

On assess of uncertainty in chaos

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Abstract: We propose an uncertainty principle for chaos, focusing on two key characteristics: alpha unpredictability and Lorenz sensitivity. This principle outlines a limitation on the relationship between two infinite sequences that underpin these concepts. It is helpful for both deterministic and stochastic dynamics, marking a significant step toward integrating these two fields. Our initial progress in this area was achieved through research on Markov chains utilizing alpha labeling. Additionally, we offer suggestions on how this principle can be used to assess the degree of chaos in specific processes. We also outline open questions regarding the relationships among various types of chaos, including a modification of the recurrence theorem.

Key-Words: - Chaos, Uncertainty principle, Alpha unpredictability, Lorenz sensitivity, Poisson stability, Ultra Poincaré chaos.

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1. The principle as a unity of opposites

Heraclitus was the first to articulate the law of the unity of opposites. Opposites exist within our minds and serve as tools for analysis. This law is connected to N.Bohr's principle of complementarity and W.Heisenberg's uncertainty principle. Consequently, it is helpful to view opposites as related to chaos. The most common manifestation of this unity is found in the concepts of divergence and convergence. In various types of chaos, these concepts are evident in Lorenz sensitivity [1, 2, 3, 4, 5, 6]. Ultra Poincaré chaos represents a new phenomenon in complexity [7, 8], characterized by alpha unpredictability and Poisson stability, which pertain to divergence and convergence, respectively.

Drawing on ideas from conventionalism [9], one could argue that opposites arise from the observation of uncertainty, and any form of it is a result of opposition. Therefore, it is essential to recognize that position and momentum are opposites, as described by Heisenberg's law. It is not surprising that the uncertainty principle related to chaos, which is suggested for discussion, is grounded in these obvious opposites of convergence and divergence.

In our research on chaos, we have identified uncertainty in dynamical chaos by employing the following inequalities:

$$h_\alpha \leq s_n d(p_n, p) \leq h_\beta, \quad (1)$$

where h_α and h_β are positive numbers, s_n represents the time sequence of divergence, p_n denotes the converging space sequence, and d is a metric. This formula has been presented in the book [7]. We believe that these inequalities will provide insights into chaos that are akin to those found in quantum mechanics. Related topics concerning alpha

unpredictability, ultra Poincaré and Lorenz chaos, artificial neural networks, differential equations, deterministic and stochastic dynamical systems [7], and the delta synchronization [8] can be explored through this proposal. In the paper, the unison of ultra Poincaré chaos for coupled models is shown, while generalized synchronization [10, 11] is absent.

The concept of the alpha unpredictable point serves as the foundation for ultra Poincaré chaos [7]. A point p in a dynamic system is considered alpha unpredictable if there exists a positive number e and sequences t_n and s_n that both increase toward infinity. In this context, let $p_n = f(t_n, p)$ converge to p , while the points $f(s_n, p_n)$ maintain a distance greater than e from $f(s_n, p)$ for all n .

The uncertainty inherent in alpha unpredictability arises from the inability to determine the infinite sequence of pairs s_n, p_n for $n = 1, 2, \dots$. Consequently, chaos cannot be completely observed in experiments and simulations. As we observe closer approximation points $f(t_n, p)$, the moments of divergence s_n become larger. Conversely, as we fine-tune a large moment s_n , identifying the corresponding p_n becomes increasingly challenging. A more precisely determined s_n results in less accuracy for p_n . Therefore, we grapple with this uncertainty, and the sequences of points p_n and moments s_n are just as theoretical as position and momentum of Heisenberg's uncertainty principle. Thus, we use the term "uncertainty" to imply that chaos, under our conditions, can only be established if complete information about the sequences t_n, s_n and p_n is collected. More specifically, we view $s_n, p_n, n = 1, 2, \dots$ as a chaotic pair. This discussion relates to alpha unpredictability in the context of ultra

