A Minimum Variance Controller for SISO Linear Time Variant Systems

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Abstract: - An optimal controller is developed for linear time variant systems described using transfer operators, where the noise response is described using a time variant moving average autoregressive model and the control response is described using a time variant autoregressive moving average model. Following the line of minimum variance control methods, this controller can achieve minimum variance output tracking without using noise variance information even when the speed of parameter variation in the system is arbitrarily fast.

Key-Words: - Linear Systems, Minimum Variance Control, Optimal Control, Stochastic Systems, Time-Varying Systems.

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1 Introduction

The minimum variance controller [1] developed by Astrom is the base of a family of very popular stochastic optimal controllers including self-tuning controller [2], generalized minimum variance controller [3] and generalized predictive controllers [4 and 5]. These controllers are easy to understand and fit easily into parameter estimation schemes for parameter adaptive control.

The minimum variance controller also provides the lower bound for output tracking error variance and is a benchmark to gauge the performance that is suboptimal. The minimum variance controller is based on transfer functions and, therefore, are for linear time invariant systems only. It cannot be extended to linear time variant systems in a straightforward way because the transfer functions do not apply to linear time variant systems. Linear time variant transfer operators [6] replaces the complex variable of the transfer functions using a one-step-delay-operator and extends the transfer function from linear time invariant systems to linear time variant plants. One of the important differences between the linear time invariant transfer operators and the linear time variant transfer operators is that the linear time variant transfer operators are not commutative with respect to multiplication.

A pseudo commutation technique was developed in 1997 [6]. It allows a linear time variant autoregressive operator to exchange position in multiplication with a linear time variant moving average operator using an equivalent transfer operator in different form but the same input and output relation. Based on this technique the first minimum variance controller for linear time variant systems with colored noise disturbances was developed in 1997 [7] based on a linear time variant controlled autoregressive moving average (CARMA) model.

In this paper we develop a minimum variance controller for different linear time variant systems that can be represented using a linear time variant model whose response to noise is a moving average autoregressive process and whose response to control variable is an autoregressive moving average process. We call this model a CMAAR model, which is different from the CARMA model because time variant transfer operators are not commutative with respect to multiplication. It will be shown that the CMAAR model represents at least an equally large class of linear time variant systems as the linear time variant CARMA model and the linear time variant minimum variance controller can ensure both closed-loop exponential stability and the minimum variance output tracking error regardless the variation speed of the plant dynamics.

2 Linear Time Variant System and Control Objective

We consider the following linear time variant CMAAR model,

$$y(k+d) = A^{-1}(k,q^{-1})B(k,q^{-1})u(k)$$
$$+C(k,q^{-1})D^{-1}(k,q^{-1})w(k+d) \quad (1)$$

where *y(k)*, *u(k)* and *w(k)* are the plant output, input, and noise. *d* is a positive integer representing time delay of the system.

$$A(k,q^{-1}) = 1 + a_1(k)q^{-1} + \cdots + a_{na}(k)q^{-na}$$

$$B(k,q^{-1}) = b_0(k) + b_1(k)q^{-1} + \cdots + b_{nb}(k)q^{-nb}$$

$$C(k,q^{-1}) = 1 + c_1(k)q^{-1} + \cdots c_{nc}(k)q^{-nc}$$

$$D(k,q^{-1}) = 1 + d_1(k)q^{-1} + \cdots d_{nd}(k)q^{-nd} \quad (2)$$

are time-varying polynomials in the one-step-delay operator q^{-1}. The polynomials are time variant because their coefficients are time varying. The one-step-delay operator is applicable to the time-varying coefficients, input, output, and noise variables as follows.

$$q^{-1}f(k)=f(k-1) \quad (3)$$

The polynomials in (2) are not transfer functions but transfer operators because q is an operator not a complex number. Consequently, we call the operators $D(k,q^{-1}), C(k,q^{-1})$, $B(k,q^{-1})$ and $A(k,q^{-1})$ transfer operators rather than transfer functions for linear time variant processes. The moving average operation for the input is described using the following linear time variant equation,

$$v(k) = B(k,q^{-1})u(k) \quad (4)$$

and we call $B(k,q^{-1})$ a linear time variant moving average operator. A zero initial condition solution of the following linear time variant autoregressive equation

