The Mechanism of Resonant Capture and Acceleration of Electrons by the Field of a Plane Electromagnetic Wave in Vacuum

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Abstract: The mechanism of electron capture and acceleration by the field of a plane electromagnetic wave in a vacuum is described. The conditions for such capture are formulated. It is shown that the main physical mechanism of resonance occurrence is oscillations of particles in the wave field with the subsequent process of phase synchronization. This mechanism is manifested in the Adler equation, which appears in the phase relations between the wave and particles and is the main component of the conditions of resonant interaction of particles and waves. A comparative analysis of the new resonant conditions with the known ones is given. The numerical results of acceleration of charged particles are in good agreement with the analytical ones.

Key-Words: acceleration, phase synchronization, wave-particle interaction, resonances

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1 Introduction

Wave-particle interactions underlie plasma physics and accelerator theory. Current trends in accelerator theory are to increase the field strengths of electromagnetic waves that interact with accelerated particles. This allows for a reduction in the size of modern accelerators. However, at field strengths exceeding 10^4 – 10^5 V/cm (depending on the duration), breakdown occurs in accelerator elements, and the material of these elements is destroyed. Therefore, current efforts by physicists are aimed at finding ways to implement mechanisms and schemes for accelerating particles in plasma or, even better, in a vacuum. There are many works devoted to the study of the features of the particle acceleration process in plasma and in a vacuum.

The simplest and most fundamental scheme of interaction of electromagnetic waves in a vacuum is the problem of particle dynamics in the field of plane electromagnetic waves in a vacuum. Many ideas about particle dynamics in such a scheme are based on the results of works [1-5]. See also [6-8]. These works consider the dynamics of charged particles in the field of a plane electromagnetic wave in a vacuum. Moreover, the wave vector of the wave has only one component $\vec{k} = \{0, 0, k_z\}$. The main result of these works is the formation of the idea that the effective interaction of charged particles with the field of such a wave and the acceleration of particles can occur only over a limited time interval. The process of particle acceleration after a certain time interval is replaced by the deceleration of particles. In the frame of reference in which the particle is at rest on average, the trajectory of the particles describes a closed figure - an eight [6].

In the works [9-10] it was found that if a plane electromagnetic wave has a wave vector that contains several components that make it up, for example $\vec{k} = \{k_x, 0, k_z\}$, then even at $k_x << k_z$ the dynamics of the particles can change qualitatively. Conditions appear that allow the charged particles to be captured and accelerated. In these works, not only was the fact of the capture and acceleration of charged particles by the field of a plane electromagnetic wave in a vacuum discovered, but the conditions for such capture and acceleration (resonant interaction conditions) were also found. The numerical results confirmed the authors' considerations.

However, the constructed mathematical model of such capture and acceleration does not describe this process fully enough. In particular, the resonance conditions found do not contain the main bifurcation parameter k_x. This parameter is contained only in the resonance weight. By the concept of resonance weight, we mean the coefficients that are included in the right-hand side of the equations for the momenta and for the energy of particles. This leads to the fact that the dynamics of particle motion determined within the framework of this model may differ significantly from that which follows from the numerical results. The resulting mathematical model in many ways resembles mathematical models of particle dynamics in cyclotron resonances. Moreover, the method for constructing these models is similar.

In this paper we will use another method for obtaining the conditions for the capture and acceleration (resonance conditions) of electrons in a vacuum by a field of plane electromagnetic waves of arbitrary polarization in a vacuum. This new approach allows us to more clearly clarify the physical content of the mechanism of resonant interaction of a plane wave with charged particles in a vacuum.

Below, in the second section, the problem statement is presented, expressions for the fields are given, and equations that determine the dynamics of the particles are presented. In the third section, the well-known problem of the dynamics of electrons in the field of a plane electromagnetic wave in a vacuum is considered. This wave has only one component of the wave vector. Rigorous analytical solutions are obtained that are valid in the laboratory system. These solutions do not contain the possibility of capturing and accelerating particles by the field of such a wave. There is only oscillatory dynamics of the particles. The particles become nonlinear oscillators.

The solutions obtained in this section are used in the next (fourth) section as a zero approximation in the small parameter $k_x << 1$. This section considers the presence of a small transverse component of the wave vector of the wave $k_x << k_z$. Analysis of the phase relationships of the wave and particle shows the presence of stable stationary phases. The conditions for the appearance and existence of these stable and stationary phases determine the conditions for the capture of particles by the wave field. These conditions are the conditions for the resonant interaction of the wave with electrons.