Poincaré chaos and the sensitivity characteristic of Lorenz chaos. Hence, we address the principle of uncertainty in chaos as a phenomenon. In the following sections, we will elaborate on why this principle applies to all types of chaos. Suppose we accept that we are dealing with the uncertainty of chaos concerning the sequences. In that case, it can be recognized that the inequalities in (1) establish a principle of uncertainty since they provide limits (estimates) for the phenomenon, which is encompassed within numerical laws. Algorithms to calculate these sequences have already been developed in [7, 8]. Sensitivity is a source of uncertainty in Lorenz chaos, which is a key ingredient in the sophisticated dynamics. In the following discussion, we will apply the sequential approach to defining sensitivity. This approach is rooted in the ideas of the German mathematician E. Heine and is used to understand limits and continuity of functions. We have previously utilized this approach in our papers to examine alpha unpredictability and to define Lorenz sensitivity for the first time in the literature. This approach is highly productive for our principal discussion. A motion $f(q, t)$ is sensitive at a point p provided that there exists a positive number e , sequences of points p_n and moments $s_n, n = 1, 2, \dots$, such that $p_n \rightarrow p$, and the distance between $f(p, s_n)$ and $f(p_n, s_n)$ is larger than e for all n . We have to look for points p_n arbitrarily close to a point p to learn sensitivity, but this makes the less observable moments, where the divergence occurs, less apparent. Moreover, if one tries to determine large moments of the divergence, then the precision for the location of corresponding points p_n decreases, and vice versa. Finally, the infinite couples $s_n, p_n, n = 1, 2, \dots$ can not be indicated ever. Thus, the principle of the uncertainty for chaos is based on Lorenz's sensitivity. The separation number e can be an uncertainty constant for ultra Poincaré and Lorenz chaos, since the smallness of the error for the initial point does not affect the separation size.

The discussions above suggest that deterministic chaos, like quantum mechanics, is related to the profound nature of reality. Furthermore, we believe that, based on the principle of chaos, one can identify analogs of wave functions and the Schrodinger equation for deterministic phenomena. We are confident that there are additional relationships between spatial and temporal sequences that should be acknowledged within the paradigm of uncertainty. What is the philosophical conclusion of our findings? The greater the uncertainties in dynamics, the more valuable they become. Uncertainty is a key factor that indicates a theoretical phenomenon is worth

investigating, as it embodies the unity of opposites. Therefore, discovering the uncertainty inherent in chaos is a significant achievement.

2. More arguments for the principle

Based on our findings, we assert that ultra Poincaré chaos represents the most advanced form of irregular dynamics in the field of science. While other types of chaos rely on dense, closed orbits as a foundational framework, we focus on unclosed, Poisson stable orbits, each of which is dense within the chaotic region. This implies that the alternatives to ultra Poincaré chaos do not depend on the principle to the same extent that alpha unpredictability does. This could potentially explain the uncertainty observed in our studies.

Next, we will address the necessity of the positive constants, h_α and h_β . One might inquire whether chaos can exist without these constraints. Below, we present two arguments to address this question.

(a) Assume that a constant h_β for chaotic dynamics cannot be identified as a finite number. In this case, there is a possibility that a pair s_n, p_n , for $n = 1, 2, \dots$, satisfies $s_n \rightarrow \infty$, while the distance $d(p_n, p)$ remains bounded away from zero. This scenario indicates that the pair cannot be chaotic. This situation highlights the necessity of the constant to rule out the presence of chaos. The final remark about the constant is that the larger h_β is, the closer the dynamics are to being non-chaotic and regular. Thus, one can say that h_β is a protection of regularity constant.

(b) Now, consider the case where there is no appropriate lower boundary, h_α . Then, you can find as a chaotic pair s_n, p_n , where s_n is a bounded sequence and $d(p_n, p)$ approaches zero. This contradicts the principle of continuous dependence on initial values in dynamical systems. It is clear that the closer h_α is to zero in (1), the more we advance toward stochastic discrete dynamics and quantum chaos. That is why, one can call h_α as a protection of randomness constant.

