The Reliability of the Nuclear Reactor Pumps Using Numerical Estimation Methods

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Abstract: - This paper presents the kernel estimation as a new estimation method for studying the reliability of the nucellar reactor pumps based on the kernel density function. In the literature, the Pareto-II distribution is one of the models, which is the best fit for the nuclear reactor pumps data. Thus, in this work the Pareto-II distribution parameters have been estimated based on the kernel and Picard methods and comparing to the Bayesian method based on the informative gamma and kernel priors via an extensive Monte Carlo simulation study. The simulation results indicated that the kernel and Picard estimation methods are more efficient than the Bayesian method. As application, the reliability of safety mechanisms of the nuclear reactor pumps based on real data have been studied, which indicated that decreasing the reliability with increasing time, which need more cooling at higher temperatures.

Keywords: - Gamma prior; Kernel estimation method; Kernel prior; Picard's method.

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1 Introduction

Pareto distribution was introduced as a model for the distribution of income. In recent years, several different forms have been studied by many authors including [10] and [15] among others. The Pareto distribution of the second kind also known as Lomax or Pearson's Type VI distribution, see [18, 19]. [5] provided the Pareto of the second kind as a good model in biomedical problems, such as survival time following a heart transplant. [12] studied annual wage data of production line workers in a large industrial firm. Many authors have studied the Pareto distribution based on complete as well as censored samples including [2], [3, 4], [11], [13, 14], [16], [17], [21], and [26, 27, 28]. Thus, in this work we introduced the kernel method as a new estimation method based on the kernel density function for estimating the Pareto-II distribution parameters and compared to the Picard and Bayesian methods, based on an extensive simulation study. The simulation results indicated that the Kernel and Picard methods are more efficient than the Bayes method based on the informative gamma and kernel priors. The probability density function and the cumulative distribution function of the Pareto distribution of the second kind are given, respectively, as follows:

$$f(x|\alpha,\beta) = \frac{\alpha}{\beta}(1+\frac{x}{\beta})^{-(\alpha+1)},\ x>0,\ \ \alpha,\beta>0, \quad (1)$$

$$F(x) = 1-(1+\frac{x}{\beta})^{-\alpha},\ x>0,\ \ \alpha,\beta>0. \quad (2)$$

$\alpha,\beta>0$ are shape and scale parameters.

Although there is not much variety in the possible basic shapes as can be found with the gamma and Weibull distributions. However, in economics, special attention is given to the determination of the parameter, which measures the degree of inequality of income, due to its hazard failure rate, which is defined as follows:

$$h(t) = \frac{\alpha}{\beta}(1+\frac{x}{\beta})^{-1}, t>0,\ \ \alpha,\beta>0. \quad (3)$$

It is a decreasing function of time making this distribution is a suitable model for components which age with time, such as the nucellar reactor pumps data.

In reliability analysis, [20] proposed a censoring scheme called Type-II progressive hybrid censoring scheme, which is a mixture of Type-II progressive and hybrid censoring schemes. However, the drawback of the progressive hybrid censoring scheme is that very few failures may occur before the time point **T**. To provide assurance of the number of observed failures as well as the time to complete the test, [8, 9] proposed the generalized progressive hybrid censoring scheme (GPHCS). This scheme modifies the progressive hybrid censoring scheme by allowing the experiment to continue beyond time **T** if the number of failures is less than **m**, which allows the experimenter to observe at least **k** failures. This scheme can be described as follows:

Consider n identical items are placed on a test with considering $\mathrm{R}_1, \mathrm{R}_2, \dots, \mathrm{R}_\mathrm{m}$ are the random removal units, which are fixed at the beginning of the experiment with $\mathrm{m} < \mathrm{n}$ such that $n = \mathrm{m} + \sum_{\mathrm{i}=1}^{\mathrm{m}} \mathrm{R}_\mathrm{i}$. The terminated time T is also fixed beforehand with the integers **k** and **m** are pre-fixed such that $\mathrm{k} < \mathrm{m}$. In general, at the time of the i^{th} failure, R_i units will be removed randomly from the remaining surviving units $\mathrm{S}_\mathrm{i} = \mathrm{n} - \mathrm{i} - \sum_{\mathrm{j}=1}^{\mathrm{i}-1} \mathrm{R}_\mathrm{j}$, where $i \in [1, \mathrm{m}]$. Thus, we have three scenarios:

1. If the time of the m^{th} failure occurs before the time point T, then the experiment will stop at the time point $X_{m:m:n}$ and all the remaining surviving units $R_m = n - m - \sum_{j=1}^{m-1} R_j$ will be removed.
2. If the m^{th}failure does not occur before the time point T and only $\boldsymbol{k}$ failures occur after the time point T, where $X_{m:m:n} > T$. Then, at the time point $X_{k:m:n}$ the experiment terminates, and all the remaining surviving units $R_k = n - k - \sum_{j=1}^{k-1} R_j$ will be removed.
3. If the m^{th} failure does not occur before the time point T and only D failures occur at the time point T, where $X_{k:m:n} < T < X_{m:m:n}$. Then, at the time point $X_{D:m:N}$ the experiment terminates, and all the remaining surviving units $R_T^* = n - D - \sum_{j=1}^{D} R_j$ will be removed.

Thus, given a generalized progressive hybrid censored sample, the likelihood function for the three different cases can be written in a unified form as follows:

$$\mathrm{L}(\underline{X}\,; \theta) = \mathrm{C} \prod_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{f}(\mathrm{x}_\mathrm{i})\, [1 - \mathrm{F}(\mathrm{x}_\mathrm{i})]^{\mathrm{R}_\mathrm{i}} [1 - \mathrm{F}(\mathrm{T})]^{\mathrm{R}_\mathrm{T}^* \delta}, \quad (4)$$

where $C = \prod_{i=1}^{n} \sum_{j=i}^{m} (R_j + 1)$,

$$\mathrm{n} = \begin{cases} \mathrm{m}, & \delta = 0 \quad , \quad \text{if} \quad \mathrm{X}_{\mathrm{k:m:n}} \leq \mathrm{X}_{\mathrm{m:m:n}} < \mathrm{T} \\ \mathrm{k}, & \delta = 0 \quad , \quad \text{if} \quad \mathrm{T} < \mathrm{X}_{\mathrm{K:m:n}} \leq \mathrm{X}_{\mathrm{m:m:n}} \,, \\ D, & \delta = 1 \quad , \quad \text{if} \quad \mathrm{X}_{\mathrm{k:m:n}} < \mathrm{T} < \mathrm{X}_{\mathrm{m:m:n}} \end{cases}$$

where $\underline{X} = (\mathrm{X}_1, \mathrm{X}_2, \dots, \mathrm{X}_\mathrm{n})$ and R_T^* is the number of surviving units that are removed at the stopping time $\mathrm{T}^* = \max\{\mathrm{X}_{\mathrm{k:m:n}}, \min\{\mathrm{X}_{\mathrm{m:m:}n}, \mathrm{T}\}\}$.

The GPHCS has been applied for some distributions, such as the Weibull distribution, see [9], inverse Weibull distribution, see [22], the exponential distribution, see [8], Rayleigh distribution, see [7]. Several lifetime distributions have been studied see [22-25].

