Electrophysical Modelling of Axon in Neuron and its Analysis

Yumi Takizawa
The Institute of Statistical Mathematics, and
The Graduate University for
the Advanced Studies, JAPAN
takizawa@ism.ac.jp

Atsushi Fukasawa
Former Professor, Chiba University, JAPAN
fukasawafuji@yahoo.co.jp

Abstract: - Electrophysical modelling of axon and its analysis are given based on voltage equilibrium and current continuity conditions. Inductance *L* and capacity *C* per unit length are obtained for lossless axon. Attenuation, phase rotation, and phase (signal) velocity are given for axon with loss. It is then presented that higher velocity and wideband signal transmission are realized by reduction of *C* and *L* by myelination and large cross section of axons. Loss and phase rotation per unit length are also obtained for axon with serial resistance *r* and parallel conductance *g*. Long distance transmission is found available by reduction of serial resistance *r* and parallel conductance *g* by voltage dependent Na^{+} channels distributed at unmyelinated parts of axons.

Key-Words: - Electrophysical modelling of axon, Myelination and unmyelination in axon, Velocity and bandwidth, Attenuation, Phase rotation.

1 Introduction

Generation of electrical signal (action potential) and its transmission (conduction) to post neurons are essential functions of a neuron. Schematic diagram of a neuron is shown in Fig. 1. Dendrite and axon work as input and output ports of a cell (neuron). Input and output of axon are at hillock and at axon terminal. Synapses work for connection chemically with previous and post neurons.

Lossless and fast signal transmission were observed in myelinated and thick axons by G. Kato, 1923[1]. An early model of axon transmission was given by G. Kato and I. Tasaki, 1934[2], 1939[3]. Generation and transmission are combined in this model. Active transmission model with higher electro-magnetic mode (*Ez* mode) was given by H.S. Gasser, 1936[4]. But any scheme was not given about activity.

Afterward variety of data on velocities were measured about various axons by A. Siegel and H. Sapru, 2014[5].

In advance, it should be pointed that early neuron models do not meet electro-physical requirement.

Novel models of neuron and axon are given separately by former papers [5-10] and this paper by the authors.

This paper present electro-physical modelling of axon transmission, without and with myelins. Electro-physical modelling and equivalent circuit are first given for the axon without myelins.

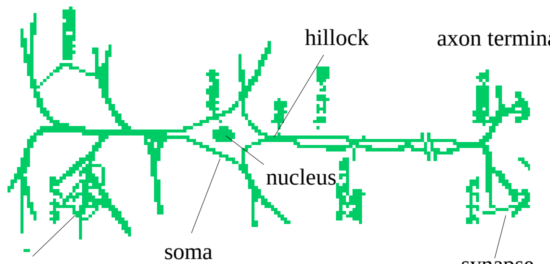

Fig. 1 Schematic diagram of a neuron.

Modelling and equivalent circuit are then given for the axon with myelins. It is estimated that myelins provide axons with fast electrical signal transmission capability by myelination.

2 Electro-physical Modelling of Unmyelinated Axon

2.1 Modelling of unmyelinated lossless axon

Electro-physical modelling of unmyelinated axon is shown in Fig. 2. Cytoplasm and external liquid are separated by the membrane. They work as an inner and an outer conductors and a dielectrics (insulator) to form a coaxial waveguide.

The cylinder coordinate system r, θ, z is used for electrical analysis. Electrical signal transmission is done by TEM (transversal electric and magnetic) mode.

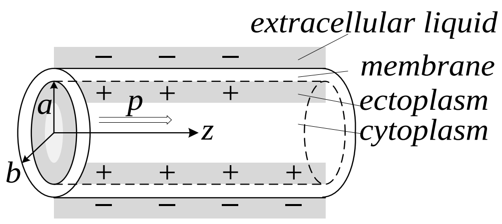

Fig. 2 Electro-physical modelling of axon.

Radial E and circular H vectors exist in cross-sectional (r, θ) plane. Poynting vector p carries signal energy (power) along z axis, where,

$$p = E \times H \qquad (1)$$

where, E, H, p are defined clockwise.

Radius of conductors a and b are shown in Fig.2. In dielectric medium, radial length dr is considered at point r.