In conclusion, the most important results are formulated and the place of the obtained results among known ones is discussed.

2 Statement of the Problem and Basic Equations

Consider a charged particle that moves in the field of a plane electromagnetic wave, which in the general case has the following components:

$$\vec{E} = \mathrm{Re}(\vec{E}_0 \exp(i\omega t - i\vec{k}\vec{r})),$$
$$\vec{H} = \mathrm{Re}\left(\frac{c}{\omega}\left[\vec{k}\vec{E}\right]\exp(i\omega t - i\vec{k}\vec{r})\right) \quad . \tag{1}$$

Where $\vec{E}_0 = E_0 \cdot \vec{\alpha}$, $\vec{\alpha} = \left\{\alpha_x, i\alpha_y, \alpha_z\right\}$ - polarization vector of the wave.

Vector equation of motion of charged particles:

$$\frac{d\vec{p}}{dt} = e\vec{E} + \frac{e}{c}\left[\frac{\vec{p}}{\gamma}\vec{H}\right] . \tag{2}$$

Without loss of generality, one can choose a coordinate system in which the wave vector of the wave has only two components k_x and k_z. For follows, it is convenient to use the following dimensionless dependent and independent variables: $\vec{p} \to \vec{p}/mc$, $\tau \to \omega t$, $\vec{r} \to \omega\vec{r}/c$. It is also convenient to use the expression for the double cross product: $\left[\vec{p}\left[\vec{k}\vec{\varepsilon}\right]\right] = \vec{k}\left(\vec{p}\cdot\vec{\varepsilon}\right) - \vec{\varepsilon}\left(\vec{k}\cdot\vec{p}\right)$.

The equations of motion in these variables will be as follows:

$$\frac{d\vec{p}}{d\tau} = \left(1 - \frac{\vec{k}\vec{p}}{\gamma}\right)\mathrm{Re}\left(\vec{\varepsilon}e^{i\psi}\right) + \frac{\vec{k}}{\gamma}\mathrm{Re}\left[\left(\vec{\mathbf{p}}\cdot\vec{p}\right)e^{i\psi}\right],$$
$$\vec{v} = \frac{d\vec{r}}{d\tau} = \frac{\vec{p}}{\gamma}, \;\; \dot{\psi} = \frac{d\psi}{d\tau} = 1 - \frac{\vec{k}\vec{p}}{\gamma} \tag{3}$$

where $\vec{\varepsilon} = \varepsilon \cdot \vec{\alpha}$, $\varepsilon = (eE_0/mc\omega)$, $\psi = \tau - \vec{k}\vec{r}$, $\vec{k}$ - is the unit vector in the direction of the wave vector, $\gamma = (1+\vec{p}^2)^{1/2}$ is the dimensionless energy of the particle (measured in units mc^2), $\vec{p}$-is the momentum of the particle.

Multiplying the equation (3) by $\vec{p}$, we obtain a useful equation that describes the change in the energy of a particle:

$$\frac{d\gamma}{d\tau} = \mathrm{Re}\left(\vec{v}\vec{\varepsilon}e^{i\psi}\right) \tag{4}$$

Equations (2) and (3) have well-known integrals:

$$\vec{p} + \mathrm{Re}\left(i\vec{\varepsilon}e^{i\psi}\right) - \vec{k}\gamma =$$
$$= \vec{p}_0 - \vec{k}\gamma_0 + \mathrm{Re}\left(i\vec{\varepsilon}e^{i\psi_0}\right) = \mathrm{const} \tag{5}$$

Index "0" denotes the values of the initial variables.

3 Precise Solutions

Let us first consider the case of particle dynamics, which is widely known [6].

For this, we will assume that the wave with which the particle interacts propagates in a vacuum along the axis z. The wave vector of such a wave has only one component $\vec{k} = \left\{0,0,k_z\right\}$, $k_z = 1$. The remaining components of the wave vector are equal to zero.

Next, in the next section, we will consider that the wave has small transverse components of the wave vector ($\vec{k} = \left\{k_x,0,k_z\right\}$, $k_x << 1$). We will take this feature of the wave into account only in the phase dynamics of particles. The dynamics of the particles will change qualitatively. Conditions for their capture in the process of long-term acceleration will appear. In mathematics, such systems are known and are called ill-conditioned systems. In physics, a qualitative change in dynamics is called a bifurcation. Therefore, the parameter $k_x << 1$ can be called a bifurcation parameter.