2 Estimation Methods

2.1 Kernel Method

We propose a simple and tractable algorithm for estimating the distribution parameters based on the bivariate kernel density estimate for the function $g(\boldsymbol{\alpha}, \boldsymbol{\beta})$ with support on $(0, \infty)$, which is defined as follows:

$$\hat{g}(\alpha, \beta) = \frac{1}{nh_1h_2} \sum_{i=1}^{n} K\left(\frac{\alpha - \alpha_i}{h_1}, \frac{\beta - \beta_i}{h_2}\right), \quad (5)$$

$h_i, i = 1,2$ are the bandwidths or smoothing parameters, which chosen such that $h_i \to 0$ and $\boldsymbol{nh_i} \to \infty$ as $n \to \infty$, where n is the sample size. The optimal choice for h_i which minimizes the mean squared errors is $h_i = 1.06 S_i n^{-0.2}$, S_i the sample standard deviations and the optimal choice for the kernel function $K(.,.)$ can be used as the bivariate standard normal distribution for the parameters α and β. The kernel estimation method has been derived as follows:

1. Generate a random sample $X = (x_1, x_2, \dots, x_n)$ from the parent distribution.
2. Bootstrapping with replacement **n** samples from the random sample in step 1 as follows:
   $X_1 = (X_{11}, \dots, X_{1n}), X_2 = (X_{21}, \dots, X_{2n}),$
   $\dots, X_n = (X_{n1}, \dots, X_{nn}).$
3. For each sample in step 2, find the MLEs for α and β. Thus, we get the following random variables: $A = (\hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_n)$ and $B = (\hat{\beta}_1, \hat{\beta}_2, \dots, \hat{\beta}_n)$:

Integrate the density function (5) with respect to θ from $\boldsymbol{\theta}_0$ to $\boldsymbol{\theta}_1$ for $\theta = (\alpha, \beta)$ as follows:

$$\int_{\theta_0}^{\theta_1} g(\theta) d\theta = \frac{1}{nh} \sum_{i=1}^{n} \int_{\theta_0}^{\theta_1} K\left(\frac{\theta - \theta_i}{h}\right) d\theta$$

Thus,

$$G(\theta_1) - G(\theta_0) = \frac{1}{nh} \sum_{i=1}^{n} \int_{\theta_0}^{\theta_1} K\left(\frac{\theta - \theta_i}{h}\right) d\theta$$

$$G(\theta_1) - G(\theta_0) = [\gamma - \frac{1}{nh} \sum_{i=1}^{n} \int_{0}^{\theta_0} K\left(\frac{\theta - \theta_i}{h}\right) d\theta]$$

$$= [n\gamma - \sum_{i=1}^{n} \int_{\frac{-\theta_i}{h}}^{\frac{\theta_0 - \theta_i}{h}} K(y)\, dy]/n$$

where $\gamma \in U(0,1)$.

Let $W(\frac{-\theta_i}{h}, \frac{\theta_0 - \theta_i}{h}) = \int_{\frac{-\theta_i}{h}}^{\frac{\theta_0 - \theta_i}{h}} K(y)\, dy$

Thus,

$$G(\theta_1) - G(\theta_0) = \frac{1}{n}\left[n\gamma - \sum_{i=1}^{n} W(\frac{-\theta_i}{h}, \frac{\theta_0 - \theta_i}{h})\right]. \quad (6)$$

It is known that the second order approximation of the first derivative, which is defined by the centered differencing, can be written as follows:

$$\frac{\mathrm{dG}(\theta_m)}{\mathrm{d}\theta} = \frac{\boldsymbol{G(\theta_1)-G(\theta_0)}}{\theta_1-\theta_0} = \mathrm{g}(\theta_\mathrm{m}), \qquad (7)$$

for $\theta_0 < \theta_\mathrm{m} < \theta_1$.
From (6) and (7) we get the integral equation

$$\tilde{\theta}_1 = \tilde{\theta}_0 + C\left[n\gamma - \sum_{i=1}^{n} W\left(\frac{-\theta_i}{h}, \frac{\theta_0-\theta_i}{h}\right)\right],$$

Thus, for n =1, 2,3, … the recurrence relation can be written as follows:

$$\tilde{\theta}_{n+1} = \tilde{\theta}_n + C\left[n\gamma - \sum_{i=1}^{n} W\left(\frac{-\theta_i}{h}, \frac{\theta_n-\theta_i}{h}\right)\right]. \quad (8)$$

The convergence of (8) is guaranteed by the condition $0 < C \le \frac{h}{nL_1}$, where $L_1 = K(0)$.

2.2 Picard's Method

Theoretically, it is known that the traditional log-likelihood function, $\mathrm{H}(\theta;\mathrm{x})$, depends on the unknown parameter $\theta = (\alpha,\beta)$ and the sample data x, which can be used to derive the MLE $\hat{\theta}(x)$ of θ, by solving the stationary equation $\frac{\partial \mathrm{H}(\theta;\mathrm{x})}{\partial\theta}|_{\hat{\theta}(x)} = 0$. Thus, based on the dependence of the MLE on the sample data, we can apply the implicit function theorem to the stationary equation by taking the total derivative with respect to any $x \in X$, with considering all partial derivatives as well as the total derivatives are assumed to be evaluated at some known value of $\hat{\theta}(x_0) = \theta_0$, we obtain the following equation:

$$\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\partial \mathrm{H}(\theta|\mathrm{X})}{\partial\theta}\right) = \frac{\partial^2 \mathrm{H}(\theta|\mathrm{X})}{\partial\theta\,\partial\mathrm{x}}|_{\theta=\hat{\theta}} + \frac{\partial^2 \mathrm{H}(\theta|\mathrm{X})}{\partial\theta^2}|_{\theta=\hat{\theta}}\frac{\mathrm{d}\hat{\theta}}{\mathrm{dx}} = 0.$$
(9)

Solving (9) we obtain the first derivative of $\hat{\theta}$ with respect to $x \in X$ at $\theta = \hat{\theta}$ as follows:

$$\frac{\mathrm{d}\hat{\theta}(x)}{\mathrm{dx}} = -\left(\frac{\partial^2 \mathrm{H}(\theta|\mathrm{X})}{\partial\theta^2}|_{\theta=\hat{\theta}}\right)^{-1}\frac{\partial^2 \mathrm{H}(\theta|\mathrm{X})}{\partial\theta\,\partial\mathrm{x}}|_{\theta=\hat{\theta}}. \quad (10)$$

Thus, we can write (10) as the following first order ordinary differential 10quation in $\hat{\theta}$:

$$\frac{\mathrm{d}\hat{\theta}(x)}{\mathrm{dx}} = \mathrm{f}(\mathrm{x},\hat{\theta}), \quad \text{at } \hat{\theta}(x_0) = \theta_0\,, \qquad (11)$$

where

$$\mathrm{f}(\mathrm{x},\hat{\theta}) = -\left(\frac{\partial^2 \mathrm{H}(\theta|\mathrm{X})}{\partial\theta^2}|_{\theta=\hat{\theta}}\right)^{-1}\frac{\partial^2 \mathrm{H}(\theta|\mathrm{X})}{\partial\theta\,\partial\mathrm{x}}|_{\theta=\hat{\theta}}.$$

The equation (11) can be solved by a numerical method such as Picard's method for finding the approximate solution given a trial set of parameter values and initial conditions. If the initial conditions are unavailable, they must be appended to the parameter $\hat{\theta}$ as quantity, which the fit is optimized.

The general iteration rule for Picard's method for the parameter α (say) for any $x \in \underline{X}$, can be derived by integrating (11) with respect to x from x_0 to x_1 as follows:

$$\hat{\alpha}(x_1) = \hat{\alpha}(x_0) + \int_{x_0}^{x_1} f(x,\hat{\alpha}_0,\hat{\beta})dx$$
$$= \alpha_0 + \int_{\beta_0}^{\beta_1} f(x,\hat{\alpha}_0,\hat{\beta})\frac{dx}{d\hat{\beta}}d\hat{\beta}$$

Using the differential equation for the parameter β $\frac{\mathrm{d}\hat{\beta}}{\mathrm{dx}} = \mathrm{g}(\mathrm{x},\hat{\alpha}_0,\hat{\beta}_0)$, we get the recurrence relation for the parameter α as follows:

$$\hat{\alpha}_{i+1} = \alpha_0 + \int_{\beta_0}^{\beta_1}\left[\frac{f(x,\hat{\alpha}_i,\hat{\beta})}{g(x,\hat{\alpha}_i\hat{\beta})}\right]d\hat{\beta}\ , \qquad (12)$$

for $i = 0,1,2,3,..$
where β_0 is the initial point and β_1 is the value for which the desired solution should be optimized.
Similarly, the Picard estimator for β can be derived from the following recurrence relation:

$$\hat{\beta}_{i+1}(\mathrm{x}) = \beta_0 + \int_{\alpha_0}^{\alpha_1}\left[\frac{\mathrm{f}(\mathrm{x},\hat{\alpha},\hat{\beta}_i)}{\mathrm{f}(\mathrm{x},\hat{\alpha},\hat{\beta}_i)}\right\}\mathrm{d}\hat{\alpha}, \qquad (13)$$

for $i = 0,1,2,3,..$
The iterative process in (12), and (13) are continued until two consecutive numerical solutions are almost the same, that is if

$$|\hat{\theta}_{i+1} - \hat{\theta}_i| < 10^{-5}, \quad i = 0,1,2,3,..\,.$$

It is important to notice that the Picard's method has been used for estimating the three-parameter Burr-XII distribution and the Pareto-II distribution, see [24, 25].