$$D(k,q^{-1})z(k+d) = w(k+d) \quad (5)$$

is denoted as

$$z(k+d) = D^{-1}(k,q^{-1})w(k+d) \quad (6)$$

$D^{-1}(k,q^{-1})$ is called a linear time variant autoregressive operator because of the autoregressive operation (5). When the autoregressive model (5) is exponentially stable we say the linear time variant autoregressive operator, $D^{-1}(k,q^{-1})$, is exponentially stable. The difference between this time variant CMAAR model and the linear time variant CARMA model is that the operation order of moving average and autoregressive process for the noise response is reversed in comparison with the CARMA model. For the CMAAR model the noise is first autoregressed and then, moving averaged. However, for the linear time variant CARMA model the noise is first moving averaged and then, autoregressed. Because of noncommutativity of linear time variant operators the CARMA model and the CMAAR model represent different time variant systems. However, the CMAAR model represents at least an equally large class of linear time variant plants as the linear time variant CARMA model because the linear time variant ARMA model $D^{-1}(k,q^{-1})C(k,q^{-1})$ corresponds to a linear time variant *nd* step observable state space model and the linear time variant MAAR model $C(k,q^{-1})D^{-1}(k,q^{-1})$ represents a linear time variant *nd* step reachable state space model [8]. However, when the system is linear time invariant the CMAAR model will reduce to a CARMA model because of the commutativity of linear time invariant transfer operators. The linear time variant CMAAR model can be rewritten as follows.

$$\begin{cases} A(k,q^{-1})v(k) = B(k,q^{-1})u(k) \\ D(k,q^{-1})z(k+d) = w(k+d) \\ y(k+d) = v(k) + C(k,q^{-1})z(k+d) \end{cases} \quad (7)$$

The first equation is a linear time variant ARMA model, the second equation is an AR model, and the third equation is an MAX model. The following assumptions are made about the CMAAR model.

i. The noise *w(k)* is an independent Gaussian noise with time variant variance and zero mean. The variance is uniformly bounded.
ii. The linear time variant autoregressive operators $D^{-1}(k,q^{-1})$, $C^{-1}(k,q^{-1})$, $B^{-1}(k,q^{-1})$ and $A^{-1}(k,q^{-1})$ are all exponentially stable.
iii. The coefficients in the linear time variant moving average operators (2) are uniformly bounded.
iv. The absolute value of $b_0(k)$ is also uniformly not zero.

These assumptions are natural extensions of those made by the original linear time invariant minimum variance controller [1] from linear time invariant plants for linear time variant systems. Degrees of $D(k,q^{-1}), C(k,q^{-1}), B(k,q^{-1})$ and $A(k,q^{-1})$ are time-varying because their coefficients are allowed to be zero. However, the delay between input and output is time invariant because of assumption iv.

Minimum Variance Control Objective

Given a desired plant output sequence $\{y^*(k+d)\}$ the objective is to design a linear time variant minimum variance controller that minimizes the following output control error variance for the linear time variant system model (1).

$$J(k+d) = E\{[y^*(k+d) - y(k+d)]^2 \,|\, Data(k)\} \tag{8}$$

where E represents mathematical expectation conditioned on Data*(k)* that is a set of all plant input and output up to and including the current time *k*.

3 Minimum Variance Controller

We apply the following right polynomial division to the noise response of the linear time variant CMAAR model (1).

$$C(k,q^{-1})D^{-1}(k,q^{-1}) = F(k,q^{-1}) + G(k,q^{-1})q^{-d}D^{-1}(k,q^{-1}) \tag{9}$$

where

$$F(k,q^{-1}) = 1 + f_1(k)q^{-1} + f_2(k)q^{-2} + \ldots + f_{d-1}q^{-d+1} \tag{10}$$

is the quotient of the polynomial right division and

$$G(k,q^{-1}) = g_0(k) + g_1(k)q^{-1} + g_2(k)q^{-2} + \cdots \tag{11}$$

the remainder. Substitute (9) into (1) we have

$$y(k+d) = U(k) + w^+ + w^- \tag{12}$$

where

$$U(k) = A^{-1}(k,q^{-1})B(k,q^{-1})u(k) \tag{13}$$

is the response for the control variable *u(k)*,

$$w^+ = F(k,q^{-1})w(k+d) = w(k+d) + f_1(k)w(k+d-1) + \ldots + f_{d-1}(k)w(k+1) \tag{14}$$

is the future noise response, and

$$w^- = G(k,q^{-1})q^{-d}D^{-1}(k,q^{-1})w(k+d) = f_d(k)w(k) + f_{d+1}(k)w(k-1) + \cdots \tag{15}$$

is the response for the current and past noises. Substituting (12) into (8) we have the following.