Therefore, we assert that for chaos to exist, the relationships expressed in equations (1) must hold true. More specifically, this means that one should be able to extract a subsequence from a chaotic pair that satisfies the principle. In other words, for a dynamical system $f(t_n, p)$ to be considered chaotic, it is essential that there exist sequences $s_n \rightarrow \infty$ and $p_n \rightarrow p$ such that the principle in equation (1) is upheld. The discussion on subsequences implies that one can say about statistical nature of the principle. A violation of the uncertainty undermines guarantees of chaos.

Let us add more uncertainty to the dynamics by implication of the principle.

a) From formula (1), it implies that

$$r_\alpha^n = \frac{h_\alpha}{s_n} \leq dp_n, p \leq \frac{h_\beta}{s_n} = r_\beta^n. \quad (2)$$

The relations (2) imply that corresponding to the divergence moments s_n points p_n are in the strip of the width $r_\beta^n - r_\alpha^n = \frac{h_\beta - h_\alpha}{s_n}$. The width tends to zero as $n \rightarrow \infty$. Thus, the points are not out of the strip, not r_α^n in the disc with radius r_β^n . Since the radius is small, this increases the uncertainty.

b) Similarly, we have that

$$R_\alpha^n = \frac{h_\alpha}{dp_n, p} \leq s_n \leq \frac{h_\beta}{dp_n, p} = R_\beta^n. \quad (3)$$

The last formula shows that for $p_n \rightarrow p$ sequence s_n is placed in the strip, whose width increases infinitely, but there are two intervals, which should be free of the moments. It is obvious that the strip's existence also increases the uncertainty and chaos.

3. How to apply the principle to measure the degree of chaos

The uncertainty principle has not been previously discussed in the literature, despite various types of chaos, such as Devaney chaos and Li-Yorke chaos, having been examined for a considerable time. The reason for this oversight seems to be the periodic motions, which are a key component in the definitions of these chaotic systems. The presence of infinitely many closed orbits in bounded domains supports the validity of inequalities (1) for these systems, and they are always present implicitly. We propose that proving the uncertainty principle should be regarded as an open problem for these types of chaos. Additionally, another open problem is to establish a connection between the Feigenbaum constant, which is derived from the period-doubling route to chaos, and the calculation of the uncertainty constant. In the context of ultra Poincaré chaos, we do not observe any prerequisites for the uncertainty principle, suggesting it may be a fundamental property of chaos.

Finally, we believe that under specific conditions, the constants associated with these chaotic systems may correlate with those defined by M. Planck and E. Hubble.

Now consider more general aspects related to (1), specifically estimations suitable for all types of complex dynamics.

(a) An increase in h_β indicates that the divergence moments s_n are tending to infinity at a much faster rate than the points p_n approach the issue point p .

This suggests that the degree of irregularity is decreasing, meaning that chaos is diminishing. Geometrically, this could imply that the widths of the bands in the simulations of the Rossler and Chua models, which enclose the trajectories, are becoming wider.

(b) As the value of h_α approaches zero, it indicates that the continuous dependence on the initial conditions in the dynamics becomes weaker. Ultimately, this can lead to stochastic processes, such as Bernoulli schemes or Markov chains, which we have previously studied, where the dependence on initial conditions disappears.

(c) To achieve a greater degree of qualitative chaos in a dynamical system, the difference between h_β and h_α must not be too large, and h_α should be separated from zero. We can examine the Chua attractor and the Rossler band in this context. By analyzing the visual representations of these sets in Figures 1 and 2, we can infer that the chaos present in the attractor is stronger than that in the band. To validate our hypothesis, further simulations are needed.

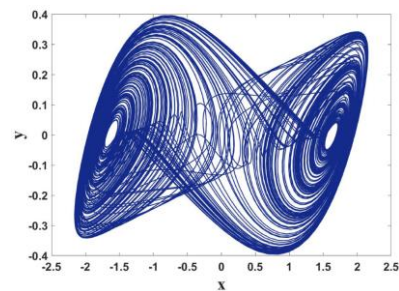


Fig.1: Chua's attractor has specific features that ensure the existence of the fundamental constants.