Looking at equation (3), it is easy to see that without loss of generality in this case it is convenient to choose such components of the particle momentum $\vec{p} = \{p_\parallel, \vec{p}_\perp\}$, $p_\parallel \parallel \vec{k}$.

Let's consider that $(\vec{k}\cdot\vec{\varepsilon}) = 0$; $(\vec{\mathbf{p}}\cdot\vec{p}) = (\varepsilon_\parallel \cdot p_\parallel + \vec{\varepsilon}_\perp \cdot \vec{p}_\perp) = (\vec{\varepsilon}_\perp \cdot \vec{p}_\perp)$; $\vec{k} = \{0, 0, k_\parallel = 1\}$ ... Then the system of equations (3) takes the form:

$$\frac{dp_\parallel}{d\tau} = \frac{1}{\gamma}\mathrm{Re}\left[(\vec{\mathbf{p}}\cdot\vec{p})e^{i\psi}\right]$$

$$\frac{d\vec{p}_\perp}{d\tau} = \left(1 - \frac{\vec{k}\vec{p}}{\gamma}\right)\mathrm{Re}\left(\vec{\varepsilon}_\perp e^{i\psi}\right) = \dot{\psi}\cdot\vec{\varepsilon}_\perp\cdot\cos\psi \quad (6)$$

After dividing the left and right sides by the derivative of the wave phase ($\dot{\psi} = d\psi / d\tau$), system (6) can be rewritten:

$$\frac{dp_\parallel}{d\psi} = \frac{1}{(\gamma\cdot\dot{\psi})}\mathrm{Re}\left[(\vec{\mathbf{p}}_\perp\cdot\vec{p}_\perp)e^{i\psi}\right]$$

$$\frac{dp_x}{d\psi} = \varepsilon_x\cdot\cos\psi \qquad \frac{dp_y}{d\psi} = -\varepsilon_y\cdot\sin\psi \qquad (7)$$

Considering that in the case under consideration $(\gamma\cdot\dot{\psi}) = const \equiv C$, we easily find the following solutions for the momentum components:

$$p_x = p_x(\psi_0) + \varepsilon_x(\sin\psi - \sin\psi_0),$$

$$p_y = p_y(\psi_0) + \varepsilon_y(\cos\psi - \cos\psi_0),$$

$$\frac{dp_\parallel}{d\psi} = \frac{1}{C}\left[\varepsilon_x p_x \cos\psi - \varepsilon_y p_y \sin\psi\right] \qquad (8)$$

For simplicity, and for further comparison with known results, we will assume that $\psi_0 = 0,\ \vec{p}(\psi_0) = 0$, and that the wave has a linear polarization ($\varepsilon_x = \varepsilon,\ \varepsilon_y = 0 = \varepsilon_z$). Then (8) can be rewritten

$$p_x = \varepsilon\sin\psi, \quad p_y = \varepsilon_y\cos\psi = 0 \quad \frac{dp_\parallel}{d\psi} = \frac{\varepsilon^2}{C}\sin\psi\cos\psi;$$

$$p_z = \frac{\varepsilon^2}{2C}\cos^2\psi = \frac{\varepsilon^2}{4C}(1 + \cos 2\psi) \qquad (9)$$

To compare the obtained results with known ones [7], solutions (9) can be supplemented with expressions for the coordinates.

$$x = -\left(\frac{\varepsilon_x}{\gamma\dot{\psi}}\right)\cos\psi, \quad z = \left(\frac{\varepsilon_\perp^2}{4(\gamma\dot{\psi})^2}\right)\left(\psi - \frac{1}{2}\sin 2\psi\right) \quad (10)$$

Here $\psi_0 = 0$.

Solutions (7) and (8) practically coincide with those given by Landau [6]. The difference is that the obtained solutions are valid in the whole space, and not only in the frame of reference in which the particles are at rest on average. Moreover, the particle trajectory describes a figure eight in the momentum space (see Figure 1). In the coordinate space, in the general case, these eights are displaced (Fig. 2).