2.3 Bayes' Method

In this section, the Bayes estimators for the parameters α and β will be derived using the informative gamma and non-parametric kernel priors.

2.3.1 Informative Gamma Prior

We consider the unknown parameters α and β have independent gamma prior distributions with the joint probability density function, which is given by:

$$\mathrm{h}(\alpha,\beta) \propto \alpha^{\mathrm{a}-1}\beta^{\mathrm{c}-1}\mathrm{e}^{-\mathrm{b}\alpha-\mathrm{d}\beta}, \qquad (14)$$

where the hyperparameters $\mathrm{a},\mathrm{b},\mathrm{c}$ and d are assumed to be known non-negative real numbers and chosen to reflect the prior belief about the unknown parameters.

2.3.2 Informative Kernel Prior

For deriving the kernel prior, we introduce the bivariate kernel density estimator for the unknown probability density function $g(\alpha,\beta)$ with support on $(0,\infty)$, which is defined as

$$\hat{\mathrm{g}}(\alpha,\beta) = \frac{1}{\mathrm{nh_1h_2}}\sum_{\mathrm{i}=1}^{n}\mathrm{K}\left(\frac{\alpha-\alpha_\mathrm{i}}{\mathrm{h_1}},\frac{\beta-\beta_\mathrm{i}}{\mathrm{h_2}}\right), \qquad (15)$$

where h_i, for i=1,2 is called the bandwidth or smoothing parameter, which is chosen such that $h_i \rightarrow$

0 and $nh_i \to \infty$ as $n \to \infty$. The role of the bandwidth h_i is to scale our kernels, if h_i is large, the density estimate could be too smooth, and if it is small, the estimate could be too variable. Unfortunately, the choice of h_i is the main problem of the kernel, where the optimal h_i is not known in general, but for a large amount of data, the mean integrated squared error of $\hat{g}(\alpha, \beta)$ is minimized when $h_i = 1.06 S_i n^{-0.2}$, S_i is the sample standard deviation of the samples. Based on the properties of the maximum likelihood estimates (MLEs) for the parameters, which converge in probability to the original parameters.
The kernel prior has been used in [1] and [22].

Thus, using the joint priors (14) and (15) with the likelihood function of the GPHCS (4), the posterior density function for the parameters α and β can be written in a unified form as follows:
$\mathrm{f}(\alpha, \beta | \underline{\mathrm{x}}) = K l(\alpha, \beta) \mathrm{L}(\bar{\mathrm{X}}; \theta)$, where

$$l(\alpha, \beta) = \mathrm{h}(\alpha, \beta) \hat{\mathrm{g}}(\alpha, \beta)$$
$$\propto \alpha^{a-1} \beta^{c-1} e^{-b\alpha - d\beta} \hat{g}_1^{p_1}(\alpha) \hat{g}_2^{p_2}(\beta),$$

is the general prior distribution function with the following special cases:

i. $p_1 = p_2 = 0$ for the informative prior (14),
ii. $p_1 = p_2 = 1$, $a = c = 1$, and $b = d = 0$ for the kernel prior (15),

Thus, the posterior density can be written as follows:

$$\mathrm{f}(\alpha, \beta | \underline{X}) = C \hat{g}_1^{p_1}(\alpha) \hat{g}_2^{p_2}(\beta) \alpha^{n+a-1} \beta^{-n+c-1}$$
$$\times \exp\{-\alpha b - \beta d - \textstyle\sum_{i=1}^{n} \ln(1 + x_i/\beta)\}$$
$$\times \exp\{-\alpha[\textstyle\sum_{i=1}^{n}(1 + R_i) \ln(1 + x_i/\beta) + \delta R_T^* \ln(1 + T/\beta)]\} \qquad (16)$$

3 Simulation Study

In this section we study the performance of the kernel, Picard and Bayes methods based on an extensive simulation study, through two criteria the average bias (AVB) and the mean squared error (MSE) as given , respectively, by:

$$\mathrm{AVB} = \tfrac{1}{\mathrm{L}} \textstyle\sum_{i=1}^{L} |\hat{\theta}_i - \theta|, \quad \mathrm{MSE} = \textstyle\sum_{i=1}^{L} (\hat{\theta}_i - \theta)^2 / L$$

$\hat{\theta}$ is the estimate of θ and L is the number of replications.

In our simulation study we choose the distribution hyperparameters of α and β as follows:
$a = c = 5$ and $b = d = 3$, also the values for the parameters have chosen as: $\alpha = (0.75, 1.5)$ and $\beta = (1.5, 2.5)$. Using the above values of the parameters for generating different samples from the Pareto-II distribution with sizes $n = 20, 40$, and 60 to represent small, moderate, and large sizes. To assess the performance of these estimates the AVB and MSEs for each one were calculated using 1000 replications.

From the simulation results in Tables 3, 4, 5 and 6, some of the points are quite clear based on these estimates and the others have been summarized in the following main points:

1. In general, the point estimates for the parameters α and β based on the kernel and Picard methods have the smallest estimated AVB and MSEs values as compared with the estimates based on Bayesian method using the informative gamma and Kernel priors.
2. The estimated values of MSE based on the Kernl prior are smaller than those based on the informative gamma prior.
3. The estimated values of MSE increase as the value of α increases and decrease as the value of β increases.
4. The estimated values of MSE decrease as the termination time of the experiment T and the sample size increase as expected.

As a conclusion, it appears that the point estimates based on the kernel and Picard methods compete and outperform the Bayesian method based on the informative gamma and kernel priors.

4 Real Data Applications

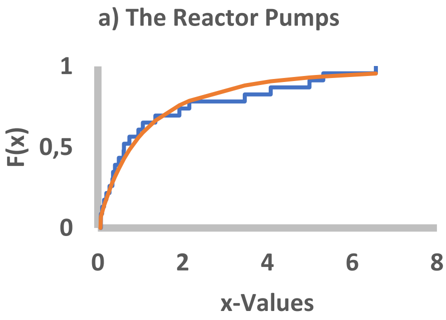
In this section, we studied two real data sets to demonstrate the performance of the proposed methods on the Pareto type-II model in practice and to illustrate that this distribution can be considered a good lifetime model for fitting some new areas of applications. We have fitted these data sets using some goodness of fit tests such as the Kolmogorov-Smirnov (K-S), Anderson-Darling (A-D), and Chi-Square (CH2) tests for significance level of 0.05.