Minimum Variance Controller Theorem

Consider the linear time variant CMAAR model (1). If the linear time variant autoregressive operators $D^{-1}(k,q^{-1}),\ C^{-1}(k,q^{-1}), B^{-1}(k,q^{-1})$ and $A^{-1}(k,q^{-1})$ are all exponentially stable the linear time variant minimum variance controller has the following form.

$$A(k-d,q^{-1})C(k-d,q^{-1})\hat{z}(k) = A(k-d,q^{-1})y(k) - B(k-d,q^{-1})u(k-d) \tag{16}$$

$$B(k,q^{-1})u(k) = A(k,q^{-1})y^*(k+d) - A(k,q^{-1})G(k,q^{-1})\hat{z}(k) \tag{17}$$

The closed loop system is exponentially stable, and the minimum variance control error covariance has the form,

$$J(k+d) = \sigma^2(\mathrm{k+d}) + \sigma^2(\mathrm{k+d-1})f_1^2(k) + \ldots + \sigma^2(\mathrm{k+1})f_{d-1}^2(k) \tag{18}$$

Proof

Substituting (9) into (1) we have

$$y(k+d) = A^{-1}(k,q^{-1})B(k,q^{-1})u(k) + F(k,q^{-1})w(k+d) + G(k,q^{-1})q^{-d}D^{-1}(k,q^{-1})w(k+d) \tag{19}$$

Comparing it with the linear time variant minimum variance controller (17) and noting the second equation in (7) we have

$$\tilde{y}(k+d) - w^{+} = G(k,q^{-1})\tilde{z}(k) \tag{20}$$

where

$$\tilde{y}(k+d) = y(k+d) - y^{*}(k+d) \tag{21}$$

is the output tracking error and

$$\tilde{z}(k) = z(k) - \hat{z}(k) \tag{22}$$

is the estimation error for *z(k)* when (16) is used. Subtracting (16) from (1) and noting the second equation in (7) we have

$$C(k,q^{-1})\tilde{z}(k) = 0 \tag{23}$$

It follows that $\tilde{z}(k)$ will exponentially decay to zero because the autoregressive operator $C^{-1}(k,q^{-1})$ is exponentially stable. Consequently, equation (20) will converge exponentially to the following equation.

$$\tilde{y}(k+d) = w^{+} \tag{24}$$

Substituting it into the minimum variance control cost we have

$$J(k+d) = E\left\{ {w^{+}}^{2} \middle| Data(k) \right\} \tag{25}$$

w^{+} is future noise component that is independent of *Data(k)* and, thus, unpredictable at time *k*. The minimum variance control variance is, therefore, (18) because the future noise is zero mean.

Noting (23), (17), (16) and (6) we have the closed loop control system for close loop stability analysis,

$$\Phi(k,q^{-1})\begin{bmatrix} y(k+d) \\ u(k) \\ \hat{z}(k+d) \\ z(k+d) \end{bmatrix} = \begin{bmatrix} 0 \\ A(k,q^{-1})y^{*}(k+d) \\ 0 \\ w(k+d) \end{bmatrix} \tag{26}$$

where $\Phi(k,q^{-1})$ is defined by the square matrix

$$\begin{bmatrix} A(k,q^{-1}) & -B(k,q^{-1}) & -A(k,q^{-1})C(k,q^{-1}) & 0 \\ 0 & B(k,q^{-1}) & A(k,q^{-1})G(k,q^{-1})q^{-d} & 0 \\ 0 & 0 & C(k,q^{-1}) & -C(k,q^{-1}) \\ 0 & 0 & 0 & D(k,q^{-1}) \end{bmatrix} \tag{27}$$

The plant output and input can be determined from the closed loop equation using the following equation.

$$\begin{bmatrix} y(k+d) \\ u(k) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} y(k+d) \\ u(k) \\ \hat{z}(k+d) \\ z(k+d) \end{bmatrix} \tag{28}$$

The autoregressive operator matrix of the closed loop system is the first matrix on the left of closed loop equation (26). It is defined by (27). Because this matrix is upper triangular and all the diagonal elements have exponential stability when used as autoregressive operators. The linear time variant closed loop system is exponentially stable.