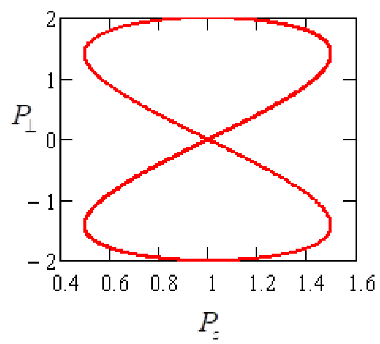

Figure 1. Particle trajectory in momentum space. $\varepsilon_x = 2,\ \varepsilon_y = \varepsilon_z = 0$ $k_x = 0.0$. Initial conditions: $x = y = z = p_x = p_y = p_z = 0$

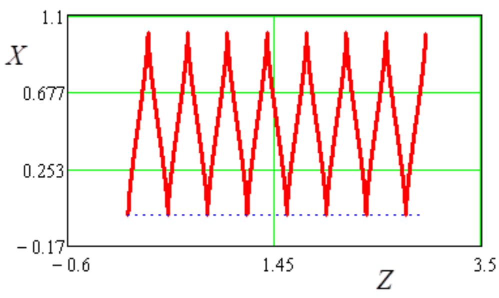

Figure 2. . Particle trajectory in coordinate space. $\varepsilon_x = 2,\ \varepsilon_y = \varepsilon_z = 0$ $k_x = 0.0$.
Initial conditions: $x = y = z = p_x = p_y = p_z = 0$

Such solutions firstly come from the works of D.M. Volkov [1] and V.I. Ritus [2]. Within the framework of classical electrodynamics, they are presented in [3,4]. Such solutions are often referred as exact solutions. These solutions, together with the results described in the monograph [6], as well as the well-known Lawson-Woodward theorem [5,7,8], have formed a widespread belief that the acceleration of charged particles by the field of electromagnetic waves in vacuum is impossible. Below we will show that these solutions do not exhaust all solutions of the problem.

4 Conditions of Particle Capture by a Field of Plane Wave in Vacuum.

In this section we will consider a small transverse component of the wave vector ($k_x << 1$). In [9-11] it was shown that taking this component of the wave vector into account can qualitatively change the dynamics of particles. Expressions were obtained for the conditions under which a plane wave in a vacuum captures electron and effectively accelerates them. These conditions show that such capture into acceleration occurs when the wave has a transverse component of the wave vector ($k_\perp \neq 0$), with a sufficiently high wave field strength, and with a sufficiently high initial longitudinal momentum.
Numerical studies show that the most important of these parameters is the presence of a transverse component of the wave number. In addition, these conditions are obtained as a result of a certain change of variables, and the conditions themselves do not explicitly contain a dependence on the magnitude of the transverse wave vector. The conditions for the emergence of any processes rarely reflect all the features of the process itself. Therefore, it can be expected that the conditions obtained in [9-11] can be change by other conditions that can more fully reflect the process of particle capture and acceleration. Let us show that such conditions can be obtained. We will start from equation (3), in which we set $k_x << 1, p_z >> 1$. The system of equations for the impulses will be practically no different from the system of equations (6):

$$\frac{dp_x}{d\tau} = \left(1 - \frac{\vec{k}\vec{p}}{\gamma}\right) \mathrm{Re}\left(\varepsilon_x e^{i\psi}\right) = \dot{\psi} \cdot \varepsilon_x \cdot \cos\psi$$

$$\frac{dp_z}{d\tau} = \frac{k_z}{\gamma} \mathrm{Re}\left[\left(\vec{\mathbf{p}} \cdot \vec{p}\right) e^{i\psi}\right] \qquad (11)$$

Where $\psi = \tau - k_z z - k_x x$, $x = -\left(\frac{\varepsilon_x}{\gamma\dot{\psi}}\right)\cos\psi$

The main difference from the system (6) is that we considered one of the transverse components of the wave vector ($k_x \neq 0$) in the expression for the wave phase. In addition, as an expression for the transverse coordinate of the particle, we took the expression that was obtained in the previous section ($k_x = 0$). Thus, using the results of the previous section, we considered that the electrons in the wave field became oscillators (nonlinear oscillators). Moreover, this fact will primarily affect the phase relationships between the wave and the particles. Therefore, below we will consider the presence of the transverse component of the wave vector only in the phase dynamics of the particles. It is easy to see that taking this value into account in other places of the equations changes practically nothing. Let us consider the expression for the phase in more detail. It can be rewritten as

$$\Phi = \tau - k_z z + k_x \left(\frac{\varepsilon_x}{\gamma\dot{\psi}}\right)\cos\Phi \qquad (12)$$

The condition for resonant interaction will be the condition of stationarity of this expression ($\Phi = const; \dot{\Phi} = 0$):

$$\dot{\Phi} = \left(1 - k_z v_z\right)\left[1 - \left(\frac{k_x \varepsilon_x}{\gamma\dot{\psi}}\right)\sin\Phi\right] \qquad (13)$$

This is the main result of this section.