True electric charges ± q are given on the inner and outer conductors per unit length. Capacitance C is calculated as follows.

$$E = \frac{q}{2\pi\varepsilon\, r} \quad \text{[V/m]} \qquad (2)$$

$$v = \int_a^b E\, dr = \int_a^b \frac{q}{2\pi\varepsilon\, r}\, dr \qquad (3)$$

$$= \frac{q}{2\pi\varepsilon} \ln\frac{b}{a} \quad \text{[V]} \qquad (4)$$

where, E, v are electric field strength in the space and the potential difference (voltage) of inner and outer liquid conductors. Capacity per unit length is given as ;

$$C = \frac{q}{v} = \frac{2\pi\varepsilon}{\ln\frac{b}{a}} \quad \text{[F/m]} \qquad (5)$$

Inductance L is calculated as follows.

$$H = \frac{\mu i}{2\pi\, r} \quad \text{[H/m]} \qquad (6)$$

$$\Phi = \int_a^b H\, dr = \int_a^b \frac{\mu i}{2\pi r}\, dr \quad \text{[Wb]} \qquad (7)$$

where, H, i are magnetic field strength in the space and the current of inner and outer liquid conductors. B and Φ are density and total flux of magnetic field strength in space between inner and outer liquid conductors.

Inductance per unit length is given as;

$$L = \frac{\Phi}{i} = \frac{\mu}{2\pi} \ln\frac{b}{a} \quad \text{[H/m]} \qquad (8)$$

It is pointed that permittivity ε and permeability μ in MKSA system of cytoplasm and outer liquid are as follows;

$$\varepsilon = \varepsilon_0 \varepsilon_r \quad \text{[F/m]} \qquad (9)$$

$$\varepsilon_0 = \frac{4}{3}\pi \times 10^{-12} \quad \text{[F/m]} \qquad (10)$$

and,

$$\mu = \mu_0 \mu_r \quad \text{[H/m]} \qquad (11)$$

$$\mu_0 = 4\pi \times 10^{-7} \quad \text{[H/m]} \qquad (12)$$

$$\mu_r = 1, \qquad (13)$$

and,

$$\varepsilon_0\, \mu_0 = \frac{1}{c_0^{\,2}} \qquad (14)$$

here, c_0 is the light velocity in vacuum.

Usual voltage of action potential of a neuron is about 100 mV. But current value is so little that the current flows almost the peripheral of inner conductor.

When the distribution is assumed exponential, effective depth δ of penetration is given as;

$$\delta = \sqrt{\frac{2}{\omega\sigma}} \quad \text{[m]} \qquad (15)$$

where, σ is conductance of inner and outer liquid.

2.2 Modelling of unmyelinated axon with loss

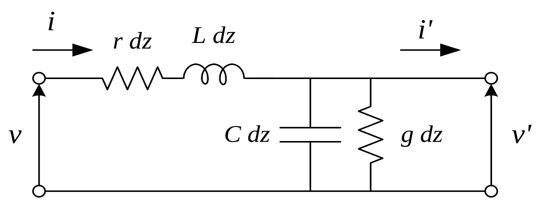

Fig. 3 Equivalent circuit of unmyelinated axon of length *dz* with loss.

The equivalent circuit of the model is shown in Fig.3. The following equations are obtained from conditions of voltage equilibrium and current continuity.

$$v = L\,dz\,\frac{di}{dt} + i\,rdz + v' \quad [\mathrm{V}] \qquad (16)$$

$$i = C\,dz\,\frac{dv'}{dt} + g\,dz\,v' + i' \quad [\mathrm{A}] \qquad (17)$$

and,

$$v - v' = dv \qquad [\mathrm{V}] \qquad (18)$$

$$i - i' = di \qquad [\mathrm{A}] \qquad (19)$$

From Eqs. (16) ~ (19),

$$\frac{dv}{dz} = L\frac{di}{dt} + i\,r \qquad [\mathrm{V/m}] \quad (20)$$

$$\frac{di}{dz} = C\frac{dv'}{dt} + g\,v' \qquad [\mathrm{A/m}] \quad (21)$$

From Eq. (20) and (21),

$$\frac{d^2v}{dz^2} = L\frac{d}{dz}\left(\frac{di}{dt}\right) + r\frac{di}{dz} \qquad (22)$$

$$\frac{d}{dt}\left(\frac{di}{dz}\right) = C\frac{d}{dt}\left(\frac{dv'}{dt}\right) + g\frac{dv'}{dt} \qquad (23)$$

From (22) and (23),

$$\frac{d^2v}{dz^2} = LC\frac{d^2v'}{dt^2} + (gL + rC)\frac{dv'}{dt} + rgv' \quad (24)$$