4.1 The Reactor Pumps Data

In this section, real data sets for the secondary reactor pumps have been analyzed to illustrate the proposed methods. An important aspect of the nuclear energy is safety. One of the most severe accidents in nuclear power generation is the loss of coolant, where the re-circulating coolant of the pressurized water reactor may flash into steam. Under such conditions, the reactor cooling pumps become unable to generate the same head as those in the single-phase flow case. Thus, the secondary reactor pump is a feed-water pump that takes feed water from the desecrator storage tank, pressured up by the booster pump, and pushes it into the steam generator through the high-pressure heater. Accordingly, the main feed pump must be a high temperature and high-pressure pump since it requires a head larger than the pressure inside the steam generator. The secondary circulation pump differs slightly in design and has been developed specifically for cooling at higher temperatures. The following data set represents the time between the failures of the secondary reactor pumps. [29] has discussed the classical and Bayesian estimation methods under the Type-II censoring scheme of this data set. The times between failures of 23 secondary reactor pumps are as follows:

2.160, 0.746, 0.402, 0.954, 0.491, 6.560, 4.992, 0.347, 0.150, 0.358, 0.101, 1.359, 3.465, 1.060, 0.614, 1.921, 4.082, 0.199, 0.605, 0.273, 0.070, 0.062, 5.320

We found the Pareto-II model to be a good fit for this dataset, as shown in Table 1 and Figure (1 a). For studying the reliability of these reactor pumps based on this dataset, we found the estimates for the parameters that represent the shape of the failures between pumps using our model to determine the behavior of the failure pumps. We noticed that the proposed method estimates for **α** lies in the interval [2, 2.2] and for **β** lies in the interval [1.5, 2.2] indicate that the above dataset is heavily right-skewed, which means the failure rate decreases with increasing time, see Figure (1b), and that means decreasing the reliability of safety mechanisms with increasing time.



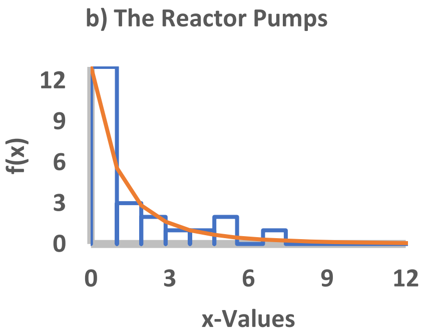


Fig. 1: a) The Empirical CDF and the fitted CDF
. b) The Histogram and the fitted PDF.

4.2 The Repair Times of Airborne Data

Airborne Satellite communications on the move (COTM) is a technology used to provide Satellite communication services to commercial and military aircrafts and UAVS (unmanned aerial vehicles) while they are in motion. This technology becoming increasingly important as the demand for real-time data and connectivity in the aviation industry. [6] used 46 observations for the repair times of airborne communication transceiver as given below:

0.20, 0.30, 0.50, 0.50, 0.50, 0.50, 0.60, 0.60, 0.70, 0.70, 0.70, 0.80, 0.80, 1.00, 1.00, 1.00, 1.00, 1.00, 1.10, 1.30, 1.50, 1.50, 1.50, 1.50, 2.00, 2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30, 3.30, 4.00, 4.00, 4.50, 4.70, 5.00, 5.40, 7.00, 7.50, 8.80, 9.00, 10.3, 22.0, 24.5.

The validity of the Pareto type-II model was checked by using Kolmogorov-Smirnov (K-S) test, Anderson-Darling (A-D) and chi-square tests, see Table 1. It is observed that the calculated values of these tests are less than the critical values and the power of the tests are greater than the significance level which equals to 0.05, see Figure (2 a). This shows that the Pareto-II model provides a good fit to the above data. For studying the reliability of the airborne communication transceiver based on this dataset, we find the estimates for the parameters that represent the shape and scale of the failures between the repaired time using our model to determine the behavior of the airborne communication transceiver. We noticed that the Runge-Kutta and Bayes estimates for **α** lies in the interval [2.3, 3.2] and for **β** lies in the interval [6.5, 7.8] indicate that the above dataset is heavily right-skewed, which means the repaired times decreases with increasing time, see Figure (2 b), and that means the repair times of the airborne communication transceiver are very significant.

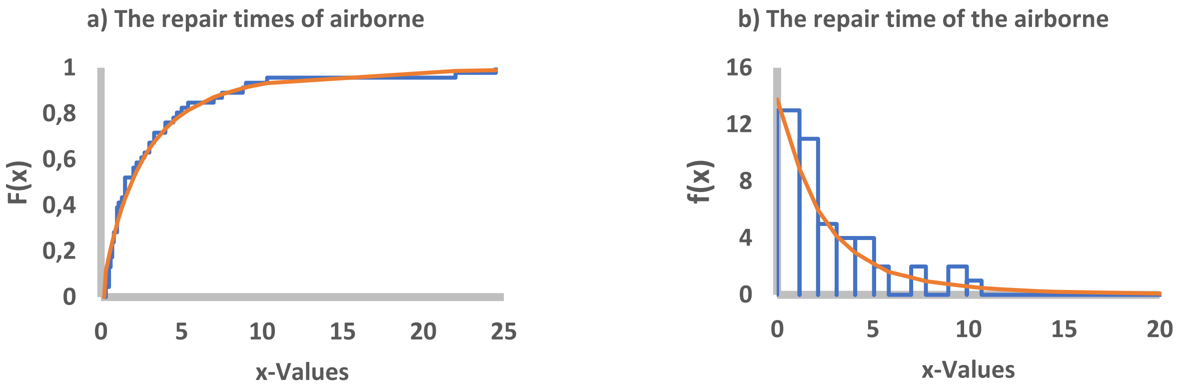


Fig. 2: a) The Empirical CDF and the fitted CDF.
b) The Histogram and the fitted PDF.

Table 1: The critical and the calculated values for the K-S, A-D, and CH2 tests and their powers (p-values). The MLE for the parameters for these data sets have been calculated.

Data	**The Tests**	**Critical values**	**Calculated values**	**p-values**	**MLES**	
					$\hat{\alpha}$	$\hat{\beta}$
Reactor Pumps data N=23	**K-S**	**0.8109**	**0.4784**	**0.7895**	**2.2406**	**2.1669**
	A-D	**0.8131**	**0.2917**	**0.6725**		
	CH2	**9.1989**	**8.7408**	**0.1484**		
Repair time data N=46	**K-S**	**0.8200**	**0.9248**	**0.0322**	**3.2519**	**7.9541**
	A-D	**0.7725**	**0.7326**	**0.0974**		
	CH2	**24.1995**	**8.6155**	**0.1487**		

Table 2: The estimate and the mean squared errors (MSE) for the parameters α and β using kernel, Picard and Bayes methods based on the GHPCS with different values of m and k.

Samples	**T**	**Par**	**Kernel Estimates**		**Picard Estimate**		**Bayes estimates**			
							Gamma prior		**Kernel prior**	
			Estimate	**MSE**	**Estimate**	**MSE**	**Estimate**	**MSE**	**Estimate**	**MSE**
$m = n/2$ and $k = m/2$										
Reactor Pumps data N=23	**0.5**	α	**2.1739**	**0.0044**	**2.0168**	**0.0503**	**1.9247**	**0.0998**	**2.1413**	**0.0986**
		β	**0.0700**	**0.0023**	**0.0329**	**0.0013**	**0.2837**	**0.0284**	**0.8598**	**0.7024**
	5.5	α	**2.1724**	**0.0047**	**2.0169**	**0.0503**	**2.0019**	**0.0569**	**2.1901**	**0.00255**
		β	**0.1002**	**0.0062**	**0.0328**	**0.0012**	**0.3405**	**0.1016**	**0.8026**	**0.6097**
Repair time data N=46	**1.5**	α	**2.999**	**0.0643**	**3.1823**	**0.0049**	**2.4734**	**0.8699**	**2.5833**	**0.8961**
		β	**0.7078**	**0.0077**	**0.8049**	**0.0085**	**0.1859**	**0.0317**	**0.1142**	**0.1129**
	4.5	α	**2.9717**	**0.0789**	**3.1808**	**0.0052**	**3.0694**	**0.8672**	**2.7785**	**0.8844**
		β	**0.1375**	**0.0034**	**0.0805**	**0.0093**	**0.1883**	**0.3251**	**0.1045**	**0.9338**
$m = 3n/4$ and $k = 3m/4.$										
Reactor Pumps data N=23	**0.5**	α	**2.1067**	**0.0018**	**2.1951**	**0.0021**	**2.0342**	**0.02**	**1.7632**	**0.2284**
		β	**0.0450**	**0.0055**	**0.0204**	**0.00014**	**0.1849**	**0.0266**	**1.0187**	**0.9942**
	5.5	α	**2.1101**	**0.0012**	**2.1957**	**0.0021**	**2.0232**	**0.0134**	**1.7370**	**0.2541**
		β	**0.0414**	**0.0039**	**0.0221**	**0.0014**	**0.2108**	**0.0358**	**1.0171**	**0.9908**
Repair time data N=46	**1.5**	α	**2.1959**	**0.0543**	**3.1552**	**0.0025**	**2.5934**	**0.8329**	**2.7535**	**0.7642**
		β	**0.6878**	**0.0057**	**0.7504**	**0.0055**	**0.1759**	**0.0017**	**0.1343**	**0.1125**
	4.5	α	**2.5617**	**0.0529**	**3.0508**	**0.0035**	**3.1594**	**0.8023**	**2.5755**	**0.7524**
		β	**0.1575**	**0.0014**	**0.0705**	**0.0053**	**0.1573**	**0.1125**	**0.1235**	**0.8438**