The main feature of condition (13) is determined by the second factor. It can be said that condition (13), considering the second factor, represents the Adler equation. The stationary points of equation (13) (the resonant value of the phase) can be determined from (13):

$$\Phi_* = \arcsin\left(\frac{\gamma\dot{\psi}}{\varepsilon_x k_x}\right) \approx \arcsin\left(\frac{1}{2\varepsilon k_x \gamma}\right) \qquad (14)$$

The width of particle capture in acceleration (the width of nonlinear resonance) can be easily determined from the same equation (13):

$$\dot{\delta} = -\delta b \cos\Phi_* \qquad (15)$$

Where $\delta = \Phi - \Phi_*$, $b \equiv \frac{k_x \varepsilon_x}{\gamma\dot{\psi}} \approx 2k_x \varepsilon_x \gamma$.

The equations that determine the momenta of particles can be written as

$$\frac{dp_z}{d\tau} \approx \varepsilon_x \cos\Phi_*$$
$$\frac{dp_x}{d\tau} = \left[(1-v_z)+k_x\right](\varepsilon_x \cos\Phi_*) \qquad (16)$$

Their solutions:

$$p_z \approx (\varepsilon_x \cos\Phi_*)\tau, \qquad p_x = \left[(1-v_z)+k_x\right](\varepsilon_x \cos\Phi_*)\tau$$

Let us compare the obtained results with the results of previous studies.

1. First, let us compare the resonance conditions (particle capture and acceleration). In the works [9-11], the process of electron capture and acceleration by the field of a plane electromagnetic wave in a vacuum was analytically and numerically discovered (apparently for the first time). The obtained results convincingly demonstrated the possibility of particle capture and acceleration. However, the obtained mathematical models did not describe the numerical results well in all details. In particular, the following expression was obtained to determine the capture conditions

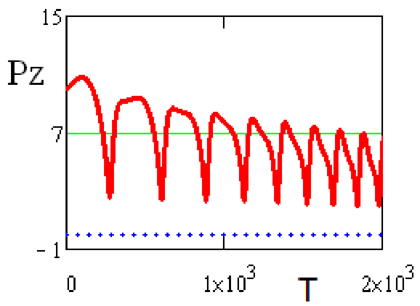

Figure 5. Time dependence of the longitudinal momentum of a particle under the following initial conditions:
$\varepsilon = 1;\ k_x = 0.1;$
$p_x = p_y = 0.1;\ p_z = 10;\ x = y = 0;\ z = 0$

$$\dot{\Phi} = (1-k_z v_z)\left[1-\left(\frac{\varepsilon}{p_\perp}\right)\sin\Phi\right] \qquad (17)$$

Formula (17) was obtained for the case of a linearly polarized wave that propagates along the axis z and that has only one component of the electric field ($\vec{E} = E_0\left\{0, i\alpha_y, 0\right\}$). From a comparison of this expression with (13), it is evident that the general structure of these conditions is the same – this is the Adler equation. The difference is that this condition (17) does not contain the main bifurcation parameter k_x. Condition (13) contains this parameter explicitly. The size of the capture region also changes.

2.The right-hand sides of the truncated equations for the momentum components in [9-11] contain Bessel functions:

$$\frac{dp_z}{d\tau} = \varepsilon \cdot v_\perp J_0(k_x p_\perp / \gamma\dot{\psi}) \cdot \cos(\Phi_*) \qquad (18)$$

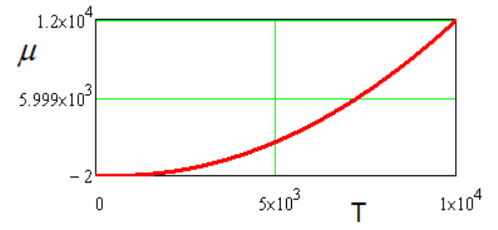

Figure 3. Argument of Bessel functions $\mu = k_x p_\perp / \gamma\dot{\psi}$, $\varepsilon = 2$ $k_x = 0.1$