When *v* is sinusoidal, it is written as,

$$v = \exp(j\omega t) \qquad [\mathrm{V}] \qquad (25)$$

$$\frac{d^2v}{dt^2} = -\omega^2 v \qquad (26)$$

v' is replaced by *v* if *dv*/*v'* is small enough.

$$\frac{d^2v}{dz^2} = \{-\omega^2 LC + j\omega(gL + rC) + rg\}\,v \quad (27)$$

$$= \left\{(j\omega\sqrt{LC})^2 + j\omega\sqrt{LC}\,\left(\frac{r}{\sqrt{\frac{L}{C}}} + \frac{g}{\sqrt{\frac{C}{L}}}\right) + rg\right\} v$$

(28)

$$\frac{d^2v}{dz^2} \approx \left\{\frac{1}{2}\left(\frac{r}{\sqrt{\frac{L}{C}}} + \frac{g}{\sqrt{\frac{C}{L}}}\right) + j\omega\sqrt{LC}\right\}^2 v \quad (29)$$

General solution of *v* on time and space is given as follows;

$$\exp(-j\omega t)\,v = v_1\exp\{-j(\omega t + \gamma z)\}$$
$$+ v_2\exp\{-j(\omega t - \gamma z)\} \quad (30)$$

It is found that v_1 and v_2 are components of signals transmitting forward and backward directions of *z*. Now, γ is transmission coefficient of transmitting wave signals, α and β are attenuation and phase (rotation) constants of γ. Attenuation α is corresponding to toransmission loss.

$$\gamma \approx \frac{1}{2}\left(\frac{r}{\sqrt{\frac{L}{C}}} + \frac{g}{\sqrt{\frac{C}{L}}}\right) + j\omega\sqrt{LC} \quad (31)$$

$$= \alpha + j\beta \qquad (32)$$

Phase velocity c is defined by time-differential of equal phase points along z axis.

$$\frac{d}{dt}(\omega t + \gamma z) = \omega + \gamma \frac{dz}{dt} = 0 \qquad (33)$$

$$c = \frac{dz}{dt} = \left| \frac{\omega}{\gamma} \right| \qquad [\text{m/s}] \qquad (34)$$

$$\approx \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \qquad [\text{m/s}] \qquad (35)$$

Transmission time T of length l is given as ;

$$T = \frac{l}{c} \qquad [\text{s}] \qquad (36)$$

where, r, g are assumed small enough compared to $\sqrt{L/C}$, $\sqrt{C/L}$ respectively. Lossless transmission is realized when r and g are zero approximately.

It is found that velocity c depend on dimensions a and b of the axon. It is also found that long transmission is available by the aid of positive ions injected in the axon.

3 Electro-physical Modelling of Myelinated Axon

3.1 Electrical scheme of myelinated axon

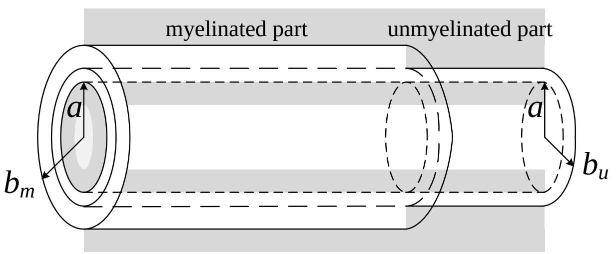

Fig. 4 Electro-physical modelling of axon with myelinated and ummyelinated parts.

Myelinated axon is composed by long myelinated and short unmyelinated parts as shown in Fig.4. A myelinated part is made by a glia cell. The distance between inner and outer conductors is expanded with multiple windings of a glia cell. Unmyelinated part appears between adjacent myelinated parts caused by adjacent glia cells.

The capacity Cm at myelin part is reduced to be less values than the capacity Cu at unmyelinated part in Fig. 5. On the other hand, inductance L is reduced as low as the cross sectional dimension increases.

Na^+ ions are injected into the axon through Na+ channels distributed at each unmyelinated part.

3.2 Characteristics of Myelinated axon

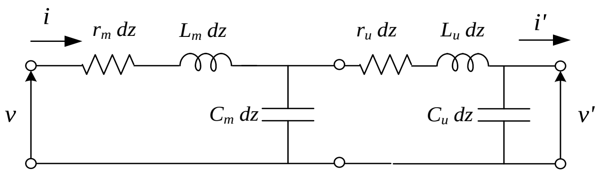

Fig. 5 Equivalent circuit of axon with myelinated and ummyelinated parts. L_m, C_m, and L_u, C_u are circuit parameters corresponding to myelinated and unmyelinated parts of an axon.