From the results in Table 1, the Pareto type-II model is a good fit for these data sets as the power of the tests (p-values) is greater than the significance level of the tests and the calculated values are less than the critical values. Figures (1 a, 2 a) display the empirical CDF and the CDF of the Pareto-type-II distribution for these data sets, which confirm the goodness of fit tests. The results in Table 2 indicated that the kernel and Picard estimates of the parameters have MSE values lower than the Bayesian estimates for both datasets. Moreover, the MSE values for the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ decrease as the values of **T, m** and **k** increase.

5 Conclusion

Generally, it is found that the point estimation based on the kernel and Picard methods outperform the Bayesian method under generalized progressive hybrid censored data. Moreover, the kernel and Picard method estimates are unbiased and considerably more efficient than the Bayesian estimates based on informative gamma and kernel priors. As a conclusion in the context of nuclear energy safety, the most severe accidents in nuclear power generation are loss-of-coolant events, where the recirculating coolant in a pressurized water reactor may flash into steam. Thus, studying the proposed real data indicated that the reliability of safety mechanisms is found to decrease with increasing time. Consequently, the reactor coolant with a pressurized water must be done permanently.

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Table 3: The Average bias (ABS) and Mean Squared Errors (MSEs) in parentheses for the parameter α using the Kerel, Picard and Bay es methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=0.75 .

n	m	k	α	β	Numerical Estimations		Bayesian Estimations	
					Kernel Method	**Picard Method**	**Gamma Prior**	**Kernel Prior**
20	10	5	0.75	1.5	0.1048(0.0144)	0.1166(0.0136)	0.1948(0.0417)	0.1872(0.0514)
				2.5	0.1062(0.0145)	0.1151(0.0133)	0.3659(0.1355)	0.1958(0.0573)
			1.5	1.5	0.2525(0.0677)	0.2354(0.0555)	0.8672(0.7534)	0.2820(0.1197)
				2.0	0.2471(0.0652)	0.2313(0.0536)	0.7953(0.6584)	0.2636(0.1055)
		8	0.75	1.5	0.0876(0.0087)	0.1154(0.0133)	0.0877(0.0116)	0.1645(0.0403)
				2.5	0.0868(0.0086)	0.1143(0.0131)	0.2307(0.0577)	0.1668(0.0432)
			1.5	1.5	0.1826(0.0346)	0.2332(0.0544)	0.5939(0.3572)	0.2533(0.0964)
				2.0	0.1844(0.0353)	0.2301(0.0530)	0.8761(0.7694)	0.2449(0.0916)
	15	8	0.75	1.5	0.0871(0.0087)	0.1154(0.0133)	0.0890(0.0125)	0.1640(0.0400)
				2.5	0.0874(0.0087)	0.1142(0.0130)	0.2308(0.0581)	0.1635(0.0423)
			1.5	1.5	0.1848(0.0354)	0.2334(0.0545)	0.5952(0.3589)	0.2554(0.0998)
				2.0	0.1842(0.0352)	0.2301(0.0530)	0.8746(0.7668)	0.2349(0.0849)
		11	0.75	1.5	0.0727(0.0058)	0.1146(0.0131)	0.1380(0.0291)	0.1496(0.0327)
				2.5	0.0739(0.0060)	0.1137(0.0129)	0.1441(0.0280)	0.1436(0.0331)
			1.5	1.5	0.1515(0.0235)	0.2317(0.0537)	0.3651(0.1451)	0.2390(0.0863)
				2.0	0.1511(0.0234)	0.2292(0.0525)	0.7085(0.5062)	0.2261(0.0755)
40	20	10	0.75	1.5	0.0779(0.0067)	0.1165(0.0136)	0.2182(0.0495)	0.1549(0.0343)
				2.5	0.0782(0.0068)	0.1148(0.0132)	0.3790(0.1444)	0.1500(0.0342)
			1.5	1.5	0.1638(0.0275)	0.2347(0.0551)	0.8898(0.7925)	0.2625(0.1052)
				2.0	0.1644(0.0277)	0.2311(0.0534)	0.7854(0.5732)	0.2422(0.0877)
		15	0.75	1.5	0.0663(0.0046)	0.1154(0.0133)	0.0732(0.0085)	0.1380(0.0275)
				2.5	0.0664(0.0046)	0.1142(0.0130)	0.2576(0.0686)	0.1285(0.0243)
			1.5	1.5	0.1328(0.0179)	0.2332(0.0544)	0.6443(0.4172)	0.2396(0.0852)
				2.0	0.1313(0.0175)	0.2302(0.0530)	0.9106(0.8301)	0.2237(0.0737)
	30	15	0.75	1.5	0.0662(0.0046)	0.1154(0.0133)	0.0725(0.0082)	0.1317(0.0258)
				2.5	0.0669(0.0047)	0.1142(0.0130)	0.2594(0.0693)	0.1305(0.0260)
			1.5	1.5	0.1320(0.0177)	0.2332(0.0544)	0.6451(0.4182)	0.2464(0.0899)
				2.0	0.1317(0.0176)	0.2299(0.0529)	0.9098(0.8285)	0.2201(0.0735)
		23	0.75	1.5	0.0595(0.0036)	0.1143(0.0131)	0.1185(0.0215)	0.1141(0.0198)
				2.5	0.0595(0.0036)	0.1136(0.0129)	0.1443(0.0258)	0.1114(0.0183)
			1.5	1.5	0.1152(0.0133)	0.2312(0.0534)	0.3232(0.1116)	0.2225(0.0707)
				2.0	0.1143(0.0132)	0.2287(0.0523)	0.6831(0.4694)	0.1959(0.0576)
60	30	15	0.75	1.5	0.0677(0.0048)	0.1163(0.0135)	0.2175(0.0487)	0.1457(0.0301)
				2.5	0.0670(0.0047)	0.1148(0.0132)	0.3813(0.1459)	0.1333(0.0256)
			1.5	1.5	0.1320(0.0177)	0.2348(0.0552)	0.8895(0.7917)	0.2618(0.0967)
				2.0	0.1331(0.0180)	0.2310(0.0534)	06835(0.5735)	0.2311(0.0781)
		23	0.75	1.5	0.0600(0.0037)	0.1153(0.0133)	0.0593(0.0055)	0.1323(0.0243)
				2.5	0.0597(0.0036)	0.1142(0.0130)	0.2527(0.0654)	0.1119(0.0181)
			1.5	1.5	0.1151(0.0133)	0.2333(0.0544)	0.6195(0.3853)	0.2428(0.0814)
				2.0	0.1151(0.0133)	0.2300(0.0529)	0.8950(0.8016)	0.2117(0.0650)
	45	23	0.75	1.5	0.0592(0.0036)	0.1154(0.0133)	0.0591(0.0056)	0.1276(0.0229)
				2.5	0.0592(0.0036)	0.1142(0.0130)	0.2507(0.0643)	0.1129(0.0185)
			1.5	1.5	0.1155(0.0134)	0.2332(0.0544)	0.6232(0.3899)	0.2412(0.0804)
				2.0	0.1154(0.0134)	0.2300(0.0529)	0.8949(0.8014)	0.2079(0.0637)
		34	0.75	1.5	0.0550(0.0031)	0.1143(0.0131)	0.1116(0.0180)	0.1012(0.0151)
				2.5	0.0553(0.0031)	0.1136(0.0129)	0.1414(0.0234)	0.0978(0.0142)
			1.5	1.5	0.1059(0.0112)	0.2313(0.0535)	0.3254(0.1109)	0.2111(0.0628)
				2.0	0.1057(0.0112)	0.2288(0.0523)	0.6810(0.4656)	0.1908(0.0533)

