s-ball of V ,

$$\mathbb{V}_s = \{\mathbb{V}_s \in \mathbb{V} : \|\mathbf{v}\| < s\}$$

for any fixed $s > 0$.

Möbius addition $\oplus_M$ in the ball Vs is a binary operation in V_s given by the equation

$$\mathbf{u} \oplus_M \mathbf{v} = \frac{(1 + \frac{2}{s^2}\mathbf{u}\cdot\mathbf{v} + \frac{1}{s^2}\|\mathbf{v}\|^2)\mathbf{u} + (1 - \frac{1}{s^2}\|\mathbf{u}\|^2)\mathbf{v}}{1 + \frac{2}{s^2}\mathbf{u}\cdot\mathbf{v} + \frac{1}{s^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2}$$

Möbius addition in the ball V_s is known in the literature as a hyperbolic translation [2, 4]. Following the discovery of the gyrocommutative gyrogroup structure in 1988 [5], Möbius hyperbolic translation in the ball V_s now deserves the title “Möbius addition” in the ball V_s, in full analogy with the standard vector addition in the space V that contains the ball.

Möbius addition in the ball V_s satisfies the gamma identity

$u.v \in V_s$

$$\gamma_{\mathbf{u}\oplus_M \mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\sqrt{1 + \frac{2}{s^2}\mathbf{u}\cdot\mathbf{v} + \frac{1}{s^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2}$$

$$\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{v}\|^2}{s^2}}}$$

$$\mathbf{u} \boxplus_M \mathbf{v} = \frac{\gamma_{\mathbf{u}}^2\mathbf{u} + \gamma_{\mathbf{v}}^2\mathbf{v}}{\gamma_{\mathbf{u}}^2 + \gamma_{\mathbf{v}}^2 - 1}$$

4. Gyrogroups Are Loops

A loop is a groupoid $(G,\oplus)$ with an identity element, 0, such that each of its two loop equations for the unknowns x and y,

$a \oplus x = b$

$y \oplus a = b$ (30)

possesses a unique solution in G for any $a, b \in G$ [3, 4].

Any gyrogroup is a loop. Indeed, if $(G,\oplus)$ is a gyrogroup then the respective unique solutions of the gyrogroup loop equations (30) are [56].

5. Möbius scalar multiplication in the Ball

Having developed the Möbius gyrogroup as a grouplike loop, we do not stop at the loop level. Encouraged by analogies gyrogroups share with groups, we now seek analogies with vector spaces as well. Accordingly, we uncover the scalar multiplication, $\otimes_M$ between a real number $r \in R$ and a vector $v \in V_s$, that a Möbius gyrogroup $(V_s, \oplus_M)$ admits, so that we can turn the Möbius gyrogroup into a Möbius gyrovector space $(V_s, \oplus_M, \otimes_M)$.

Definition 4. (Möbius Scalar Multiplication).

Let $(V_s, \oplus_M)$ be a Möbius gyrogroup. Then its corresponding Möbius gyrovector space $(V_s, \oplus_M, \otimes_M)$ involves the Möbius scalar multiplication

$r \otimes_M v = v \otimes_M r$ in V_s given by the equation

$$r \otimes_M \mathbf{v} = s\frac{\left(1 + \frac{\|\mathbf{v}\|}{s}\right)^r - \left(1 - \frac{\|\mathbf{v}\|}{s}\right)^r}{\left(1 + \frac{\|\mathbf{v}\|}{s}\right)^r + \left(1 - \frac{\|\mathbf{v}\|}{s}\right)^r}\frac{\mathbf{v}}{\|\mathbf{v}\|}$$

$$= s\tanh(r \tanh^{-1}\frac{\|\mathbf{v}\|}{s})\frac{\mathbf{v}}{\|\mathbf{v}\|}$$

where $r \in R$, $v \in V_s$ $v \neq 0$; and $r \otimes_M 0 = 0$.

6. Möbius Gyroline and More

In full analogy with straight lines in the standard vector space approach to Euclidean geometry, let us consider the gyroline equation in the ball V_s

$$L_{AB} := A \oplus (\ominus A \oplus B) \otimes t$$

$t \in R$, $A, B \in V_s$, in a Möbius gyrovector space $(V_s, \oplus_M, \otimes_M)$.

The Möbius Gyroline LAB through the points A and B

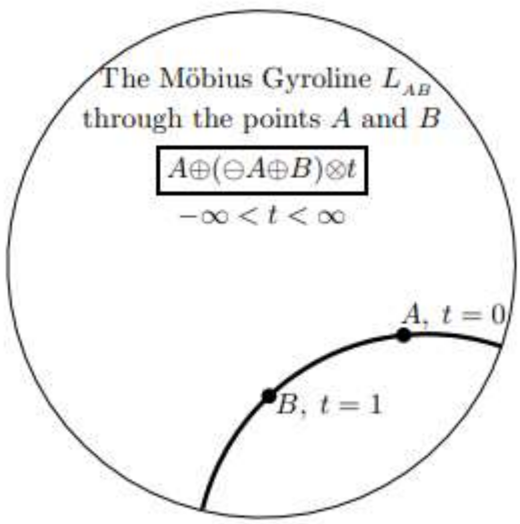

Fig.1In "gyroformalism", hyperbolic geometric expressions take the graceful forms of their Euclidean counterparts.