Myelinated axon is shown in Fig. 5. L_m and C_m are inductances and capacitance of the myelinated part. L_n and C_n are inductances and capacitance of unmyelinated part.

The following relations are pointed as follows;

$$Lm = Lu = L \qquad (37),$$

$$Cm << Cu = C \qquad (38),$$

and,

$$lm >> lu \qquad (39)$$

then,

$$\omega_C\,(myelinated) >> \omega_C\,(\text{un}myelinated) \qquad (40)$$

It is found that transmission of wideband signals through long axons is realized by myelination of axon. Then sharp pulses are realized to transmit for long distance by myelination.

It is concluded that larger information per unit time is realized by myelination.

Wideband transmission is realized by the effect of reduction in L and C.

Long distance transmission is realized by reduction of attenuation by increase of conductance.

3.3 Cutoff angular frequency

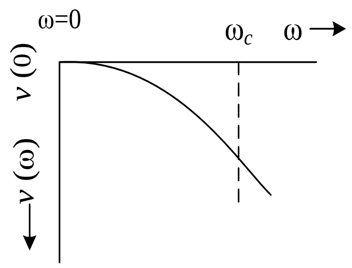

Fig. 6 Frequency characteristics and cut off ω_C.

When transmission coefficient γ is complex number;

$$\gamma(\omega) = \alpha(\omega) + j\beta(\omega) \qquad (41)$$

$\alpha(\omega)$ and $\beta(\omega)$ are attenuation and phase shift of signal transmission.

Signal transmission characteristics depends on the frequency ω. The frequency characteristics of signal transmission is expressed by Fig 6.

At the end of line $z = l$,

$$v(\omega, l) = v(0, l)\exp\{-(\alpha + j\beta)\omega\} \quad (42).$$

ω_C is the cutoff angular frequency at the point, where amplitude decreases equal to $1/e$ of $v(0)$. e is the basis of natural logarithm.

$$\left|v(\omega_c, l)\right| = \exp\{-\frac{1}{e}v(0, l)\} \qquad (43)$$

$$\omega_c = 1/\sqrt{LC} \qquad (44)$$

It is found that cutoff angular frequency ω_C increases depending on product values of LC.

3.4 Long distance transmission

It was found by G. Kato that Na^+ channels distribute at discrete points of unmyelinated part of axon. It shows that ε could be controlled by influx and efflux of positive ions. If positive ions are taken much into axon, effective ε decreases lower, and reduction of transmission time τ of signal.

Many ion channels could exist at large diameter of axon. It is concluded that the signal transmission (phase) velocity becomes higher as increase of size of cross section of axon.

4 Higher Mode Transmission with the Axon

If wavelength λg of transmitting wave signals is longer enough compared a half of spacing $b - a$, only Transversal Electric and Magnetic (TEM) mode is transmitted. TEM mode is the dominant mode of coaxial line.

If wavelength λg is shorter than a half of spacing $b - a$, higher mode is excited from the dominant mode.

$$b - a > \frac{1}{2}\lambda_g \qquad (45)$$

$$\lambda_g = \frac{1}{\sqrt{\varepsilon}}\lambda_0 \qquad (46)$$

where, λ_0, λ_g are the wave length of signal in free space and in liquid.

Typical higher mode Ez is excited inside the line. Longitudinal electric vector Ez occurs along z axis. However if higher mode is brought from dominant TEM mode, not only signal power varies but also higher mode works as noise against signal, it causes data error and reduction of reliability of communication systems.

The electro-saltatory transmission along an axon with myelin was presented by I. Tasaki (1939)[7]. He proposed that rapid transmission with saltatory effect, which means leaping transmission of current between myelins outside the axon. This provides unreasonable result of signal transmission not inside but outside the axon.

The electro-saltatory transmission implies signal transmission by Ez component provides not only low reliable transmission but also severe crosstalk among parallel axons in a neural system.

5 Conclusion

The fundamental characteristics of axons are first analyzed in this paper.

It is clarified that signal transmission (phase) velocity in axon depends on inductance L and capacitance C of axon which are defined by myelination and cross-sectional dimension of axon.

It is also clarified that wideband signal and long distance transmission are realized by myelination, and large cross-sectional dimension (thick) of axons

It is also presented that increase of conduction of the axon is brought by injection of positive ions through ion channels distributed at unmyelinated part of axon.

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