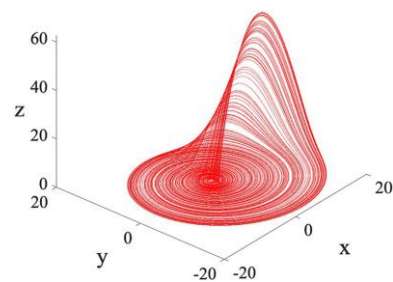


Fig.2: Rossler's band has qualitative characteristics that are sufficient to illustrate the principle of uncertainty.

4 What are the consequences for science?

It is clear that the mentioned relations can be valuable for the qualitative theory of chaos and its manifestation in other fields of science. Potential issues are discussed below.

It is essential to highlight that the recurrence theorem [12] specifically addresses Poisson stability and is recognized as a fundamental result in ergodicity. A key assumption of the theorem is the preservation of volume. We believe that this does not contradict the concept of alpha unpredictability; however, we do not think that alpha unpredictability can be derived solely from this assumption. Therefore, the recurrence theorem can be adapted to incorporate alpha unpredictability, along with the uncertainty inherent in chaos and ergodicity. As a result, both convergence and divergence will be observed within the recurrence, allowing the theorem to reach its complete form. Additionally, it is important to note that no previous research has considered what happens to trajectories between moments of convergence, but our study begins to address this question.

We assert that this principle is essential to any form of chaos exhibiting Lorenz sensitivity. This includes Li-Yorke and Devaney chaos, as well as those described by period-doubling bifurcation diagrams. Consequently, the principle serves as a necessary condition, and the definitions are not complete without it. It is of significant interest to confirm the presence of uncertainty based on specific characteristics such as dense closed orbits and bifurcation diagrams. We believe that the dynamical reasons underlying the Feigenbaum constant can be useful for the uncertainty constants outlined.

The principle of uncertainty provides a necessary complement to the question of what deterministic chaos is. It is not enough to define the existence of converging and diverging sequences; these sequences must also satisfy specific inequalities as outlined in (1). In our view, the definitions of Lorenz chaos and ultra Poincare chaos fail to clearly distinguish deterministic chaos from regular dynamics on one side and stochastic processes on the other. By incorporating the principle of uncertainty, we can confidently assert what qualifies as chaos.

Alpha labeling is a new mathematical structure developed in our research to study complexities [7]. In our papers, this structure has been applied to chaos, fractals, and random processes. We believe that uncertainty constants can be determined for dynamics that are reduced to alpha labeling.

We hope that the principle of uncertainty within the dynamics of alpha labeling will be helpful in understanding uncertainty in stochastic processes with finite state spaces, such as Markov chains, and that it can later be extended to continuous random

processes. Additional applications are anticipated in the realm of quantum chaos, where relative restrictions stem from M. Planck's constant. If the qualitative characteristics introduced in our studies are applied to deterministic, stochastic, and quantum processes, it could represent a formal unification of scientific disciplines.

A key problem is to clarify the selection of uncertainty constants. Are they unique to each chaotic dynamic, or can they be derived from various fields such as chemistry, biology, or physics? How do these constants relate to quantum or cosmological constants? Is it possible to calculate them based on the Feigenbaum constant when considering the period-doubling route to chaos?

It is crucial to understand that there is no Bayesian probability within this principle since there is no uncertainty in the probability itself. Nevertheless, our study considers stochastic motions that include deterministic chaotic elements. This pertains to the sequences of convergence and divergence that we introduce with respect to alpha unpredictability.

To better reflect realistic dynamics, we propose using scenarios that can be developed from abstractions in Conway's Game of Life [13] and the dynamics of multiple quantum trajectories as analyzed by Feynman [14], particularly in relation to chaos unpredictability.

In conclusion, we express strong confidence that, based on alpha unpredictability and alpha labeling [7], the principle of uncertainty in relation to chaos can make a significant contribution to research on the complexities of the world.

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