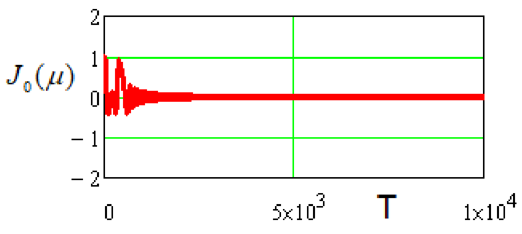

Figure 4. The Bessel functions themselves decrease rapidly
$\mu = k_x p_\perp / \gamma\dot{\psi}$, $\varepsilon = 2$ $k_x = 0.1$

However, such a feature does not manifest itself in numerical studies. Indeed, as can be seen in Figure 3, the arguments of the Bessel functions grow rapidly. The Bessel functions themselves decrease rapidly (see Fig.4). Therefore, the acceleration process quickly ceases. This contradicts the numerical results. Formulas (16) describe the numerical results much better (see Fig.6)

It should be noted that not all particles captured by the wave field are accelerated by this field. Some of them may be slowed down. As an example, Figures 5 and 6 show the time dependence of the longitudinal momentum of two particles. These particles differ only in their initial position: $z = 0$ Fig. 5 and $z = 1$ Fig. 6. The greater the wave force parameter, the fewer the slowed down particles.

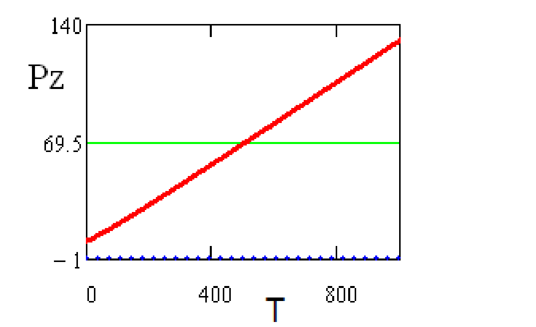

Figure 6. Time dependence of the longitudinal momentum of a particle under the following initial conditions:
$\varepsilon = 1;\ k_x = 0.1;$
$p_x = p_y = 0.1;\ p_z = 10;\ x = y = 0;\ z = 1$

It is of interest to study in more detail the condition of phase synchronism of the wave with particles (13). This condition was obtained under the condition that the transverse component of the wave vector is small, then we will assume that $k_x << 1$. We will also assume that the initial components of the particle momentum satisfy the inequalities $p_\perp << 1,\ p_z > 1$. In this case, the condition of the initial capture of particles by the wave field $k_x \varepsilon > \gamma\dot{\psi}$ will take the form $k_x \varepsilon > 1/2p_z \approx 1/2\gamma$. If this condition is not met, then complete capture does not occur (see Fig. 7). However, the presence of a non-zero transverse component of the wave vector leads to a significant increase in the oscillation amplitudes ($p_{z\max} \approx 80$). Note that at $k_x = 0$ the maximum value of the pulse is only $p_{z\max} \approx \varepsilon^2 \approx 0.25$.

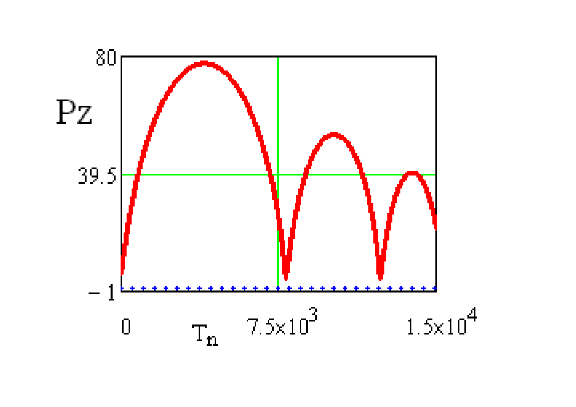

Figure 7. Time dependence of the longitudinal momentum of a particle under the following initial conditions: $\varepsilon = 0.5,\ p_z(0) = 5,\ k_x = 0.1,$
$\varepsilon k_x = 0.05,\ \gamma\dot{\psi} \approx \frac{1}{2p_z} = 0.1,\ \ \varepsilon k_x < \gamma\dot{\psi}$

Condition (13) can be easily satisfied by increasing the value of the initial longitudinal momentum. Thus, Figure 8 shows the dynamics of a particle whose initial longitudinal momentum is twice as large ($p_z(0) = 10$). It is evident that the particle is captured and accelerates throughout the entire calculation time. A further increase in the initial momentum does not change anything qualitatively.