Remarks

1. We developed a linear time variant minimum variance controller for stochastic optimal control of the linear time variant CMAAR model. The difference between this model and the standard linear time variant CARMA model is in the order of the autoregressive and moving average operation in the noise response. The linear time variant CARMA model is defined as follows.

   $$y(k+d) = A^{-1}(k,q^{-1})B(k,q^{-1})u(k) + A^{-1}(k,q^{-1})C(k,q^{-1})w(k+d) \tag{29}$$

   Because the linear time variant operators are not commutative with respect to multiplication and/or division the linear time variant CMAAR model is different from the linear time variant CARMA model. In the linear time variant CMAAR model the noise response is represented by a linear time variant MAAR model that can be represented using an *nd* step reachable canonical state space form [8]. In the linear time variant CARMA model the noise response is a linear time variant ARMA model that can be represented using a linear time variant *na* step observable canonical form [8]. In this sense the linear time variant CMAAR model represents at least an equally wide class of linear time variant plants as the standard linear time variant CARMA model. The fact that a linear time variant system can be described using a linear time variant CMAAR model does not mean it can be described using a linear time variant CARMA model.

2. Equation (26) is the closed loop linear time variant system and the square matrix $\Phi(k, q^{-1})$ in the left most position of the closed loop equation represents the linear time variant autoregressive operation. Because the matrix is triangular as shown in (27) the closed loop exponential stability is determined by the diagonal elements when they are used in autoregressive operation. Their corresponding autoregressive operators are $D^{-1}(k, q^{-1})$, $C^{-1}(k, q^{-1}), B^{-1}(k, q^{-1})$ and $A^{-1}(k, q^{-1})$. Noting assumption ii we know that the closed loop system is exponentially stable.

4 Simulation

We consider the 2-step-ahead stochastic optimal control of the linear time variant CMAAR model, where

$$A(k, q^{-1}) = 1 + a(k)q^{-1}$$

$$B(k, q^{-1}) = 1 + b(k)q^{-1}$$

$$C(k, q^{-1}) = 1 + c(k)q^{-1}$$

$$D(k, q^{-1}) = 1 + d(k)q^{-1} \qquad (30)$$

with

$$a(k) = \begin{cases} 0.5(1 - 0.9e^{-k}) & 25i < k \leq 25(i+1) \\ -0.5(1 - 0.9e^{-k}) & 25(i-1) < k \leq 25i \end{cases}$$

$$b(k) = 0.4(1 + sin(0.5\pi k))$$

$$c(k) = \begin{cases} 0.7\frac{k}{k+1} & 15i < k \leq 15(i+1) \\ -0.7\frac{k}{k+1} & 15(i-1) < k \leq 15i \end{cases}$$

$$d(k) = 0.6\cos(0.4\pi k)) \qquad (31)$$

In this CMAAR model the noise $w(k)$ is an independent Gaussian white noise that has unit variance and zero mean. Moreover, the linear time variant autoregressive operators $D^{-1}(k, q^{-1})$, $C^{-1}(k, q^{-1})$, $B^{-1}(k, q^{-1})$ and $A^{-1}(k, q^{-1})$ are all exponentially stable because the absolute values of the parameters *d(k), c(k), b(k)* and *a(k)* are all uniformly less than unit. The minimum variance control performance is given in **Fig. 1**. It shows that the output of the linear time variant system is driven to follow the desired plant output, the square wave, even when the parameters (31) of the plant are changing rapidly

5 Conclusion

We developed a linear time variant optimal controller for minimum variance control of stochastic linear time variant plants described by the CMAAR model that is different from the linear time variant CARMA model for linear time variant processes. However, linear CMAAR model represents an equally large class of linear time variant processes as the linear time variant CARMA model does. The control objective is for minimum variance output tracking. This minimum variance controller does not require the pseudo commutation for overcoming the noncommutativity of linear time variant transfer operators and is able to ensure minimum variance output tracking and exponential stability in closed loop control.

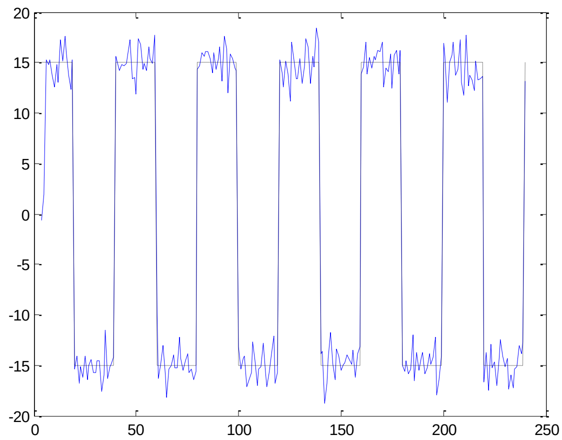

Fig 1. The desired plant output versus the real plant output. The real plant output is represented by the solid line and desired plant output is represented using the dotted line.

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