Table 4: The Average bias (ABS) and Mean Squared Errors (MSEs) in parentheses for the parameter α using the Kernel , Picard and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=1.5.

n	**m**	**k**	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	**Numerical Estimations**		**Bayesian Estimations**	
					Kernel Method	**Picard Method**	**Gamma Prior**	**Kernel Prior**
20	**10**	**5**	**0.75**	**1.5**	**0.0972(0.0128)**	**0.1572(0.0247)**	**0.1948(0.0417)**	**0.3379(0.1157)**
				2.5	**0.0986(0.0129)**	**0.1303(0.0170)**	**0.3659(0.1355)**	**0.3947(0.1570)**
			1.5	**1.5**	**0.2437(0.0633)**	**0.4169(0.1758)**	**0.8672(0.7534)**	**0.9838(0.9686)**
				2.0	**0.2384(0.0609)**	**0.3228(0.1043)**	**0.7943(0.6845)**	**0.8463(0.7684)**
		8	**0.75**	**1.5**	**0.0800(0.0074)**	**0.1198(0.0144)**	**0.0877(0.0116)**	**0.2190(0.0520)**
				2.5	**0.0792(0.0073)**	**0.1038(0.0108)**	**0.2307(0.0577)**	**0.2823(0.0826)**
			1.5	**1.5**	**0.1670(0.0291)**	**0.2903(0.0844)**	**0.5939(0.3572)**	**0.7880(0.6231)**
				2.0	**0.1687(0.0297)**	**0.2577(0.0665)**	**0.8761(0.7694)**	**0.8942(0.8012)**
	15	**8**	**0.75**	**1.5**	**0.0796(0.0074)**	**0.1195(0.0143)**	**0.0890(0.0125)**	**0.2207(0.0530)**
				2.5	**0.0798(0.0075)**	**0.1037(0.0108)**	**0.2308(0.0581)**	**0.2825(0.0830)**
			1.5	**1.5**	**0.1692(0.0299)**	**0.2904(0.0844)**	**0.5952(0.3589)**	**0.7889(0.6245)**
				2.0	**0.1685(0.0296)**	**0.2578(0.0665)**	**0.8746(0.7668)**	**0.8928(0.7988)**
		11	**0.75**	**1.5**	**0.0652(0.0047)**	**0.1051(0.0111)**	**0.1380(0.0291)**	**0.1389(0.0253)**
				2.5	**0.0664(0.0049)**	**0.0967(0.0094)**	**0.1441(0.0280)**	**0.2071(0.0481)**
			1.5	**1.5**	**0.1388(0.0198)**	**0.2617(0.0685)**	**0.3651(0.1451)**	**0.6346(0.4072)**
				2.0	**0.1383(0.0197)**	**0.2265(0.0513)**	**0.7085(0.5062)**	**0.7515(0.5678)**
40	**20**	**10**	**0.75**	**1.5**	**0.0664(0.0049)**	**0.1368(0.0187)**	**0.2182(0.0495)**	**0.3649(0.1340)**
				2.5	**0.0706(0.0056)**	**0.1139(0.0130)**	**0.3790(0.1444)**	**0.4178(0.1751)**
			1.5	**1.5**	**0.1481(0.0226)**	**0.3323(0.1105)**	**0.8898(0.7925)**	**0.3764(0.1045)**
				2.0	**0.1484(0.0227)**	**0.2876(0.0828)**	**0.7946(0.6835)**	**0.3524(0.1342)**
		15	**0.75**	**1.5**	**0.0605(0.0039)**	**0.1115(0.0124)**	**0.0732(0.0085)**	**0.2503(0.0647)**
				2.5	**0.0602(0.0039)**	**0.1009(0.0102)**	**0.2576(0.0686)**	**0.3154(0.1009)**
			1.5	**1.5**	**0.1211(0.0149)**	**0.2825(0.0798)**	**0.6443(0.4172)**	**0.8477(0.7195)**
				2.0	**0.1196(0.0145)**	**0.2364(0.0559)**	**0.9106(0.8301)**	**0.9510(0.9051)**
	30	**15**	**0.75**	**1.5**	**0.0610(0.0039)**	**0.1115(0.0124)**	**0.0725(0.0082)**	**0.2501(0.0645)**
				2.5	**0.0611(0.0040)**	**0.1008(0.0102)**	**0.2594(0.0693)**	**0.3168(0.1017)**
			1.5	**1.5**	**0.1203(0.0147)**	**0.2831(0.0802)**	**0.6451(0.4182)**	**0.8482(0.7203)**
				2.0	**0.1200(0.0146)**	**0.2361(0.0558)**	**0.9098(0.8285)**	**0.9502(0.9036)**
		23	**0.75**	**1.5**	**0.0530(0.0029)**	**0.1003(0.0101)**	**0.1185(0.0215)**	**0.1378(0.0236)**
				2.5	**0.0529(0.0029)**	**0.0911(0.0083)**	**0.1443(0.0258)**	**0.2144(0.0494)**
			1.5	**1.5**	**0.1021(0.0105)**	**0.2324(0.0540)**	**0.3232(0.1116)**	**0.6179(0.3845)**
				2.0	**0.1014(0.0104)**	**0.2070(0.0429)**	**0.6831(0.4694)**	**0.7517(0.5669)**
60	**30**	**15**	**0.75**	**1.5**	**0.0615(0.0040)**	**0.1224(0.0150)**	**0.2175(0.0487)**	**0.3691(0.1368)**
				2.5	**0.0612(0.0040)**	**0.1047(0.0110)**	**0.3813(0.1459)**	**0.4250(0.1810)**
			1.5	**1.5**	**0.1205(0.0148)**	**0.2945(0.0868)**	**0.8895(0.7917)**	**0.3645(0.10452)**
				2.0	**0.1213(0.0150)**	**0.2606(0.0680)**	**0.6758(0.5735)**	**0.3524(0.1352)**
		23	**0.75**	**1.5**	**0.0535(0.0029)**	**0.1030(0.0106)**	**0.0593(0.0055)**	**0.2410(0.0596)**
				2.5	**0.0532(0.0029)**	**0.0948(0.0090)**	**0.2527(0.0654)**	**0.3148(0.1002)**
			1.5	**1.5**	**0.1021(0.0105)**	**0.2507(0.0629)**	**0.6195(0.3853)**	**0.8390(0.7046)**
				2.0	**0.1020(0.0105)**	**0.2198(0.0483)**	**0.8950(0.8016)**	**0.9474(0.8981)**
	45	**23**	**0.75**	**1.5**	**0.0528(0.0029)**	**0.1029(0.0106)**	**0.0591(0.0056)**	**0.2414(0.0599)**
				2.5	**0.0528(0.0029)**	**0.0948(0.0090)**	**0.2507(0.0643)**	**0.3132(0.0991)**
			1.5	**1.5**	**0.1025(0.0106)**	**0.2508(0.0629)**	**0.6232(0.3899)**	**0.8415(0.7088)**
				2.0	**0.1023(0.0106)**	**0.2201(0.0484)**	**0.8949(0.8014)**	**0.9473(0.8979)**
		34	**0.75**	**1.5**	**0.0481(0.0023)**	**0.0952(0.0091)**	**0.1116(0.0180)**	**0.1321(0.0211)**
				2.5	**0.0482(0.0024)**	**0.0877(0.0077)**	**0.1414(0.0234)**	**0.2138(0.0480)**
			1.5	**1.5**	**0.0922(0.0085)**	**0.2198(0.0483)**	**0.3254(0.1109)**	**0.6211(0.3878)**
				2.0	**0.0920(0.0085)**	**0.1950(0.0380)**	**0.6810(0.4656)**	**0.7560(0.5728)**