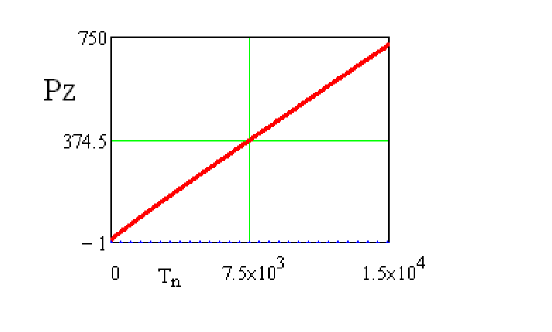

Figure 8. Time dependence of the longitudinal momentum of a particle under the following initial conditions:
$\varepsilon = 0.5,\ p_z(0) = 10,\ k_x = 0.1,\ \ \varepsilon k_x = 0.05;$
$\gamma\dot{\psi} \approx \frac{1}{2p_z} = 0.05;\ \ \varepsilon k_x = \gamma\dot{\psi}$

5 Conclusion

Let us note and discuss the most important results. The most important is the derivation of formula (13), which determines the conditions of phase synchronism of the wave and particles. In another way, similar expressions in structure were obtained in works [9-11]. However, these conditions did not include the main bifurcation parameter k_x, the appearance of which qualitatively changes the dynamics of particles - they can be captured by the wave field and effectively exchange energy with it.

Above, the main attention was paid to the acceleration of particles. However, as can be seen in Figure 5, particles can also give up their energy to the wave. This depends on the initial position of the particle relative to the wave phase. Additional analysis shows that almost all particles from the decelerating phase pass into the accelerating phase. Only the times of this transition differ.

Let us briefly formulate the mechanism of particle capture and acceleration: 1. When $k_x = 0$ the particles in the wave field become oscillators (see formulas (9) and (10)) 2. When the wave acquires a transverse component of the wave vector $k_x \neq 0$, the phase relations of the wave and particle acquire an additional term $k_x x$. 3. The conditions for resonant interaction will be the condition of phase stationarity $\dot{\psi}_N = 0$ (see (13)). This condition takes the form of the Adler equation [12,13]. Namely the appearance this additional term in expression for phase allows resonance to appear $\dot{\psi}_N = 0$.

This mechanism of occurrence of resonances is like the mechanism of occurrence of cyclotron resonances. It should be noted that all cyclotron resonances (except autoresonance) occur only if the wave vector has a transverse component ($k_x \neq 0$). Indeed, in a magnetic field, particles are transformed into oscillators. If the wave has a transverse component of the wave vector, then cyclotron rotation leads to an additional term included in the wave phase. It is this term that allows the stationarity of the phase to be realized (the condition of cyclotron resonances). Thus, the above-described resonant interactions of electrons with a plane wave in a vacuum and cyclotron resonances differ from each other only by the way born of the oscillators.

A few words should be said about the large number of works devoted to the study (theoretical and experimental) of the dynamics of electrons in the field of laser radiation (see, for example, [14-16] and the literature cited there). The structure of laser radiation can retain the influence of the features of the emitters for a long time (at large distances). This primarily applies to beams of the Gaussian type. The wave fronts of such radiation flows approach a plane wave front only at large distances from the source. In theory, such beams are described in most cases by the parabolic approximation. Such models describe many features of laser radiation well. However, it is easy to show that the spectrum of such models contains slow components ($v_{ph} < c$). Let us show this. The initial wave equation is $\Delta E + k^2 E = 0$, where $k = \omega / c$. We will solve the wave equation in the form $E = u(x, y, z)\exp(ikz)$. We substitute this solution into the wave equation and neglect the second derivative with respect to compared to the first derivative. The result will be the following equation, which is parabolic type

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2ik\frac{\partial u}{\partial z} = 0$$

This equation is the basis from which the Gaussian beam and higher beam modes are obtained. It is easy to see that the spectrum of this equation has slow components. To do this, it is enough to substitute the solution in the form $u(x, y, z) = b\exp\left[i(k_x x + k_y y + k_z z\right]$ into this equation. As a result, after simple transformations, we find $E \square \exp\left[i\left(2k \pm k\sqrt{2}\right)\right]$. These are slow components. There are no such components in the original wave equation. For this reason, when using such models in studying acceleration processes, it is necessary to take this feature into account.

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