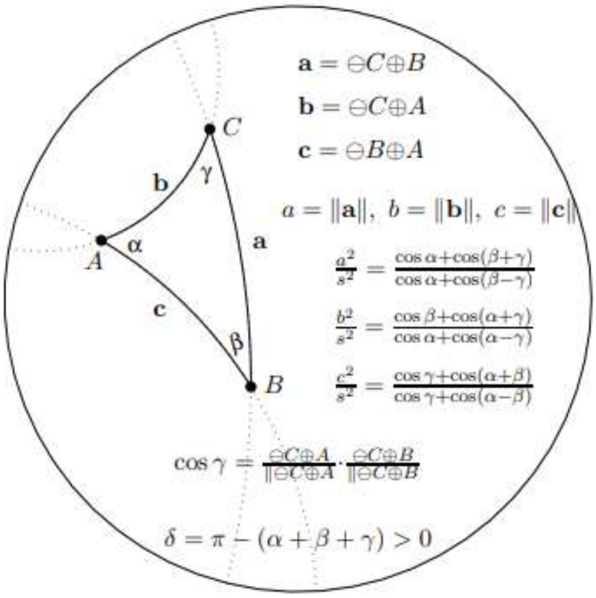

Fig.2 Möbius gyrotriangle and its standard notation and identities in a Möbius gyrovector space $(V_s, \oplus_M, \otimes_M)$.

In Euclidean geometry vector addition coincides with the parallelogram addition law. In contrast, in hyperbolic geometry gyrovector addition, given by Möbius addition, and the Möbius gyroparallelogram addition law are distinct.

7. Einstein Operations in the Ball

Definition 5. (Einstein Addition in the Ball).

Let V be a real inner product space and let V_s be the s-ball of V,

$$\mathbb{V}_s = \{\mathbf{v} \in \mathbb{V} : \|\mathbf{v}\| < s\}$$

(4) where $s > 0$ is an arbitrarily fixed constant (that represents in physics the vacuum speed of light c).

Einstein addition $\oplus_E$ is a binary operation in V_s given by the equation

$$\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1 + \frac{\mathbf{u} \cdot \mathbf{v}}{s^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}} \mathbf{v} + \frac{1}{s^2} \frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}} (\mathbf{u} \cdot \mathbf{v}) \mathbf{u} \right\}$$

where γ_u is the gamma factor, in V_s, and where $*$ and $\|*\|$ are the inner product and norm that the ball V_s inherits from its space V.

We may note that the Euclidean 3-vector algebra was not so widely known in 1905 and, consequently, was not used by Einstein. Einstein calculated in his founding paper [12] the behavior of the velocity components parallel and orthogonal to the relative velocity between inertial systems, which is as close as one can get without vectors to the vectorial version (4). Seemingly structureless, Einstein velocity addition could not play in Einstein's special theory of relativity a central role. Indeed, Borel's attempt to "repair" the seemingly "defective" Einstein velocity addition in the years following 1912 is described in [16]. Fortunately, however, there is no need to "repair" the Einstein velocity addition law since, like Möbius addition in the ball, Einstein addition in the ball is a gyrocommutative gyrogroup operation, which gives rise to the Einstein ball gyrogroups $(V_s, \oplus_E)$ and gyrovector spaces $(V_s, \oplus_E, \otimes_E)$ [5,8]. Furthermore, Einstein's gyration turns out to be the Thomas precession of relativity physics [5], so that Thomas precession is a kinematic effect rather than a dynamic effect as it is usually portrayed [58]. A brief history of the discovery of Thomas precession is presented in [3].

The gamma factor is related to Einstein addition by the gamma identity

$$\gamma_{\mathbf{u} \oplus_E \mathbf{v}} = \gamma_{\mathbf{u}} \gamma_{\mathbf{v}} \left(1 + \frac{\mathbf{u} \cdot \mathbf{v}}{s^2} \right)$$

This gamma identity provided the historic link between Einstein's special theory of relativity and the hyperbolic geometry of Bolyai and Lobachevsky, as explained in [6].

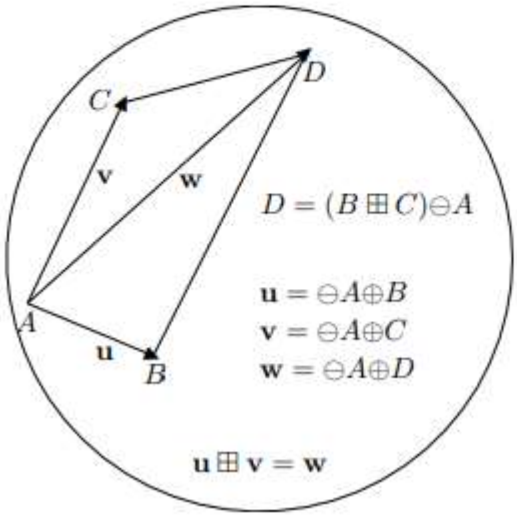

Fig.3 The Einstein gyroparallelogram addition law of relativistically admissible velocities.

The classical interpretation of particle aberration is obvious in terms of the triangle law of Newtonian velocity addition (which is the common vector addition in Euclidean geometry), as demonstrated graphically. The relativistic interpretation of particle aberration is, however, less obvious.

8 Conclusion

Gyrogroups are suitable generalization of groups, whose origin is described in [7, 8]. They share remarkable analogies with groups. In fact, every group forms a gyrogroup under the same operation. Many of classical theorems in group theory also hold for gyrogroups, including the Lagrange theorem [4], the fundamental isomorphism theorems [5], and the Cayley theorem [5] . Gyrogroup actions and related results, such as the orbit-stabilizer theorem, the orbit decomposition theorem, and the Burnside lemma have been studied in [6]. The present note deals with a connection between groups and gyrogroups, namely with the gyrogroup associated to any group central by a 2-Engel group. We determine conditions for such a gyrogroup to be gyrocommutative and for such two gyrogroups to be isomorphic.

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