Table 5: The Average bias (ABS) and Mean Square Errors (MSEs) in parentheses
For the parameter β using the Kernel, Picard and Bayes methods with
m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=0.75.

n	m	k	α	β	Numerical Estimations		Bayesian Estimations	
					Kernel Method	**Picard Method**	**Gamma Prior**	**Kernel Prior**
20	**10**	**5**	**0.75**	**1.5**	**0.1811(0.0349)**	**0.2281(0.2134)**	**0.5066(0.2599)**	**0.4501(0.2063)**
				2.5	**0.2278(0.0541)**	**0.3517(0.4971)**	**0.8407(0.7153)**	**0.6029(0.3687)**
			1.5	**1.5**	**0.1805(0.0348)**	**0.1723(0.4733)**	**0.5020(0.2533)**	**0.4543(0.2149)**
				2.0	**0.2283(0.0544)**	**0.2604(0.1384)**	**0.8336(0.6982)**	**0.6246(0.4048)**
		8	**0.75**	**1.5**	**0.1676(0.0298)**	**0.2572(0.2977)**	**0.5002(0.2579)**	**0.4256(0.1857)**
				2.5	**0.2117(0.0465)**	**0.3846(0.2228)**	**0.8252(0.7006)**	**0.5626(0.3214)**
			1.5	**1.5**	**0.1632(0.0285)**	**0.1693(0.0875)**	**0.4953(0.2490)**	**0.4274(0.1901)**
				2.0	**0.2089(0.0455)**	**0.2849(0.1991)**	**0.8276(0.6966)**	**0.5792(0.3465)**
	15	**8**	**0.75**	**1.5**	**0.1666(0.0294)**	**0.2498(0.3301)**	**0.4951(0.2524)**	**0.4226(0.1831)**
				2.5	**0.2122(0.0468)**	**0.5758(0.3836)**	**0.8280(0.7071)**	**0.5652(0.3243)**
			1.5	**1.5**	**0.1649(0.0290)**	**0.1741(0.2041)**	**0.4986(0.2529)**	**0.4264(0.1884)**
				2.0	**0.2107(0.0463)**	**0.2623(0.0809)**	**0.8298(0.7020)**	**0.5735(0.3376)**
		11	**0.75**	**1.5**	**0.1539(0.0252)**	**0.2983(0.9408)**	**0.4744(0.2343)**	**0.4003(0.1651)**
				2.5	**0.1972(0.0404)**	**0.7714(0.4362)**	**0.7999(0.6722)**	**0.5347(0.2916)**
			1.5	**1.5**	**0.1507(0.0241)**	**0.1741(0.0566)**	**0.4862(0.2441)**	**0.4047(0.1691)**
				2.0	**0.1907(0.0378)**	**0.2844(0.1016)**	**0.8153(0.6895)**	**0.5423(0.3009)**
40	**20**	**10**	**0.75**	**1.5**	**0.1486(0.0236)**	**0.2675(0.1215)**	**0.4933(0.2491)**	**0.4086(0.1700)**
				2.5	**0.1960(0.0400)**	**0.3783(0.4909)**	**0.8170(0.6823)**	**0.5437(0.2991)**
			1.5	**1.5**	**0.1457(0.0229)**	**0.1602(0.0464)**	**0.5006(0.2541)**	**0.4118(0.1749)**
				2.0	**0.1953(0.0399)**	**0.2898(0.9622)**	**0.8286(0.6944)**	**0.5541(0.3160)**
		15	**0.75**	**1.5**	**0.1415(0.0213)**	**0.2317(0.0909)**	**0.4760(0.2375)**	**0.3876(0.1552)**
				2.5	**0.1851(0.0355)**	**0.3895(0.2576)**	**0.7941(0.6650)**	**0.5099(0.2646)**
			1.5	**1.5**	**0.1369(0.0201)**	**0.1694(0.1176)**	**0.4847(0.2419)**	**0.3830(0.1510)**
				2.0	**0.1796(0.0336)**	**0.2659(0.0840)**	**0.8067(0.6701)**	**0.5091(0.2649)**
	30	**15**	**0.75**	**1.5**	**0.1406(0.0211)**	**0.2523(0.3083)**	**0.4729(0.2345)**	**0.3846(0.1527)**
				2.5	**0.1849(0.0355)**	**0.3928(0.3228)**	**0.7889(0.6500)**	**0.5087(0.2628)**
			1.5	**1.5**	**0.1374(0.0202)**	**0.1644(0.0350)**	**0.4877(0.2448)**	**0.3845(0.1523)**
				2.0	**0.1782(0.0331)**	**0.2619(0.0739)**	**0.8080(0.6697)**	**0.5197(0.2774)**
		23	**0.75**	**1.5**	**0.1342(0.0191)**	**0.2895(0.3265)**	**0.4257(0.1968)**	**0.3514(0.1307)**
				2.5	**0.1742(0.0314)**	**0.4648(0.9003)**	**0.7106(0.5443)**	**0.4681(0.2253)**
			1.5	**1.5**	**0.1279(0.0174)**	**0.1695(0.0303)**	**0.4430(0.2087)**	**0.3533(0.1304)**
				2.0	**0.1726(0.0308)**	**0.3000(0.1719)**	**0.7381(0.5777)**	**0.4762(0.2326)**
60	**30**	**15**	**0.75**	**1.5**	**0.1396(0.0208)**	**0.2116(0.1285)**	**0.4909(0.2482)**	**0.3930(0.1574)**
				2.5	**0.1839(0.0352)**	**0.3457(0.2159)**	**0.8014(0.6611)**	**0.5117(0.2649)**
			1.5	**1.5**	**0.1360(0.0199)**	**0.1770(0.2226)**	**0.4908(0.2446)**	**0.3820(0.1498)**
				2.0	**0.1775(0.0330)**	**0.2543(0.0707)**	**0.8277(0.6958)**	**0.5209(0.2774)**
		23	**0.75**	**1.5**	**0.1297(0.0179)**	**0.2257(0.1007)**	**0.4535(0.2188)**	**0.3616(0.1360)**
				2.5	**0.1731(0.0311)**	**0.3718(0.2124)**	**0.7581(0.6067)**	**0.4767(0.2317)**
			1.5	**1.5**	**0.1264(0.0171)**	**0.1572(0.0252)**	**0.4691(0.2281)**	**0.3533(0.1285)**
				2.0	**0.1656(0.0286)**	**0.2634(0.0757)**	**0.7920(0.6565)**	**0.4801(0.2367)**
	45	**23**	**0.75**	**1.5**	**0.1293(0.0178)**	**0.2156(0.0501)**	**0.4477(0.2125)**	**0.3577(0.1328)**
				2.5	**0.1716(0.0305)**	**0.3637(0.1518)**	**0.7515(0.5944)**	**0.4734(0.2281)**
			1.5	**1.5**	**0.1259(0.0169)**	**0.1579(0.0259)**	**0.4789(0.2401)**	**0.3580(0.1327)**
				2.0	**0.1685(0.0295)**	**0.2615(0.0715)**	**0.7818(0.6352)**	**0.4762(0.2319)**
		34	**0.75**	**1.5**	**0.1320(0.0183)**	**0.2760(0.3729)**	**0.3993(0.1750)**	**0.3278(0.1146)**
				2.5	**0.1651(0.0282)**	**0.4333(0.2535)**	**0.6686(0.4915)**	**0.4398(0.2002)**
			1.5	**1.5**	**0.1284(0.0174)**	**0.1668(0.0292)**	**0.4220(0.1911)**	**0.3279(0.1122)**
				2.0	**0.1640(0.0278)**	**0.2919(0.3082)**	**0.7040(0.5310)**	**0.4425(0.2003)**

Table 6: The Average bias (ABS) and Mean Square Errors (MSEs) in parentheses for the parameter β using the Kernel, Picard and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=1.5.

n	**m**	**k**	**α**	**β**	**Numerical Estimations**		**Bayesian Estimations**	
					Kernel Method	**Picard Method**	**Gamma Prior**	**Kernel Prior**
20	**10**	**5**	**0.75**	**1.5**	**0.2532(0.0681)**	**0.1596(0.0255)**	**0.6966(0.4854)**	**0.3873(0.1501)**
				2.5	**0.3791(0.1507)**	**0.2598(0.0675)**	**0.2839(0.0808)**	**0.1844(0.0341)**
			1.5	**1.5**	**0.2560(0.0701)**	**0.1554(0.0241)**	**0.7033(0.4946)**	**0.4550(0.2071)**
				2.0	**0.3857(0.1560)**	**0.2556(0.0653)**	**0.2748(0.0756)**	**0.1518(0.0231)**
		8	**0.75**	**1.5**	**0.1509(0.0240)**	**0.1555(0.0242)**	**0.7963(0.6346)**	**0.4262(0.1820)**
				2.5	**0.2174(0.0490)**	**0.2557(0.0654)**	**0.1725(0.0302)**	**0.1205(0.0147)**
			1.5	**1.5**	**0.1492(0.0237)**	**0.1531(0.0234)**	**0.8091(0.6548)**	**0.5107(0.2609)**
				2.0	**0.2175(0.0492)**	**0.2532(0.0641)**	**0.1570(0.0247)**	**0.0785(0.0062)**
	15	**8**	**0.75**	**1.5**	**0.1500(0.0238)**	**0.1554(0.0242)**	**0.7963(0.6347)**	**0.4263(0.1822)**
				2.5	**0.2177(0.0492)**	**0.2556(0.0654)**	**0.1726(0.0303)**	**0.1206(0.0147)**
			1.5	**1.5**	**0.1507(0.0241)**	**0.1531(0.0235)**	**0.8096(0.6556)**	**0.5112(0.2614)**
				2.0	**0.2196(0.0502)**	**0.2532(0.0641)**	**0.1570(0.0247)**	**0.0785(0.0062)**
		11	**0.75**	**1.5**	**0.1064(0.0119)**	**0.1545(0.0239)**	**0.8674(0.7536)**	**0.4559(0.2089)**
				2.5	**0.1517(0.0237)**	**0.2545(0.0648)**	**0.0955(0.0100)**	**0.0774(0.0063)**
			1.5	**1.5**	**0.1055(0.0117)**	**0.1525(0.0232)**	**0.8907(0.7936)**	**0.5498(0.3024)**
				2.0	**0.1487(0.0228)**	**0.2526(0.0638)**	**0.0706(0.0051)**	**0.0282(0.0008)**
40	**20**	**10**	**0.75**	**1.5**	**0.1050(0.0116)**	**0.1559(0.0243)**	**0.8586(0.7374)**	**0.4762(0.2270)**
				2.5	**0.1651(0.0282)**	**0.2561(0.0656)**	**0.1031(0.0107)**	**0.0775(0.0060)**
			1.5	**1.5**	**0.1142(0.0139)**	**0.1532(0.0235)**	**0.8648(0.7478)**	**0.5419(0.2937)**
				2.0	**0.1680(0.0293)**	**0.2533(0.0642)**	**0.0950(0.0091)**	**0.0483(0.0023)**
		15	**0.75**	**1.5**	**0.0798(0.0066)**	**0.1548(0.0240)**	**0.9588(0.9197)**	**0.5352(0.2868)**
				2.5	**0.1132(0.0131)**	**0.2550(0.0650)**	**0.0121(0.0383)**	**0.0124(0.0003)**
			1.5	**1.5**	**0.0785(0.0065)**	**0.1525(0.0233)**	**0.9681(0.9373)**	**0.6152(0.3786)**
				2.0	**0.1130(0.0131)**	**0.2526(0.0638)**	**0.0140(0.0002)**	**0.0239(0.0006)**
	30	**15**	**0.75**	**1.5**	**0.0800(0.0067)**	**0.1548(0.0240)**	**0.9590(0.9199)**	**0.5354(0.2870)**
				2.5	**0.1137(0.0132)**	**0.2550(0.0650)**	**0.0120(0.0002)**	**0.0125(0.0003)**
			1.5	**1.5**	**0.0792(0.0065)**	**0.1526(0.0233)**	**0.9686(0.9382)**	**0.6158(0.3792)**
				2.0	**0.1129(0.0130)**	**0.2526(0.0638)**	**0.0139(0.0002)**	**0.0239(0.0006)**
		23	**0.75**	**1.5**	**0.0587(0.0035)**	**0.1544(0.0239)**	**0.7694(0.3265)**	**0.6003(0.3626)**
				2.5	**0.0847(0.0073)**	**0.2546(0.0648)**	**0.1035(0.0113)**	**0.0560(0.0035)**
			1.5	**1.5**	**0.0572(0.0033)**	**0.1523(0.0232)**	**0.2362(0.1546)**	**0.7030(0.4945)**
				2.0	**0.0844(0.0072)**	**0.2524(0.0637)**	**0.1273(0.0163)**	**0.1027(0.0106)**
60	**30**	**15**	**0.75**	**1.5**	**0.0793(0.0066)**	**0.1539(0.0237)**	**0.9604(0.9224)**	**0.5439(0.2960)**
				2.5	**0.1143(0.0134)**	**0.2540(0.0645)**	**0.0085(0.0001)**	**0.0092(0.0001)**
			1.5	**1.5**	**0.0801(0.0067)**	**0.1521(0.0231)**	**0.9660(0.9332)**	**0.6078(0.3694)**
				2.0	**0.1136(0.0133)**	**0.2521(0.0636)**	**0.0128(0.0002)**	**0.0178(0.0003)**
		23	**0.75**	**1.5**	**0.0582(0.0035)**	**0.1533(0.0235)**	**0.8586(0.7374)**	**0.6215(0.3866)**
				2.5	**0.0850(0.0073)**	**0.2534(0.0642)**	**0.1031(0.0107)**	**0.0635(0.0041)**
			1.5	**1.5**	**0.0578(0.0034)**	**0.1517(0.0230)**	**0.8648(0.7478)**	**0.7005(0.4908)**
				2.0	**0.0838(0.0071)**	**0.2518(0.0634)**	**0.0950(0.0191)**	**0.0973(0.0095)**
	45	**23**	**0.75**	**1.5**	**0.0580(0.0034)**	**0.1533(0.0235)**	**0.9588(0.9197)**	**0.6211(0.3862)**
				2.5	**0.0846(0.0072)**	**0.2534(0.0642)**	**0.0121(0.0002)**	**0.0645(0.0043)**
			1.5	**1.5**	**0.0578(0.0034)**	**0.1518(0.0230)**	**0.9681(0.9373)**	**0.7005(0.4907)**
				2.0	**0.0846(0.0073)**	**0.2518(0.0634)**	**0.0140(0.0002)**	**0.0973(0.0095)**
		34	**0.75**	**1.5**	**0.0488(0.0024)**	**0.1531(0.0235)**	**0.9590(0.9199)**	**0.6941(0.4836)**
				2.5	**0.0713(0.0051)**	**0.2532(0.0641)**	**0.0120(0.0002)**	**0.1286(0.0168)**
			1.5	**1.5**	**0.0483(0.0024)**	**0.1516(0.0230)**	**0.9686(0.9382)**	**0.7946(0.6317)**
				2.0	**0.0714(0.0051)**	**0.2517(0.0633)**	**0.0139(0.0356)**	**0.1717(0.0295)**